TRANSFORMED GOVERNING EQUATIONS FOR TRIANGULAR GRID
STRUCTURE USED IN THERMAL-HYDRAULIC SUBCHANNEL ANALYSIS

by

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October 1976

Base Technology

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ABSTRACT

The governing equations for flow and temperature distribution in rod bundles utilizing the boundaries of the triangular mesh system are developed in the present report. The coordinate transformation is non-orthogonal and uses the natural boundaries of the mesh. Three linearly independent equations of momentum and energy are formulated in the present development.

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I. Introduction

The governing equations for flow and temperature distributions in rod bundles are based on a lumped parameter approach developed in the past by Zermik [1], Bowring [2], Rowe [3,4,5], Wantland [6] and Sha and Schmitt [7] for various grid structures or subchannel arrangements. The purpose of this development is to present a consistent set of equations utilizing a natural coordinate system. The natural coordinate system utilizes the boundaries of triangular mesh coordinate axes as shown in Fig. 1. The coordinate system is non-orthogonal; however, it allows for the natural mesh boundaries, well established in the field of rod bundle thermal hydraulics, to be used. The concept of using a coordinate transformation from the usual rectangular system of coordinates versus one that more readily utilizes the physical geometrical constraints has long been recognized in continuum field theory. The development of such a transformation in rod bundle thermal hydraulics provides significant advantages over presently developed systems. The use of natural or intrinsic coordinates allows for physical variables to be defined or calculated across triangular mesh interfaces. These can then be more easily compared with well established experimental data which rely heavily on triangular mesh considerations. In addition, the most significant advantage occurs in the formulation provided by three linearly independent momentum equations, one in the axial direction and two in the direction specified by the boundaries of the triangular mesh. Although THI-3D offers an overall system momentum balance the use of the present formulation provides for a local conservation of momentum for each of the individual triangular control volumes without redundancy or change of control volume. This alleviates the concern of possible momentum excess or shortage locally associated with formulations presently in use which change control volume from a

Fig. 1. Triangular Subchannel Arrangement

triangle to a rectangle and use a one-dimensional approach to balance the transverse momentum.

II. Coordinate Transformation

In a given three-dimensional region the rectangular coordinates (x, y, z) of any point can be expressed as functions of (u_1, u_2, u_3) so that

$$x = x(u_1, u_2, u_3), \quad y = y(u_1, u_2, u_3), \quad z = z(u_1, u_2, u_3) \quad (1)$$

where u_1, u_2, u_3 , are three independent, continuous, single valued functions.

Conversely, these can be expressed as functions of the rectangular coordinates:

$$u_1 = f_1(x, y, z), \quad u_2 = f_2(x, y, z), \quad u_3 = f_3(x, y, z) \quad (2)$$

The functions in (1) and (2) are assumed to be single-valued and to have continuous derivatives so that the correspondence between the rectangular system (x, y, z) and the transformed system (u_1, u_2, u_3) is unique. The appropriate set of eqs. (1) or (2) defines a transformation of coordinates. If

$\underline{r} = x \underline{i} + y \underline{j} + z \underline{k}$ is a position vector of a point P as shown, then correspondingly $\hat{\underline{r}} = \hat{\underline{r}}(u_1, u_2, u_3)$ will represent a position vector in the transformed system. The tangent vectors to u_1, u_2, u_3 , are given by

$$\underline{\alpha}_1 = \frac{\partial \hat{\underline{r}}}{\partial u_1}, \quad \underline{\alpha}_2 = \frac{\partial \hat{\underline{r}}}{\partial u_2}, \quad \underline{\alpha}_3 = \frac{\partial \hat{\underline{r}}}{\partial u_3}, \quad (3)$$

The unit tangent vectors are given by

$$\underline{e}_1 = \frac{\partial \hat{\underline{r}}}{\partial u_1} / \left| \frac{\partial \hat{\underline{r}}}{\partial u_1} \right|, \quad \underline{e}_2 = \frac{\partial \hat{\underline{r}}}{\partial u_2} / \left| \frac{\partial \hat{\underline{r}}}{\partial u_2} \right|, \quad \underline{e}_3 = \frac{\partial \hat{\underline{r}}}{\partial u_3} / \left| \frac{\partial \hat{\underline{r}}}{\partial u_3} \right| \quad (4)$$

