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RAY TRACING IN AXISYMMETRIC
TOROIDAL COORDINATES

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RAY TRACING IN AXISYMMETRIC TOROIDAL COORDINATES[†]

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ABSTRACT

A method is presented for determining the trajectory of a particle moving along a straight line orbit in axisymmetric toroidal coordinate systems. These geometries occur in problems associated with neutral, neutron and radiation transport studies in tokamak fusion devices. Numerical solutions of the equations describing the trajectory are performed in geometric configurations generated by the solution of the plasma equilibrium problem. An example problem of the deposition of a pencil beam of high energy neutral particles in a tokamak plasma which has a non-circular cross section due to the necessity of incorporating a divertor and the desirability of operating at a high plasma energy density is described.

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I. Introduction

Radiation and neutral particle transport are important mechanisms of energy transport in tokamaks. Although these processes are not intimately associated with the topology of the magnetic field, as is the transport of ions and electrons, they are strongly affected by its presence because the magnetic flux defines surfaces on which the properties of the plasma are approximately constant. Consequently, emissivities, absorptivities and other properties of the medium that are required in the solution of transport problems are constant on the nested set of axisymmetric toroidal flux surfaces in the tokamak. Their presence introduces significant toroidal effects which in many instances cannot be ignored. The importance of these effects increases as the shape of the flux surfaces departs from simple circles and as the toroidal curvature increases. Consequently, analyses of such problems as the spatial distribution of the deposition of high energy neutral beam particles based on circular approximations⁽¹⁾ will not be valid in more advanced tokamak systems in which highly non-circular flux surfaces are present. Examples of these are fusion reactors, which require divertors for the control of boundary effects and systems that are designed to operate at high β (ratio of plasma pressure to magnetic field energy density) such as the Doublet⁽²⁾ and the recently proposed Flux Conserving Tokamak.⁽³⁾

The analysis of neutral transport problems (we use the term, neutral transport, generically to differentiate from charged particle transport and include neutrons and photons in this category) in tokamak plasmas is based on solution of the integral transport equation

$$I(s) = I(0) e^{-\int_0^s \kappa ds'} + \int_0^s \epsilon e^{-\int_{s'}^s \kappa ds''} ds', \quad (1)$$

in which the integration is performed along the trajectory of the neutral particle. The properties of the medium that appear in equation 1, are known functions of the plasma properties which in turn are defined on the flux surfaces of the plasma; consequently, the solution of equation 1 requires that a means be found to determine in what manner these properties vary along the particle's trajectory. We present such a method in this paper.

The trajectories of the neutral particles are straight lines, and consequently they travel along the geodesics of the three-dimensional Euclidean space, E_3 , in which the problem is defined. The method that we have developed solves the equations defining a geodesic curve of E_3 in axisymmetric toroidal coordinates, thereby determining what flux surface the particle is on and consequently, what the properties of the medium are at each point of the trajectory. In fact, the method allows the coordinate

surfaces to be defined directly by the numerical solution of the toroidal equilibrium problem and is thus ideally suited to application in plasma studies based on these solutions and for incorporation in flux coordinate plasma transport codes.^(4,5,6)

The spatial distribution of the deposition of energetic neutral beam particles is one of several problems treated by equation 1 and for which our method is useful. Other examples are the determination of the distribution of neutron loading on the first wall of a fusion reactor,⁽⁷⁾ and the calculation of the transport of cyclotron radiation in tokamaks.⁽⁸⁾ As the geodesic directions or particle paths are the characteristic directions of the transport equation, this solution also facilitates the application of the method of characteristics⁽⁹⁾ to the solution of the transport equation in axisymmetric toroidal geometries.⁽¹⁰⁾

In section II the geodesic equations in axisymmetric toroidal coordinates are derived. This derivation is not specific to flux surface geometries but is applicable to any coordinate system that is generated by rotating a nest of smooth curves about an axis that does not intersect the nest. We will restrict our attention to flux coordinate geometries (section III) and outline the numerical solution of the resulting set of ordinary differential equations. Section IV is devoted to an illustrative problem, the deposition of an injected pencil beam of high energy neutral particles in a tokamak plasma of highly non-circular cross

section, and we conclude, (section V), with some comments on the application of this method to other problems.

