Techniques for Computing the Numerical Values of the Matrix Elements of the Collision Operator in the Collisional Plasma Model

H. K. Meier
W. I. van Rij
C. O. Beasley, Jr.

MASTER

OAK RIDGE NATIONAL LABORATORY

OPERATED BY UNION CARBIDE CORPORATION FOR THE ENERGY RESEARCH AND DEVELOPMENT ADMINISTRATION

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

Printed in the United States of America: Available from
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road, Springfield, Virginia 22161
Price: Printed Copy \$4.00; Microfiche \$2.25

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the Energy Research and Development Administration/United States Nuclear Regulatory Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

Contract No. W-7405-eng-26

THERMONUCLEAR DIVISION

TECHNIQUES FOR COMPUTING THE NUMERICAL VALUES OF THE
MATRIX ELEMENTS OF THE COLLISION OPERATOR IN
THE COLLISIONAL PLASMA MODEL

H. K. Meier, W. I. van Rij, and C. O. Beasley, Jr.
Oak Ridge National Laboratory, Oak Ridge, Tennessee 37830 .

MARCH 1976

NOTICE
This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Energy Research and Development Administration, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

NOTICE This document contains information of a preliminary nature and was prepared primarily for internal use at the Oak Ridge National Laboratory. It is subject to revision or correction and therefore does not represent a final report.

OAK RIDGE NATIONAL LABORATORY
Oak Ridge, Tennessee 37830
operated by
UNION CARBIDE CORPORATION
for the
ENERGY RESEARCH AND DEVELOPMENT ADMINISTRATION

THIS PAGE
WAS INTENTIONALLY
LEFT BLANK

TABLE OF CONTENTS

Abstract	v
Definitions and Previously Derived Results	1
A Recursive Method for the Computation of	
$R_M \begin{bmatrix} l & l' & l'' \\ n & n' & n'' \end{bmatrix}, M = 0, 1, 2$	5
Expansions	9
References	21
Appendix: Derivation of the Recursion Relations	23
Distribution	29

THIS PAGE
WAS INTENTIONALLY
LEFT BLANK

TECHNIQUES FOR COMPUTING THE NUMERICAL VALUES OF THE
MATRIX ELEMENTS OF THE COLLISION OPERATOR IN
THE COLLISIONAL PLASMA MODEL*

by

H. K. Meier, W. I. van Rij, and C. O. Beasley, Jr.

ABSTRACT

A previous report¹ described the collision operator used in the CPM (Collisional Plasma Model), and presented a rigorous derivation of its matrix elements. This report is concerned with practical techniques for obtaining the numerical values of the matrix elements. It is intended primarily for those who must implement or understand a computer simulation based on the CPM.

* Research sponsored by the United States Energy Research and Development Administration under contract with the Union Carbide Corporation.

I. Definitions and Previously Derived Results

The Boltzmann equation for the species f^r takes the form

$$\begin{aligned} \partial_t f^r + \vec{v} \cdot \partial_{\vec{x}} f^r + e_r/M_r (\vec{E} + \vec{v}/c \times \vec{B}) \cdot \partial_{\vec{v}} f^r \\ = \sum_s C^{rs} [f^r, f^s], \end{aligned} \quad (1)$$

where the r.h.s. (the collision term) consists of a sum over species s .

In the CPM f^r is represented by a set of amplitudes, $f_{\ell mn}^r(\vec{x}, t)$, such that f^r itself is approximated by

$$f^r = \sum_{\ell=0}^{LMAX} \sum_{m=-\ell}^{\ell} \sum_{n=0}^{NMAX} f_{\ell mn}^r \cdot (\rho_r^{1/2} u)^{\ell} L_n^{\ell+1/2} (\rho_r u^2) e^{-1/2 \rho_r u^2} Y_{\ell m}(\alpha, \beta). \quad (2)$$

The $L_n^{\ell+1/2}$'s are generalized Laguerre polynomials and the $Y_{\ell m}$'s are spherical harmonics. The vector $\vec{u}$ is a displaced velocity vector, i.e.,

$$\vec{u} = \vec{v} - \vec{V}_r(\vec{x}, t), \quad (3)$$

where $\vec{V}_r$ represents an average or a drift velocity. The (u, α, β) are the spherical coordinates of $\vec{u}$ in a coordinate system which in general depends on $(\vec{x}, t)$. $\rho_r(\vec{x}, t)$ is a scaling factor. Its value is arbitrary but for good convergence it should be approximately M_r/kT_r .