For the transformation under consideration, a simple rotation of the y axis to u_2 , eqs. (1) and (2) respectively become

$$x = u_1 \quad y = u_2 \csc \alpha + u_1 \cot \alpha \quad z = u_3 \quad (5)$$

$$u_1 = x \quad u_2 = -x \cos \alpha + y \sin \alpha \quad u_3 = z \quad (6)$$

The unitary base vectors given by eq. (3) become

$$\underline{\alpha}_1 = \underline{i} + \cot \alpha \underline{j}$$

$$\underline{\alpha}_2 = \csc \alpha \underline{j} \quad (7)$$

$$\underline{\alpha}_3 = \underline{k}$$

The angles θ_{12} , θ_{23} , θ_{31} , between the coordinate curves u_1 , u_2 , u_3 , are given

$$\text{by} \quad \cos \theta_{12} = \frac{g_{12}}{\sqrt{g_{11} g_{22}}} \quad \cos \theta_{23} = \frac{g_{23}}{\sqrt{g_{22} g_{33}}} \quad \cos \theta_{31} = \frac{g_{31}}{\sqrt{g_{33} g_{11}}} \quad (8)$$

where

$$g_{pq} = \underline{\alpha}_p \cdot \underline{\alpha}_q = \frac{\partial \underline{r}}{\partial u_p} \cdot \frac{\partial \underline{r}}{\partial u_q} = \frac{\partial x}{\partial u_p} \frac{\partial x}{\partial u_q} + \frac{\partial y}{\partial u_p} \frac{\partial y}{\partial u_q} + \frac{\partial z}{\partial u_p} \frac{\partial z}{\partial u_q}$$

$$p, q = 1, 2, 3 \quad (9)$$

For the transformation given by eqs. (5) and (6)

$$\begin{aligned} g_{11} &= \underline{\alpha}_1 \cdot \underline{\alpha}_1 = 1 + \cot^2 \alpha = \csc^2 \alpha & g_{31} &= \underline{\alpha}_3 \cdot \underline{\alpha}_1 = 0 \\ g_{12} &= \underline{\alpha}_1 \cdot \underline{\alpha}_2 = \cos \alpha \csc^2 \alpha & g_{32} &= \underline{\alpha}_3 \cdot \underline{\alpha}_2 = 0 \\ g_{13} &= \underline{\alpha}_1 \cdot \underline{\alpha}_3 = 0 & g_{33} &= \underline{\alpha}_3 \cdot \underline{\alpha}_3 = 1 \\ g_{21} &= \underline{\alpha}_2 \cdot \underline{\alpha}_1 = \cos \alpha \csc^2 \alpha \\ g_{22} &= \underline{\alpha}_2 \cdot \underline{\alpha}_2 = \csc^2 \alpha \\ g_{23} &= \underline{\alpha}_2 \cdot \underline{\alpha}_3 = 0 \end{aligned} \quad (10)$$

Therefore the angles are

$$\cos \theta_{12} = \cos \alpha \quad \cos \theta_{23} = 0 \quad \cos \theta_{31} = 0 \quad (11)$$

From eq. (9) it is possible to determine the Jacobian which is defined by

$J = \sqrt{g}$ where g is

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} \quad (12)$$

$$J = \csc \alpha \quad (13)$$

III. Arc Lengths, Areas and Volume

It is now possible to determine arc lengths, areas and volume associated with the transformed system.

Arc Lengths

$$\begin{aligned} ds_1 &= \sqrt{g_{11}} du_1 = \csc \alpha du_1, \\ ds_2 &= \sqrt{g_{22}} du_2 = \csc \alpha du_2, \end{aligned} \quad (14)$$

$$ds_3 = \sqrt{g_{33}} du_3 = du_3$$

Area in plane $u_1 u_2$

$$dA_3 = \sqrt{g_{11} g_{22} - (g_{12})^2} du_1 du_2 = \csc \alpha du_1 du_2 \quad (15)$$

$$dA_2 = \sqrt{g_{11} g_{33} - (g_{13})^2} \quad du_1 \quad du_3 = \csc \alpha \quad du_1 \quad du_3 \quad (16)$$

$$dA_1 = \sqrt{g_{22} g_{33} - (g_{23})^2} \quad du_2 \quad du_3 = \csc \alpha \quad du_2 \quad du_3 \quad (17)$$

The volume of an element is given by

$$dV = \sqrt{g} \quad du_1 \quad du_2 \quad du_3 = \csc \alpha \quad du_1 \quad du_2 \quad du_3 \quad (18)$$