II. Derivation of the Geodesic Equations

The derivation is based on the minimization of a quadratic functional representing the distance between two points in E_3 .⁽¹¹⁾ We begin with the expression for the element of arc length in cylindrical coordinates,

$$ds^2 = dr^2 + dz^2 + r^2 d\phi^2 \quad (2)$$

and introduce the axisymmetric toroidal coordinates (ρ, θ, ϕ) by the transformation

$$\begin{pmatrix} dr \\ dz \end{pmatrix} = \begin{pmatrix} r_\rho & r_\theta \\ z_\rho & z_\theta \end{pmatrix} \begin{pmatrix} d\rho \\ d\theta \end{pmatrix} , \quad (3)$$

which we write as

$$d\underline{v} = \underline{\underline{A}} \, d\underline{u} \quad (4)$$

(The coordinate systems are illustrated in Fig. 1).

Under this transformation the element of arc becomes

$$ds^2 = \langle A \underline{du}, A \underline{du} \rangle + r^2 d\phi^2, \quad (5)$$

where $\langle \quad \rangle$ indicates an inner product. The quadratic functional to be minimized is

$$L = \int_{S_0}^1 \left[\langle A \frac{du}{ds}, A \frac{du}{ds} \rangle + r^2 \left(\frac{d\phi}{ds} \right)^2 \right]^{1/2} ds. \quad (6)$$

The Euler equations for $\underline{u}$ and ϕ which minimize L are⁽¹¹⁾

$$\frac{d}{ds} Q_v - Q_v = 0, \quad (7)$$

where

$$Q = \langle A \frac{du}{ds}, A \frac{du}{ds} \rangle + r^2 \left(\frac{d\phi}{ds} \right)^2 \quad (8)$$

and v is either of the argument functions $\underline{u}$ or ϕ - dot indicates differentiation with respect to the parameter S . The auxillary condition $Q=1$ has been used without loss of generality. Differentiation with respect to ϕ gives,

$$r^2 \frac{d\phi}{ds} = k \quad (9)$$

where k is a constant along the trajectory, and the equation for $\underline{u}$ is found to be

$$\underline{A} \frac{d^2 \underline{u}}{ds^2} + \frac{d\underline{A}}{ds} \frac{d\underline{u}}{ds} - r(\underline{A}^T)^{-1} \frac{\partial r}{\partial \underline{u}} \left(\frac{d\phi}{ds} \right)^2 = 0 \quad (10)$$

We note that

$$\frac{\partial}{\partial \underline{u}} = \underline{A} \frac{\partial}{\partial \underline{v}} \quad , \quad (11)$$

and consequently the third term in equation 10 becomes

$$- \underline{q} \equiv - \begin{pmatrix} r \left(\frac{d\phi}{ds} \right)^2 \\ 0 \end{pmatrix} \quad (12)$$

Equation 10 may be integrated directly in the (r, ϕ, z) cylindrical coordinates. That is we write equation 10 as

$$\frac{d^2 \underline{v}}{ds^2} = \begin{pmatrix} k^2 / r^3 \\ 0 \end{pmatrix} \quad , \quad (13)$$

which may be decomposed into

$$\frac{d^2 z}{ds^2} = 0 \quad , \quad (14)$$

and

$$\frac{d^2 r}{ds^2} = \frac{k^2}{r^3} \quad (15)$$

Equations 14 and 15 are integrated to give

$$Z = \alpha S + \beta, \quad (16)$$

and

$$c_1 r^2 = (c_1 S - c_1 c_2)^2 + k^2, \quad (17)$$

where α, β, c_1 and c_2 are integration constants.

If the transformation between r and z and ρ and θ can be prescribed analytically in the form

$$r = r(\rho, \theta) \quad (18a)$$

$$Z = Z(\rho, \theta) \quad (18b)$$

then equations 16 and 17 form an implicit specification of ρ and θ as functions of s . Unfolding these equations to explicit form is a considerable undertaking even for the simplest geometries⁽¹²⁾ and for complicated geometries may not be possible analytically. To make matters worse, in most cases of interest here, the inverse transformation to (18) is known instead, (for example the flux function Ψ will be given as the solution of the equilibrium problem, see section III). Returning therefore to the (ρ, θ, ϕ) system of coordinates we have

$$\frac{d}{ds} \left(\underline{A} \frac{du}{ds} \right) = \underline{q} \quad (19)$$

in which we substitute equation (17) to find

$$\frac{d}{ds} \left(\underline{A} \frac{du}{ds} \right) = \left(\begin{array}{c} k^2 \left[\frac{c_1}{(c_1 s - c_1 c_2)^2 + k^2} \right]^{3/2} \\ 0 \end{array} \right) \quad (20)$$

Equation 20 is integrated once yielding

$$\underline{A} \frac{du}{ds} = \underline{I} \quad , \quad (21)$$

where

$$\underline{I} = \left(\begin{array}{c} (c_1 s - c_1 c_2) \sqrt{\frac{c_1}{(c_1 s - c_1 c_2)^2 + k^2}} \\ \alpha \end{array} \right) . \quad (22)$$

In the next section we will outline our numerical solution of equation 21 and its application to the solution of equation 1, the transport equation.