The Landau form of the Coulomb collision term is used, i.e., we take

$$C^{rs} [f^r, f^s] = \Gamma_{rs} \left(M_r/M_s C_1^{rs} [f^r, f^s] + C_2^{rs} [f^r, f^s] \right), \quad (4)$$

where

$$\Gamma_{rs} = 4\pi \left(\frac{e_r e_s}{M_r} \right)^2 \ln \Lambda, \quad (5)$$

$$\Lambda = \frac{3(kT_e)^{3/2}}{2\sqrt{\pi} N_e^{1/2} e^3}, \quad (6)$$

$$C_1^{rs} [f^r, f^s] = - \partial_{v_i} \left[f^r(\vec{v}) \partial_{v_i} \int d^3v' \frac{f^s(\vec{v}')}{|\vec{v} - \vec{v}'|} \right], \quad (7)$$

and

$$C_2^{rs} [f^r, f^s] = C_1^{rs} [f^r, f^s] \quad (8)$$

$$+ \frac{1}{2} \partial_{v_i} \partial_{v_j} \left[f^r(\vec{v}) \partial_{v_i} \partial_{v_j} \int d^3v' |\vec{v} - \vec{v}'| f^s(\vec{v}') \right].$$

In this expression e_r and e_s are the electric charges for species r and s , M_r and M_s their masses, T_e and N_e the electron temperature and density respectively, and e is the absolute value of the electron charge. The summation convention is used in the above expressions.

If expression (2) is substituted for f^r in Eq. (1), and one uses the orthogonality properties of the expansion functions one may obtain the Boltzmann equation in the CPM representation. It has the form

$$\begin{aligned} \partial_t f_{lmn}^r + \sum_{\ell'm'n'} K_{lmn, \ell'm'n'}^r f_{\ell'm'n'}^r \\ = \sum_s \sum_{\ell'm'n'} \sum_{\ell''m''n''} C^{rs} \left[\begin{matrix} \ell & \ell' & \ell'' \\ m & m' & m'' \\ n & n' & n'' \end{matrix} \right] f_{\ell'm'n'}^r f_{\ell''m''n''}^s \end{aligned} \quad (9)$$

Corresponding to expression (4), C^{rs} can be written as

$$C^{rs} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ m & m' & m'' \end{smallmatrix} \right] = \Gamma_{rs} \left\{ \frac{M_r}{M_s} C_1^{rs} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ m & m' & m'' \end{smallmatrix} \right] + C_2^{rs} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ m & m' & m'' \end{smallmatrix} \right] \right\} \quad (10)$$

Several properties of the collision operator were proved in a previous report.¹ It was shown that $C_1^{rs} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ m & m' & m'' \end{smallmatrix} \right]$ and $C_2^{rs} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ m & m' & m'' \end{smallmatrix} \right]$ can be written as

$$C_1^{rs} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ m & m' & m'' \end{smallmatrix} \right] = \lambda^{\ell''/2} Z \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ m & m' & m'' \end{smallmatrix} \right] S_1 \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right] \quad (11)$$

and

$$C_2^{rs} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ m & m' & m'' \end{smallmatrix} \right] = \lambda^{\ell''/2} Z \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ m & m' & m'' \end{smallmatrix} \right] S_2 \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right] \quad (12)$$

where

$$Z \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ m & m' & m'' \end{smallmatrix} \right] = \int d\Omega Y_{\ell m}^*(\alpha, \beta) Y_{\ell' m'}(\alpha, \beta) Y_{\ell'' m''}(\alpha, \beta). \quad (13)$$

In addition, $S_1 \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right]$, $S_2 \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right]$ were shown to be functions of a single variable

$$\lambda \equiv \frac{\rho_s}{\rho_r}, \quad (14)$$

and may be expressed in terms of three functions $R_0 \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right]$, $R_1 \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right]$, and $R_2 \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right]$, the expressions being

$$S_1 \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right] = 2 \left\{ R_0 \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right] + \frac{1}{\lambda} (\ell + 2n - \ell' - 2n') R_1 \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right] \right\} \quad (15)$$

and

$$\begin{aligned}
S_2 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} &= R_0 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} \\
&- \frac{1}{\lambda^2} \left\{ (\ell+2n-\ell'-2n')(\ell+2n-\ell'-2n'+1) R_2 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} \right. \\
&+ 2(n'+1) R_2 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n'+1 & n'' \end{bmatrix} \\
&\left. - 2n R_2 \begin{bmatrix} \ell & \ell' & \ell'' \\ n-1 & n' & n'' \end{bmatrix} \right\} \quad (16)
\end{aligned}$$