IV. Derivative Transformations

The derivative transformation from x, y, z to u_1, u_2, u_3 can be obtained through successive use of the chain rule

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial u_1} \frac{\partial u_1}{\partial x} + \frac{\partial}{\partial u_2} \frac{\partial u_2}{\partial x} + \frac{\partial}{\partial u_3} \frac{\partial u_3}{\partial x} \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial u_1} \frac{\partial u_1}{\partial y} + \frac{\partial}{\partial u_2} \frac{\partial u_2}{\partial y} + \frac{\partial}{\partial u_3} \frac{\partial u_3}{\partial y} \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial u_1} \frac{\partial u_1}{\partial z} + \frac{\partial}{\partial u_2} \frac{\partial u_2}{\partial z} + \frac{\partial}{\partial u_3} \frac{\partial u_3}{\partial z} \end{aligned} \quad (19)$$

where

$$\begin{aligned}
 \frac{\partial u_1}{\partial x} &= \frac{1}{J} \frac{\partial(y,z)}{\partial(u_2,u_3)} & \frac{\partial u_1}{\partial y} &= \frac{1}{J} \frac{\partial(z,x)}{\partial(u_2,u_3)} & \frac{\partial u_1}{\partial z} &= \frac{1}{J} \frac{\partial(x,y)}{\partial(u_2,u_3)} \\
 \frac{\partial u_2}{\partial x} &= \frac{1}{J} \frac{\partial(y,z)}{\partial(u_3,u_1)} & \frac{\partial u_2}{\partial y} &= \frac{1}{J} \frac{\partial(z,x)}{\partial(u_3,u_1)} & \frac{\partial u_2}{\partial z} &= \frac{1}{J} \frac{\partial(x,y)}{\partial(u_3,u_1)} \\
 \frac{\partial u_3}{\partial x} &= \frac{1}{J} \frac{\partial(y,z)}{\partial(u_1,u_2)} & \frac{\partial u_3}{\partial y} &= \frac{1}{J} \frac{\partial(z,x)}{\partial(u_1,u_2)} & \frac{\partial u_3}{\partial z} &= \frac{1}{J} \frac{\partial(x,y)}{\partial(u_1,u_2)}
 \end{aligned} \quad (20)$$

Evaluating eq. (5) we have

$$\frac{\partial u_1}{\partial x} = 1 \quad \frac{\partial u_2}{\partial x} = -\cos \alpha \quad \frac{\partial u_2}{\partial y} = \sin \alpha \quad (21)$$

$$\frac{\partial u_3}{\partial z} = 1 \quad \frac{\partial u_1}{\partial y} = \frac{\partial u_1}{\partial z} = \frac{\partial u_2}{\partial z} = \frac{\partial u_3}{\partial x} = \frac{\partial u_3}{\partial y} = 0 \quad (22)$$

Substituting eqs. (21), (22) into eq. (19) we obtain

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u_1} - \cos \alpha \frac{\partial}{\partial u_2}$$

$$\frac{\partial}{\partial y} = \sin \alpha \frac{\partial}{\partial u_2} \quad (23)$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial u_3}$$

By applying the chain rule to eq. (23) the second derivatives can be evaluated to give the following

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial u_1^2} - 2 \cos \alpha \frac{\partial^2}{\partial u_1 \partial u_2} + \cos^2 \alpha \frac{\partial^2}{\partial u_2^2} \quad (24)$$

$$\frac{\partial^2}{\partial y^2} = \sin^2 \alpha \frac{\partial^2}{\partial u_2^2}$$

$$\frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial u_3^2}$$

V. Velocity Determination

The velocity of the fluid at a point in a coordinate system x^k is $\frac{dx^k}{dt}$. In a coordinate system $\bar{x}^j$ the velocity is $\frac{d\bar{x}^j}{dt}$. So we have

$$\frac{d\bar{x}^j}{dt} = \frac{\partial \bar{x}^j}{\partial x^k} \frac{dx^k}{dt} \quad (25)$$

Let the velocities for $x^k = x^k(x, y, z)$ be designated as $u = \frac{dx}{dt}$,

$v = \frac{dy}{dt}$ and $w = \frac{dz}{dt}$ whereas for $\bar{x}^j = \bar{x}^j(u_1, u_2, u_3)$

the velocities will be designated as $q_1 = \frac{du_1}{dt}$, $q_2 = \frac{du_2}{dt}$

and $q_3 = \frac{du_3}{dt}$.