III. Solution for u in Flux Geometries

Our principal concern is in performing the indicated integrations over the particle path length, s , which is a single valued parameter that increases monotonically along the particle path. In some geometries the substitution of

an alternately defined parameter, that is, single valued except at a finite number of isolated points will facilitate this integration. Ideally, the parameter of choice is the flux coordinate Ψ itself (or some other appropriate flux surface label) since n , σ and the source functions are defined as functions of Ψ . This choice however does not allow for a simple analytical solution except in some particular cases and instead we use a numerical solution procedure. Rome et al. have performed an analysis much in this manner,⁽¹⁾ choosing the distance between the principal axis of the torus and the position along the trajectory for a parameter; their analysis treated a beam injected in a plane parallel to the plane $Z=0$.

Our approach is to stratify the interval over which the integrals in equation 1 are performed in such a manner that each incremental interval, Δs_i , is associated with a predetermined increment of flux $\Delta \Psi_i$ about a characteristic value $\tilde{\Psi}_i$. For suitably small $\Delta \Psi_i$ we evaluate the properties of the plasma at $\tilde{\Psi}_i$ and presume them constant over the interval. The quadrature can then be performed using any of several standard techniques. The first of equations 21, written as

$$d\psi = (a_{11} I_1 + a_{12} I_2) ds, \quad (23)$$

where $\underline{a} \equiv \underline{A}^{-1}$, provides the prescription for finding the quadrature intervals Δs_i . We restrict our attention to the

case in which the ρ coordinate is identified with the poloidal flux, Ψ , found as the solution of the MHD equilibrium problem,⁽¹³⁾ and the θ coordinate is chosen such that the component of the magnetic field in the θ direction is the poloidal field. In this case,

$$\underline{a} = \begin{pmatrix} r B_z & -r B_r \\ B_r/B_\theta & B_z/B_\theta \end{pmatrix}, \quad (24)$$

where B_z , B_r and B_θ are the z , r and total poloidal components of the magnetic field. Substituting in equation 23,

$$d\Psi = -r(B_z I_1 - I_2 B_r) ds. \quad (25)$$

The solution of the trajectory equation in the (r, z, ϕ) cylindrical system allows us to evaluate Ψ at any point from the solution of the MHD equilibrium problem and consequently B_r , B_z and B_θ as well. Once a starting point has been chosen, equation (23) determines the next quadrature point at which the new value of $\bar{\psi}$ is found. In practice the integrals in equation (1) are evaluated simultaneously - only a single traverse of the trajectory is required for problems in which the plasma is purely absorbing, (deposition of an energetic neutral beam) or purely emitting, (distribution of the neutron wall load distribution).

The expression,

$$r(B_z I_1 - B_r I_2)$$

will vanish if $r=0$ which is physically inaccessible or at points at which either

$$\frac{B_r}{B_z} = \frac{I_1}{I_2} \quad (26)$$

or

$$B_r^2 + B_z^2 = B_\theta^2 = 0 \quad (27)$$

We call such points turning points of the trajectory. They are characterized by the trajectory being tangent to a flux surface, (equation 26), or passing through a magnetic axis, (equation 27), and in the (ψ, θ) plane by a reversal in direction relative to ψ , i.e., from increasing ψ to decreasing ψ or visa-versa. In practice a particular trajectory may have a maximum of three turning points, (excluding $r=0$), if the flux surfaces are everywhere convex and must have a minimum of one in all cases. Physically, the turning point occurs in a vicinity in which the path length within a particular flux shell is long, and in practice we locate the zero and span the interval from the preceding to next flux surface intersection.

The second of equations 21,

$$\frac{d\theta}{ds} = \frac{1}{B_\theta} \left[B_r I_1 + B_z I_2 \right] \quad (28)$$

is integrated to find the θ values associated with each point at which the ray in question intersects the flux surfaces, and from $\Psi(s)$ and $\theta(s)$ a locus of intersection points may be constructed as in Fig. 2.