$R_0 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix}$, $R_1 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix}$, $R_2 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix}$ are defined to be

$$R_0 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} = \frac{(-1)^{n''} \pi n! \Gamma(\ell''+n''+\frac{3}{2})}{\Gamma(\ell+n+\frac{3}{2}) \Gamma(\ell''+\frac{3}{2})} \sum_{j=0}^{n''} \frac{(-1)^j 2^j}{j! (n''-j)!} Q \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & j \end{bmatrix} \quad (17a)$$

$$R_1 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} = \frac{(-1)^{n''} \pi n! \Gamma(\ell''+n''+\frac{3}{2})}{\Gamma(\ell+n+\frac{3}{2}) \Gamma(\ell''+\frac{3}{2})} \sum_{j=0}^{n''} \frac{(-1)^j 2^j}{j! (n''-j)! (\ell''+j+\frac{1}{2})} Q \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & j-1 \end{bmatrix} \quad (17b)$$

and

$$R_2 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} = \frac{(-1)^{n''} \pi n! \Gamma(\ell''+n''+\frac{3}{2})}{\Gamma(\ell+n+\frac{3}{2}) \Gamma(\ell''+\frac{3}{2})} \sum_{j=0}^{n''} \frac{(-1)^j 2^j}{j! (n''-j)! (\ell''+j+\frac{1}{2}) (\ell''+j-\frac{1}{2})} Q \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & j-2 \end{bmatrix} \quad (17c)$$

where

$$Q \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & j \end{bmatrix} = \int_0^\infty dx x^{1/2(\ell+\ell'+\ell''+1)} e^{-(1+\lambda/2)x} \quad (18)$$

$$L_n^{\ell+1/2}(x) L_{n'}^{\ell'+1/2}(x) \Phi(-j, \ell''+\frac{3}{2}; \frac{\lambda}{2} x)$$

Φ , which is known as Humbert's symbol, is a confluent hypergeometric function. Integer values of j are required from -2 to $NMAX''$, where $NMAX''$ is the largest value of N'' desired.

II. A Recursive Method for the Computation of

$$R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix}, M = 0, 1, 2$$

Outlined below is a set of recurrence expressions which in principle can be used to compute the R functions. Unfortunately this method is numerically unstable and leads to very erroneous results if the subscripts ($\ell, \ell', \ell'', n, n', n''$) become very large (unless the work is done to very high precision). The method is, however, useful as a starting point for practical computations, and has been used successfully with extended precision subroutines to generate the collision operator that was used initially in the CPM simulations.

The method is outlined in this section; the derivation will be found in the appendix.

The method starts by defining a new set of quantities. They are

$$Q \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & j \end{bmatrix}_k = \int_0^\infty dx \, x^{\ell''+k+1/2} e^{-(1+\lambda/2)x} \cdot L_n^{\ell+1/2}(x) L_{n'}^{\ell'+1/2}(x) \Phi(-j, \ell'' + \frac{3}{2}; \frac{\lambda}{2} x) \quad (19)$$

$$R_0 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix}_k = \frac{(-1)^{n''} \pi n! \Gamma(\ell''+n''+\frac{3}{2})}{\Gamma(\ell+n+\frac{3}{2}) \Gamma(\ell''+\frac{3}{2})} \sum_{j=0}^{n''} \frac{(-1)^j 2^j}{j! (n''-j)!} Q \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & j \end{bmatrix}_k \quad (20a)$$

$$R_1 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix}_k = \frac{(-1)^{n''} \pi n! \Gamma(\ell''+n''+\frac{3}{2})}{\Gamma(\ell+n+\frac{3}{2}) \Gamma(\ell''+\frac{3}{2})} \sum_{j=0}^{n''} \frac{(-1)^j 2^j}{j! (n''-j)! (\ell''+j+\frac{1}{2})} Q \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & j-1 \end{bmatrix}_k \quad (20b)$$

and

$$R_2 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix}_k = \frac{(-1)^{n''} \pi n! \Gamma(\ell''+n''+\frac{3}{2})}{\Gamma(\ell+n+\frac{3}{2}) \Gamma(\ell''+\frac{3}{2})} \sum_{j=0}^{n''} \frac{(-1)^j 2^j}{j! (n''-j)! (\ell''+j+\frac{1}{2}) (\ell''+j-\frac{1}{2})} Q \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & j-2 \end{bmatrix}_k \quad (20c)$$