By expanding eq. (25) and using the defined velocity components eq. (25) in general becomes

$$\begin{aligned} q_1 &= u \frac{\partial u_1}{\partial x} + v \frac{\partial u_1}{\partial y} + w \frac{\partial u_1}{\partial z} \\ q_2 &= u \frac{\partial u_2}{\partial x} + v \frac{\partial u_2}{\partial y} + w \frac{\partial u_2}{\partial z} \\ q_3 &= u \frac{\partial u_3}{\partial x} + v \frac{\partial u_3}{\partial y} + w \frac{\partial u_3}{\partial z} \end{aligned} \quad (26)$$

Substituting eqs. (21) and (22) the velocity components in the transformed plane become

$$\begin{aligned} q_1 &= u \\ q_2 &= -\cos \alpha u + \sin \alpha v \\ q_3 &= w \end{aligned} \quad (27)$$

VI. Conservation of Mass

Using the transformations developed in the above sections the equation for the conservation of mass can now be transformed into the new coordinate system. The continuity equation for a rectangular coordinate system can be

expressed as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underline{V} = 0 \quad (28)$$

for a volume element dV . Taking each term separately we have

$$\frac{\partial \rho}{\partial t} dV = \frac{\partial \rho}{\partial t} \csc \alpha du_1 du_2 du_3 \quad (29)$$

$$\frac{\partial \rho u}{\partial x} dV = \frac{\partial \rho q_1}{\partial u_1} \csc \alpha du_1 du_2 du_3 \quad (30)$$

$$- \cos \alpha \frac{\partial \rho q_1}{\partial u_2} \csc \alpha du_1 du_2 du_3$$

$$\frac{\partial \rho v}{\partial y} dV = \cos \alpha \frac{\partial \rho q_1}{\partial u_2} \csc \alpha du_1 du_2 du_3 \quad (31)$$

$$+ \frac{\partial \rho q_2}{\partial u_2} \csc \alpha du_1 du_2 du_3$$

$$\frac{\partial \rho w}{\partial z} dV = \frac{\partial \rho q_3}{\partial u_3} \csc \alpha du_1 du_2 du_3 \quad (32)$$

Substituting eqs. (29), (30), (31), and (32) into eq. (28) and dividing through by du_3 we obtain

$$\begin{aligned} \frac{\partial \rho}{\partial t} dA_3 + \frac{\partial \rho q_1}{\partial u_1} \csc \alpha du_2 du_3 \frac{du_1}{du_3} + \frac{\partial \rho q_2}{\partial u_2} \csc \alpha du_1 du_3 \frac{du_2}{du_3} \\ + \frac{\partial \rho q_3}{\partial u_3} \csc \alpha du_1 du_2 = 0 \end{aligned} \quad (33)$$

$$\text{Let } m_1 = \rho q_1 \csc \alpha du_2 du_3 = \rho q_1 dA_1 \quad (34)$$

$$m_2 = \rho q_2 \csc \alpha du_1 du_3 = \rho q_2 dA_2 \quad (35)$$

$$m_3 = \rho q_3 \csc \alpha du_1 du_2 = \rho q_3 dA_3 \quad (36)$$

Eq. (33) can be rewritten as

$$\frac{\partial \rho}{\partial t} dA_3 + \frac{\partial m_1}{\partial u_1} \frac{du_1}{du_3} + \frac{\partial m_2}{\partial u_2} \frac{du_2}{du_3} + \frac{\partial m_3}{\partial u_3} = 0 \quad (37)$$

Further, if we define mass fluxes per unit length except in the axial direction, then eq. (37) can be reduced to the same form as the mass conservation equation used in THI-3D in finite difference form

$$\frac{\Delta \rho}{\Delta t} \Delta A_3 + \Delta \bar{m}_1 + \Delta \bar{m}_2 + \frac{\Delta m_3}{\Delta u_3} = 0 \quad (38)$$

where

$$\bar{m}_1 = \frac{m_1}{\Delta u_3} \quad \text{and} \quad \bar{m}_2 = \frac{m_2}{\Delta u_3}$$

VII. Energy Equation

The general form of the energy equation based on constant thermal conductivity and heat capacity can be written as

$$\begin{aligned} & \frac{\partial \rho h}{\partial t} dV + \frac{\partial \rho u h}{\partial x} dV + \frac{\partial \rho v h}{\partial y} dV + \frac{\partial \rho w h}{\partial z} dV \\ &= \frac{k}{c_p} dV \left[\frac{\partial^2 h}{\partial z^2} + \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right] + q'' dV - \frac{q'_d}{A_d} dV \quad (39) \end{aligned}$$