In using this method for investigating the deposition of neutral beam particles, it is also desirable to know the pitch angle relative to the magnetic field lines at each point on the locus (neutral particle trajectory). This information is necessary for the investigation of the orbit dynamics of the energetic ions produced by ionization and charge exchange events. The pitch angle is defined as the angle between the propagation vector of the ray and the unit vector in the direction of the magnetic field. Consequently, this angle, ω , is defined by the inner product of the vector that has components $\frac{d\psi}{ds}$, $\frac{d\theta}{ds}$, $\frac{d\phi}{ds}$ and the magnetic field vector which has components B_θ and B_ϕ in the (ψ, θ, ϕ) coordinate system. It is easily shown that

$$\cos \omega = \frac{(B_r I_1 + B_z I_2) + k B_\phi}{\sqrt{B_\theta^2 + B_\phi^2}} \quad (29)$$

IV. Neutral Particle Deposition - An Illustrative Example

The technique presented in the previous section is well suited to the job of determining the deposition profile of a pencil beam of neutral particles injected into a tokamak of non-circular cross section. As an example consider the flux contours of the plasma of a double-null divertored tokamak⁽¹⁴⁾ shown in Fig. 3. These flux contours have been generated by the numerical solution of the MHD equilibrium problem using the Princeton Equilibrium Code.⁽¹⁵⁾ The pressure profile associated with this equilibrium is also shown in Fig. 3. For the deposition problem the plasma is assumed to be purely absorbing and equation 1 is written as:

$$\Phi = \Phi_0 e^{-\int_0^s n \sigma ds'} \quad (30)$$

The total absorption cross section is due to degradation of the beam flux by impact ionization and by charge exchange. For the case of beam energies in excess of 100 keV and plasma ion temperatures below 10 keV a suitable fit for the cross section data is given by⁽¹⁶⁾

$$\sigma = 8.29 \times 10^{-12} \ln (.166 E)/E ; E > 150 \text{ keV} \quad (31a)$$

$$\ln \sigma = - 2.006 (\ln E)^2 + 8.156 \ln E - 15.128; E \leq 150 \text{ keV} \quad (31b)$$

where E is the energy of the injected particle.

If an injection energy of 200 keV is assumed the 1/2 energy and 1/3 energy components of the beam which can account for as much as 45.5% of the injected power will not meet the requirements of equations (31). In this case the relative velocity is averaged over an assumed Maxwellian distribution for the plasma and the Maxwellian averaged cross section is obtained.⁽¹⁶⁾ Beam deposition profiles have been obtained for a variety of injection angles ranging from normal injection, (i.e., in the planes $\phi = \text{constant}$, $Z=0$) to injection tangent to the magnetic axis and from locations off the equatorial plane. As the pressure is increased the magnetic axis shifts outward, and the flux surfaces are further distorted from a circular configuration. Figures 4 and 5 give beam deposition profiles for two values of $\langle\beta\rangle$ for the plasma (ratio of average pressure to magnetic field energy density), and several injection geometries.

We see from this illustrative example that a variety of deposition profiles are obtainable and that the design of the neutral beam injection scheme can be tailored to suit the particular requirements of the desired plasma within a range of options. Alternatively the positioning of neutral injectors may be dictated by physical accessibility requirements, and we are in a position to determine the effect of changing the location and angle of injection on the resulting deposition profile.

V. Concluding Remarks

The analysis of the previous section was confined to the consideration of a pencil beam; the spatial and angular distribution of actual beams have not been considered. These effects can be treated by summing over an ensemble of pencil beams, and as the computational expense on the CDC-7600 is ~1 sec, this process is not prohibitive.

Other problems can also be treated using the ray tracing method developed here. Perhaps the most well known of these is the determination of the spatial and angular distribution of the neutron wall loading on the first wall of a fusion reactor.⁽⁷⁾ Other applications of this method are the calculation of the emission of synchrotron radiation from a hot plasma and in determining the characteristics of the transport-equation, (direction of neutron streaming), relative to toroidal flux surfaces. The former problem becomes increasingly important at high field and high temperature while the latter is particularly relevant in applying transport equation solvers to the analysis of neutral transport problems in Tokamak Plasmas.

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Fig. 1. Toroidal coordinates (ρ, θ, ϕ) ; $\hat{n}$ a unit vector normal to the constant ρ surface.

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 Fig. 2. Neutral particle trajectory in (ψ, θ) space.

Fig. 3. Flux surfaces, (in const. ϕ plane), pressure profile and current profile of a diverted tokamak.

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Fig. 4. Fraction deposition profiles: 1 - Tangential (to magnetic axis) injection; 2 - Injection at 45° from radial direction; 3 - Radial injection; 4 - Skew injection.

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 Fig. 5. Fraction deposition profiles: 1 - Tangential (to magnetic axis) injection; 2 - Injection at 45° from radial direction; 3 - Radial injection; 4 - Skew injection.