Note that $R_M \left[\begin{matrix} \ell & \ell' & \ell'' \\ n & n' & j \end{matrix} \right]_k$ is the same function of the $Q \left[\begin{matrix} \ell & \ell' & \ell'' \\ n & n' & j \end{matrix} \right]_k$'s that $R_M \left[\begin{matrix} \ell & \ell' & \ell'' \\ n & n' & j \end{matrix} \right]$ is of the $Q \left[\begin{matrix} \ell & \ell' & \ell'' \\ n & n' & j \end{matrix} \right]$'s. Furthermore, by defining k_0 to be

$$k_0 = \frac{1}{2} (\ell + \ell' - \ell'') , \quad (21)$$

it follows from expression (18) that

$$Q \left[\begin{matrix} \ell & \ell' & \ell'' \\ n & n' & j \end{matrix} \right]_{k_0} = Q \left[\begin{matrix} \ell & \ell' & \ell'' \\ n & n' & j \end{matrix} \right] ,$$

and hence,

$$R_M \left[\begin{matrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{matrix} \right]_{k_0} = R_M \left[\begin{matrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{matrix} \right] . \quad (22)$$

One of the properties of $Z \left[\begin{matrix} \ell & \ell' & \ell'' \\ m & m' & m'' \end{matrix} \right]$ as defined by expression (13), is that

$$Z \left[\begin{matrix} \ell & \ell' & \ell'' \\ m & m' & m'' \end{matrix} \right] \neq 0$$

if and only if

$$|\ell - \ell'| \leq \ell'' \leq \ell + \ell' ,$$

and then only if $(\ell + \ell')$ and ℓ'' are either both even or both odd.

Consequently the only values of k_0 of interest are for integer k_0 with

$$0 \leq k_0 \leq \text{MAXIMUM of } \ell \text{ and } \ell' . \quad (23)$$

The recursive method consists of the following steps:

1) As shown in the appendix

$$R_M \left[\begin{matrix} \ell & \ell' & \ell'' \\ 0 & 0 & -1 \end{matrix} \right]_0 = 0 \quad (24a)$$

and

$$R_M \left[\begin{matrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{matrix} \right]_0 = \frac{\pi \Gamma(\ell'' + \frac{3}{2} - M)}{\Gamma(\ell + \frac{3}{2}) (1 + \frac{\lambda}{2})^{\ell'' + 3/2 - M}} . \quad (24b)$$

Starting with these values and using the relation

$$R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_0 \quad (25)$$

$$= -\frac{1}{n''} \left[(\ell'' + 2n'' - \frac{1}{2}) - (\ell'' + n'' + \frac{1}{2} - M) \frac{2}{(1 + \frac{\lambda}{2})} \right] R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-1 \end{bmatrix}_0$$

$$- \frac{1}{n''} (\ell'' + n'' - \frac{1}{2}) \left(\frac{\lambda-2}{\lambda+2} \right) R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-2 \end{bmatrix}_0 ,$$

$$R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_0 \text{ is obtained for } n'' = 1, 2, \dots, NMAX''.$$

2) Having obtained $R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_0$, we use the expression

$$R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_k \quad (26)$$

$$= \left[(\ell'' + 2k - \frac{1}{2}) - (\ell'' + k + n'' - M + \frac{1}{2}) \frac{\lambda}{2 + \lambda} \right] R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_{k-1}$$

$$- (\ell'' + n'' + \frac{1}{2}) \frac{\lambda}{(2 + \lambda)} R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-1 \end{bmatrix}_{k-1}$$

$$- \frac{(k-1)(\ell'' + k - \frac{1}{2})}{(1 + \frac{\lambda}{2})} R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_{k-2} ,$$

to determine $R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_k$ for $k = 1, 2, \dots, k_0 + NMAX + NMAX'$,

although only $k = k_0, k_0 + 1, \dots, k_0 + NMAX + NMAX'$ need be saved.