Again taking each term separately we have

$$\frac{\partial \rho h}{\partial t} dV = \frac{\partial \rho h}{\partial t} \csc \alpha du_1 du_2 du_3 \quad (40)$$

$$\frac{\partial \rho u h}{\partial x} dV = \left[\frac{\partial}{\partial u_1} - \cos \alpha \frac{\partial}{\partial u_2} \right] \rho q_1 h \csc \alpha du_1 du_2 du_3 \quad (41)$$

$$\frac{\partial \rho v h}{\partial y} dV = \sin \alpha \frac{\partial}{\partial u_2} \left[\cot \alpha q_1 + \csc \alpha q_2 \right] \rho h \csc \alpha du_1 du_2 du_3 \quad (42)$$

$$\frac{\partial \rho w h}{\partial z} dV = \frac{\partial}{\partial u_3} \rho q_3 h \csc \alpha du_1 du_2 du_3 \quad (43)$$

$$\frac{\partial^2 h}{\partial x^2} dV = \left[\frac{\partial^2}{\partial u_1^2} - 2 \cos \alpha \frac{\partial^2}{\partial u_1 \partial u_2} + \cos^2 \alpha \frac{\partial^2}{\partial u_2^2} \right] h \csc \alpha du_1 du_2 du_3 \quad (44)$$

$$\frac{\partial^2 h}{\partial x^2} dV = \sin^2 \alpha \frac{\partial^2}{\partial u_2^2} h \csc \alpha du_1 du_2 du_3 \quad (45)$$

$$\frac{\partial^2 h}{\partial y^2} dV = \frac{\partial^2}{\partial u_3^2} h \csc \alpha du_1 du_2 du_3 \quad (46)$$

Substituting eqs.(40), (41), (42), (43), (44), (45), and (46), into eq. (39) and using the definitions of the mass fluxes in the previous section, the energy equation can be written as

$$\begin{aligned} & \frac{\partial \rho h}{\partial t} + \frac{\sin \alpha d\Lambda_2}{dV} \frac{\partial \bar{m}_1 h}{\partial u_1} + \frac{\sin \alpha d\Lambda_1}{dV} \frac{\partial \bar{m}_2 h}{\partial u_2} + \frac{du_3}{dV} \frac{\partial \bar{m}_3 h}{\partial u_3} \\ &= \frac{k}{c_p} \left[\frac{\partial^2 h}{\partial u_1^2} - 2 \cos \alpha \frac{\partial^2 h}{\partial u_1 \partial u_2} + \frac{\partial^2 h}{\partial u_2^2} + \frac{\partial^2 h}{\partial u_3^2} \right] + q''' - \frac{q'_d}{A_d} \end{aligned} \quad (47)$$

VIII. Axial Momentum Equation

The axial momentum equation can be written as

$$\begin{aligned} & \frac{\partial \rho w}{\partial t} dV + \frac{\partial \rho w^2}{\partial z} dV + \frac{\partial \rho u w}{\partial x} dV + \frac{\partial \rho v w}{\partial y} dV \\ &= - \rho g dV - \frac{\partial p(A dz)}{\partial z} + F_z \end{aligned} \quad (48)$$

where F_z represents a general friction term whose form or approximation thereof must be specified.

Taking each item separately

$$\frac{\partial \rho w}{\partial t} dV = \frac{\partial \rho q_3}{\partial t} \csc \alpha du_1 du_2 du_3 \quad (49)$$

$$\frac{\partial \rho w^2}{\partial z} dV = \frac{\partial \rho q_3^2}{\partial u_3} \csc \alpha du_1 du_2 du_3 \quad (50)$$

$$\frac{\partial \rho u w}{\partial x} dV = \left[\frac{\partial}{\partial u_1} - \cos \alpha \frac{\partial}{\partial u_2} \right] \rho q_1 q_3 \csc \alpha du_1 du_2 du_3 \quad (51)$$

$$\frac{\partial \rho v w}{\partial y} dV = \sin \alpha \frac{\partial}{\partial u_2} q_3 \left[\rho (\cot \alpha q_1 + \csc \alpha q_2) \right] \csc \alpha du_1 du_2 du_3 \quad (52)$$