3) With $R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_k$ computed, the relations

$$R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & -1 & n'' \end{bmatrix}_k = 0 , \quad (27a)$$

$$\text{and } R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & n' & n'' \end{bmatrix}_k = \frac{1}{n'} \left\{ (\ell' + 2n' - \frac{1}{2}) R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & n'-1 & n'' \end{bmatrix}_k \right. \quad (27b)$$

$$- R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & n'-1 & n'' \end{bmatrix}_{k+1}$$

$$\left. - (\ell' + n' - \frac{1}{2}) R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & n'-2 & n'' \end{bmatrix}_k \right\}$$

are used to find $R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & n' & n'' \end{bmatrix}_k$ for $k = k_0, k_0 + 1, \dots, k_0 + NMAX$, and for $n' = 1, 2, \dots, NMAX'$. Note that each application of expression (27) reduces by 1 the number of known k values since the index $k + 1$ appears on the r.h.s. of (27b).

4) With $R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & n' & n'' \end{bmatrix}_k$ known, the relation

$$R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ m & m' & m'' \end{bmatrix}_k = \frac{1}{(\ell + n + \frac{1}{2})} \left\{ (\ell + 2n - \frac{1}{2}) R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ n-1 & n' & n'' \end{bmatrix}_k \right. \quad (28)$$

$$- R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ n-1 & n' & n'' \end{bmatrix}_{k+1}$$

$$\left. - (n - 1) R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ n-2 & n' & n'' \end{bmatrix}_k \right\}$$

is used for finding $R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix}_k$. Except for finding higher n values, only $k = k_0$ is of interest, since according to (25) this gives $R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix}$.

5) Finally, $S_1 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix}$ and $S_2 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix}$ are obtained from (15) and (16).

III. Expansions

There are three ranges of values of λ that are of particular physical interest. In an ion or impurity Boltzmann equation the electron contribution to the collision term has a very small λ ; (λ is $1/1836[T_I/T_e]$ for ordinary hydrogen, and is still smaller for heavier ions). Conversely, in the electron Boltzmann equation the ion or impurity contribution to the collisions has a very large λ ; (λ is, for example, $1836[T_e/T_i]$ for hydrogen, and it will be larger for heavier ions). For any kind of self collision term $\lambda \equiv 1$, and for ion-impurity collision terms it will be of order unity. This section presents in detail three expansions for the collision operator which are valid respectively for small, medium and large values of λ .

A. Small λ expansion.

For this case $R_M \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right]_k$ is expanded in a power series in z so that

$$R_M \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right]_k = \frac{1}{(1 + \frac{\lambda}{2})^{\ell'' + 3/2 - M}} \sum_{j=0}^{\infty} R_{Mj} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right]_k z^j, \quad (29)$$

where

$$z = \frac{\lambda}{1 + \frac{\lambda}{2}} \quad (30)$$

A comparison of the above with expression (24) shows that for $j = 0$

$$R_{M0} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{smallmatrix} \right]_0 = \frac{\pi \Gamma(\ell'' + 3/2 - M)}{\Gamma(\ell + 3/2)} \quad (31a)$$

while for all other values of j

$$R_{Mj} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{smallmatrix} \right]_0 = 0 \quad (31b)$$

Furthermore, for all j

$$R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & -1 \end{bmatrix}_0 = 0 \quad (31c)$$

When eq. (28) is written in terms of z rather than λ we obtain

$$R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_0 = \frac{1}{n''} \left\{ \left[\left(\ell'' + \frac{3}{2} - 2M \right) - \left(\ell'' + n'' + \frac{1}{2} - M \right) z \right] R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-1 \end{bmatrix}_0 \right. \\ \left. + \left(\ell'' + n'' - \frac{1}{2} \right) (1 - z) R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-2 \end{bmatrix}_0 \right\}$$

The substitution of the expansion defined by (29) into this results in the recurrence formula

$$R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_0 = \frac{1}{n''} \left\{ \left(\ell'' + \frac{3}{2} - 2M \right) R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-1 \end{bmatrix}_0 \right. \\ \left. - \left(\ell'' + n'' + \frac{1}{2} - M \right) R_{Mj-1} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-1 \end{bmatrix}_0 \right. \\ \left. + \left(\ell'' + n'' - \frac{1}{2} \right) R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-2 \end{bmatrix}_0 \right. \\ \left. - \left(\ell'' + n'' - \frac{1}{2} \right) R_{Mj-1} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-2 \end{bmatrix}_0 \right\} \quad (32)$$

which with eq. (31) allows the computation of $R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_0$ for all n'' and j .