$$\frac{\partial \rho A_z dz}{\partial z} = \frac{\partial \rho}{\partial u_3} \csc \alpha du_1 du_2 du_3 \quad (53)$$

$$\begin{aligned} \frac{1}{dA_3} \frac{\partial m_3}{\partial t} + \frac{1}{dA_3} \left[\frac{\partial m_3 q_3}{\partial u_3} + \frac{\partial \bar{m}_1 q_3}{\partial u_1} du_1 + \frac{\partial \bar{m}_2 q_3}{\partial u_2} du_2 \right] \\ = - \rho g - \frac{1}{dA_3} \frac{\partial p dA_3}{\partial u_3} + \frac{F_{u_3}}{dA_3 du_3} \end{aligned} \quad (54)$$

where F_{u_3} represents F_z transformed into the new coordinate system.

IX. Transverse Momentum Equation

Starting with the x-direction momentum equation in the following form

$$\frac{\partial \rho u}{\partial t} dV + \frac{\partial \rho u^2}{\partial x} dV + \frac{\partial \rho u v}{\partial y} dV + \frac{\partial \rho u w}{\partial z} dV = - \frac{\partial p dA_x}{\partial x} dx + F_x \quad (55)$$

Each term separately transforms into the following

$$\frac{\partial \rho u}{\partial t} dV = \frac{\partial \rho q_1}{\partial t} \csc \alpha du_1 du_2 du_3 \quad (56)$$

$$\frac{\partial \rho u^2}{\partial x} dV = \left[\frac{\partial}{\partial u_1} - \cos \alpha \frac{\partial}{\partial u_2} \right] \rho q_1^2 \csc \alpha du_1 du_2 du_3 \quad (57)$$

$$\frac{\partial \rho uv}{\partial y} dV = \sin \alpha \frac{\partial}{\partial u_2} \rho q_1 \left[\cot \alpha q_1 + \csc \alpha q_2 \right] \csc \alpha du_1 du_2 du_3 \quad (58)$$

$$\frac{\partial \rho uw}{\partial z} dV = \frac{\partial}{\partial u_3} \rho q_1 q_3 \csc \alpha du_1 du_2 du_3 \quad (59)$$

$$\frac{\partial p dA_x}{\partial x} dx = \left[\frac{\partial}{\partial u_1} - \cos \alpha \frac{\partial}{\partial u_2} \right] p \csc \alpha du_1 du_2 du_3 \quad (60)$$

Expanding and substituting eqs. (56) (57) (58) (59) and (60) into eq (55) we have

$$\begin{aligned} \frac{\partial \bar{m}_1}{\partial t} + \frac{\partial \bar{m}_1 q_1}{\partial u_1} + \frac{\partial \bar{m}_1 q_2}{\partial u_2} + \frac{\partial \bar{m}_1 q_3}{\partial u_3} = - \frac{1}{du_3} \frac{\partial p dA_1}{\partial u_1} \\ + \frac{\cos \alpha}{du_3} \frac{\partial p dA_1}{\partial u_2} + \frac{F_{u_1}}{du_1 du_3} \end{aligned} \quad (61)$$

The y - momentum equation is written as

$$\frac{\partial \rho v}{\partial t} dV + \frac{\partial \rho uv}{\partial x} dV + \frac{\partial \rho v^2}{\partial y} dV + \frac{\partial \rho vw}{\partial z} dV = - \frac{\partial p dA_y}{\partial y} dy + F_y \quad (62)$$

and again each term separately transforms into the following

$$\frac{\partial \rho v}{\partial t} dV = \frac{\partial}{\partial t} \rho (\cot \alpha q_1 + \csc \alpha q_2) \csc \alpha du_1 du_2 du_3 \quad (63)$$

$$\frac{\partial \rho uv}{\partial x} dV = \left(\frac{\partial}{\partial u_1} - \cos \alpha \frac{\partial}{\partial u_2} \right) \rho q_1 (\cot \alpha q_1 + \csc \alpha q_2) \csc \alpha du_1 du_2 du_3 \quad (64)$$

$$\frac{\partial \rho v^2}{\partial y} dV = \sin \alpha \frac{\partial}{\partial u_2} \rho \left[\cot \alpha q_1 + \csc \alpha q_2 \right]^2 \csc \alpha du_1 du_2 du_3 \quad (65)$$

$$\frac{\partial \rho vw}{\partial z} dV = \frac{\partial}{\partial u_3} \rho q_3 \left[\cot \alpha q_1 + \csc \alpha q_2 \right] \csc \alpha du_1 du_2 du_3 \quad (66)$$

$$\frac{\partial p dA_y}{\partial y} dy = \sin \alpha \frac{\partial}{\partial u_2} p \csc \alpha du_1 du_2 du_3 \quad (67)$$