Similarly by writing eq. (26) in terms of z we obtain

$$R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_k = \left[\left(\ell'' + 2k - \frac{1}{2} \right) - \frac{1}{2} \left(\ell'' + k + n'' - M + \frac{1}{2} \right) z \right] R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_{k-1} \\ - \frac{1}{2} \left(\ell'' + n'' + \frac{1}{2} \right) z R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-1 \end{bmatrix}_{k-1} \\ - (k-1) \left(\ell'' + k - \frac{1}{2} \right) \left(1 - \frac{1}{2} z \right) R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_{k-2},$$

and the substitution of expression (29) into this leads to

$$\begin{aligned}
 R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_k &= (\ell'' + 2k - \frac{1}{2}) R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_{k-1} \\
 &- \frac{1}{2} (\ell'' + k + n'' - M + \frac{1}{2}) R_{Mj-1} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_{k-1} \\
 &- \frac{1}{2} (\ell'' + n'' + \frac{1}{2}) R_{Mj-1} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-1 \end{bmatrix}_{k-1} \\
 &- (k-1) (\ell'' + k - \frac{1}{2}) R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_{k-2} \\
 &+ \frac{1}{2} (k-1) (\ell'' + k - \frac{1}{2}) R_{Mj-1} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_{k-2}
 \end{aligned} \tag{33}$$

Which is the expression required for the computation of all

$R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_k$. The recursive relations (27) and (28), used for computing $R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & n' & n'' \end{bmatrix}_k$ and $R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix}_k$ respectively, have no explicit λ dependence. As a consequence $R_{M,j} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & n' & n'' \end{bmatrix}_k$ and $R_{M,j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix}_k$ can also be found with these relations by substituting $R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_k$ for $R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & n' & n'' \end{bmatrix}_k$.

$S_1 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix}$ and $S_2 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix}$ can be expressed as

$$S_1 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} = \frac{1}{(1 + \frac{\lambda}{2})^{\ell'' + 3/2}} \sum_{j=-1}^{\infty} S_{1j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} z^j, \tag{34a}$$

and

$$S_2 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} = \frac{1}{(1 + \frac{\lambda}{2})^{\ell'' + 3/2}} \sum_{j=-2}^{\infty} S_{2j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} z^j. \tag{34b}$$

Using expressions (15), (16), and (29) it is easy to obtain the results that

$$S_{1j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} = 2 R_{0j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} k_0 \quad (35a)$$

$$+ 2 (\ell + 2n - \ell' - 2n') R_{1,j+1} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} k_0 ,$$

and

$$S_{2j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} = R_{0j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} k_0 \quad (35b)$$

$$- (\ell + 2n - \ell' - 2n') (\ell + 2n - \ell' - 2n' + 1) R_{2,j+2} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} k_0$$

$$- 2 (n' + 1) R_{2,j+2} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' + 1 & n'' \end{bmatrix} k_0$$

$$+ 2n R_{2,j+2} \begin{bmatrix} \ell & \ell' & \ell'' \\ n - 1 & n' & n'' \end{bmatrix} k_0$$

B. Moderate λ Expansion

In this region define

$$z = \frac{(\lambda - 2)}{(\lambda + 2)} , \quad (36)$$

and take

$$R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} k = \frac{1}{(1 + \frac{\lambda}{2})^{\ell'' + 3/2 - M}} \sum_{j=0}^{\infty} R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} z^j \quad (37)$$

This particular form for z was chosen because it occurs frequently in various algebraic expressions used in connection with the collision operator (expression 28, for example), and it has been observed

empirically that the behavior of the various functions being computed tends to change, as a function of λ , at λ equal to 2.

As in case A, expression (24) is the starting point. The comparison of (24) with (36) shows that

$$R_{M0} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{bmatrix}_0 = \frac{\pi \Gamma(\ell'' + 3/2 - M)}{\Gamma(\ell + 3/2)} \quad (38a)$$

with all other

$$R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{bmatrix}_0 = 0 \quad (38b)$$

and of course

$$R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & -1 \end{bmatrix}_0 = 0 \quad (38c)$$

for all j .