Expanding and substituting eqs. (63) (64) (65) (66) and (67) into (62) we obtain

$$\begin{aligned}
 & \frac{\partial \bar{m}_2}{\partial t} + \cos \alpha \frac{\partial \bar{m}_1}{\partial t} \frac{du_1}{du_2} + \cos \alpha \frac{\partial \bar{m}_1 q_1}{\partial u_1} \frac{du_1}{du_2} + \frac{\partial \bar{m}_2 q_1}{\partial u_1} + \cos \alpha \frac{\partial \bar{m}_2 q_1}{\partial u_2} \\
 & + \frac{\partial \bar{m}_2 q_2}{\partial u_2} + \cos \alpha \frac{\partial \bar{m}_1 q_3}{\partial u_3} \frac{du_1}{du_2} + \frac{\partial \bar{m}_2 q_3}{\partial u_3} = - \frac{\sin^2 \alpha}{du_3} \frac{\partial p dA_2}{\partial u_2} + \frac{F_{u_2}}{dA_1} \quad (68)
 \end{aligned}$$

X. Rotation of Coordinates

Because of the triangular mesh requirement it is necessary to look at a rotation of coordinates so that the system has a coordinate at all times congruent with a horizontal base. This is shown in the figure below

The axes u_1 and u_2 are rotated through an angle $\theta = 180$ to be positioned along u_1' and u_2' respectively. Hence,

$$u_1 = -u_1', \quad u_2 = -u_2', \quad u_3 = u_3' \quad (69)$$

It then follows from eqs. (5) and (6)

$$u_1' = -x, \quad u_2' = x \cos \alpha - y \sin \alpha, \quad u_3' = z \quad (70)$$

and

$$x = -u_1', \quad y = -(\csc \alpha u_2' + \cot \alpha u_1'), \quad z = u_3' \quad (71)$$

The transformation from the unprimed to the primed coordinates requires the following substitutions to be made:

$$\begin{aligned}
\frac{\partial}{\partial u_1} &= -\frac{\partial}{\partial u'_1} & \frac{\partial^2}{\partial u_1^2} &= \frac{\partial^2}{\partial u'^2_1} \\
\frac{\partial}{\partial u_2} &= -\frac{\partial}{\partial u'_2} & \frac{\partial^2}{\partial u_2^2} &= \frac{\partial^2}{\partial u'^2_2} \\
\frac{\partial}{\partial u_3} &= \frac{\partial}{\partial u'_3} & \frac{\partial^2}{\partial u_3^2} &= \frac{\partial^2}{\partial u'^2_3}
\end{aligned} \tag{72}$$

$$\begin{aligned}
dA_1 &= \csc \alpha \left| du'_2 du'_3 \right| = dA'_1 \\
dA_2 &= \csc \alpha \left| du'_1 du'_3 \right| = dA'_2 \\
dA_3 &= \csc \alpha \left| du'_1 du'_2 \right| = dA'_3
\end{aligned} \quad dV = \csc \alpha du'_1 du'_2 du'_3 = dV' \tag{73}$$

Defining the velocities as shown

$$\begin{aligned}
q'_1 &= -q_1 \\
q'_2 &= -q_2 \\
q'_3 &= q_3
\end{aligned} \tag{74}$$

The mass fluxes are

$$\begin{aligned}
m_1 &= -\rho q'_1 \csc \alpha \left| du'_2 du'_3 \right| = -m'_1 & \bar{m}'_1 &= \frac{m'_1}{du'_3} \\
m_2 &= -\rho q'_2 \csc \alpha \left| du'_1 du'_3 \right| = -m'_2 & \bar{m}'_2 &= \frac{m'_2}{du'_3} \\
m_3 &= \rho q'_3 \csc \alpha \left| du'_1 du'_2 \right| = m'_3
\end{aligned} \tag{75}$$

First if we look at the conservation of mass eq. (37)

$$\frac{\partial \rho}{\partial t} A_3 + \frac{\partial m_1}{\partial u_1} \frac{du_1}{du_3} + \frac{\partial m_2}{\partial u_2} \frac{du_2}{du_3} + \frac{\partial m_3}{\partial u_3} = 0 \quad (37)$$

Substituting the previously primed quantities eq. (37) becomes

$$\frac{\partial \rho}{\partial t} A'_3 + \frac{\partial m'_1}{\partial u'_1} \frac{du'_1}{du'_3} + \frac{\partial m'_2}{\partial u'_2} \frac{du'_2}{du'_3} + \frac{\partial m'_3}{\partial u'_3} = 0 \quad (37')$$

We note that eq (37') takes exactly the same form as eq. (37) in the unprimed coordinates. Since the conservation of mass equation has the same form it is expected that the remaining governing equations will all follow form. Although eq. (37) was repeated here for completeness sake the other equation will not be and they will only be identified by number referring to the previous part of the text. Hence, only the primed version of the energy and momentum equations will be presented.

The energy equation, (47) becomes

$$\begin{aligned} \frac{\partial \rho h}{\partial t} + \frac{\sin \alpha dA'_2}{dV'} \frac{\partial \bar{m}'_1 h}{\partial u'_1} + \frac{\sin \alpha dA'_1}{dV'} \frac{\partial \bar{m}'_2 h}{\partial u'_2} + \frac{du'_3}{dV'} \frac{\partial m'_3 h}{\partial u'_3} = \\ \frac{k}{c_p} \left[\frac{\partial^2 h}{\partial u'^2_1} - 2 \cos \alpha \frac{\partial^2 h}{\partial u'_1 \partial u'_2} + \frac{\partial^2 h}{\partial u'^2_2} + \frac{\partial^2 h}{\partial u'^2_3} \right] + q''' - \frac{q'_d}{A'_d} \end{aligned} \quad (47')$$

The axial momentum equation becomes

$$\begin{aligned} \frac{1}{dA'_3} \frac{\partial m'_3}{\partial t} + \frac{1}{dA'_3} \left[\frac{\partial m'_3 q'_3}{\partial u'_3} + \frac{\partial \bar{m}'_1 q'_3}{\partial u'_1} du'_1 + \frac{\partial \bar{m}'_2 q'_3}{\partial u'_2} du'_2 \right] \\ = -pg - \frac{1}{dA'_3} \frac{\partial p dA'_3}{\partial u'_3} + \frac{Fu'_3}{dV'} \end{aligned} \quad (54')$$

The transverse momentum equations become

$$\begin{aligned} \frac{\partial \bar{m}'_1}{\partial t} + \frac{\partial \bar{m}'_1 q'_1}{\partial u'_1} + \frac{\partial \bar{m}'_1 q'_2}{\partial u'_2} + \frac{\partial \bar{m}'_1 q'_3}{\partial u'_3} = - \frac{1}{du'_3} \frac{\partial pdA'_1}{\partial u'_1} \\ - \frac{\cos \alpha}{du'_3} \frac{\partial pdA'_1}{\partial u'_2} - \frac{Fu'_1}{du'_1 du'_3} \end{aligned} \quad (61')$$

$$\begin{aligned} \frac{\partial \bar{m}'_2}{\partial t} - \cos \alpha \frac{\partial \bar{m}'_1}{\partial t} \frac{du'_1}{du'_2} + \cos \alpha \frac{\partial \bar{m}'_1 q'_1}{\partial u'_1} \frac{du'_1}{du'_2} + \frac{\partial \bar{m}'_2 q'_1}{\partial u'_1} \\ + \cos \alpha \frac{\partial \bar{m}'_2 q'_1}{\partial u'_2} + \frac{\partial \bar{m}'_2 q'_2}{\partial u'_2} + \cos \alpha \frac{\partial \bar{m}'_1 q'_3}{\partial u'_3} \frac{du'_1}{du'_2} + \frac{\partial \bar{m}'_2 q'_3}{\partial u'_3} = \\ - \frac{\sin^2 \alpha}{du'_3} \frac{\partial pdA'_2}{\partial u'_2} - \frac{Fu'_2}{dA'_1} \end{aligned} \quad (68')$$

All of the governing equation retain their original form as in the unprimed coordinate system. The lone exceptions being the minus sign in equations (61') and (68') associated with the friction or shear stress terms. However, since the model or the form to be used for these terms has not been explicitly defined the minus sign can be accounted for in the model, therefore making all of the equations identical in form. This identical appearance offers a distinct advantage in programming.

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