In terms of z , expression (25) becomes

$$\begin{aligned} R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_0 \\ = -\frac{1}{n''} \left[(n''+M-1) + (\ell''+n''-M+1/2)z \right] R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-1 \end{bmatrix}_0 \\ - \frac{1}{n''} (\ell''+n''-1/2)z R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-2 \end{bmatrix}_0 \end{aligned}$$

This with (37) results in

$$\begin{aligned} R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_0 = -\frac{1}{n''} (n''+M-1) R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-1 \end{bmatrix}_0 \\ - \frac{1}{n''} (\ell''+n''-M+1/2) R_{Mj-1} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-1 \end{bmatrix}_0 \\ - \frac{1}{n''} (\ell''+n''-1/2) R_{Mj-1} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-2 \end{bmatrix}_0 \end{aligned} \quad (39)$$

which is the relation required for the computation of all $R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_0$.

Likewise, writing (26) in terms of z yields

$$\begin{aligned} R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_k &= \left[\frac{1}{2}(\ell''+3k-n''-\frac{3}{2}+M) - \frac{1}{2}(\ell''+k+n''-M+\frac{1}{2})z \right] R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_{k-1} \\ &- \frac{1}{2}(\ell''+n''+\frac{1}{2}) (1+z) R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-1 \end{bmatrix}_{k-1} \\ &- \frac{1}{2}(k-1) (\ell''+k-\frac{1}{2})(1-z) R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_{k-2}, \end{aligned}$$

and the substitution of (37) into this equation gives

$$\begin{aligned} R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_k &= \frac{1}{2} (\ell''+3k-n''-\frac{3}{2}+M) R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_{k-1} \\ &- \frac{1}{2} (\ell''+k+n''-M+\frac{1}{2}) R_{Mj-1} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_{k-1} \\ &- \frac{1}{2} (\ell''+n''+\frac{1}{2}) R_{M,j} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-1 \end{bmatrix}_{k-1} \\ &- \frac{1}{2} (\ell''+n''+\frac{1}{2}) R_{M,j-1} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-1 \end{bmatrix}_{k-1} \\ &- \frac{1}{2} (k-1) (\ell''+k-\frac{1}{2}) R_{M,j} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_{k-2} \\ &+ \frac{1}{2} (k-1) (\ell''+k-\frac{1}{2}) R_{M,j-1} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_{k-2}. \end{aligned} \tag{40}$$

So that $R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_k$ for all n'' and k can be computed.

Since relations (27) and (28) do not contain λ , $R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_k$ can be replaced by $R_{M,j} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_k$ to obtain the expressions required for determining

all $R_{M,j} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & n' & n'' \end{bmatrix}_k$, and $R_{M,j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix}_{k_0}$.

$S_1[]$ and $S_2[]$ will also be expanded in a power series in z with

$$S_1 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} = \frac{1}{\lambda(1 + \lambda/2)^{\ell''+1/2}} \sum_{j=0}^{\infty} S_{1j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} z^j, \quad (41a)$$

and

$$S_2 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} = \frac{1}{\lambda^2(1 + \lambda/2)^{\ell''-1/2}} \sum_{j=0}^{\infty} S_{2j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} z^j. \quad (41b)$$

The substitution of (37) into (15) and (16) results in

$$\begin{aligned} \epsilon_1 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} &= \frac{2}{\lambda(1 + \lambda/2)^{\ell''+1/2}} \sum_{j=0}^{\infty} \left\{ (1+z) R_{0j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} k_0 \right. \\ &\quad \left. + (\ell+2n-\ell'-2n') R_{1j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} k_0 \right\} z^j \end{aligned}$$

and

$$\begin{aligned} S_2 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} &= \frac{1}{\lambda^2(1 + \lambda/2)^{\ell''-1/2}} \sum_{j=0}^{\infty} \left\{ (1+z)^2 R_{0j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} k_0 \right. \\ &\quad - (\ell+2n-\ell'-2n') (\ell+2n-\ell'-2n'+1) R_{2j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} k_0 \\ &\quad - 2(n'+1) R_{2j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n'+1 & n'' \end{bmatrix} k_0 \\ &\quad \left. + 2n R_{2j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n-1 & n' & n'' \end{bmatrix} k_0 \right\} z^j, \end{aligned}$$

where the identity $\lambda/(1 + \lambda/2) = 1 + z$, was used. The comparison of (41) with the above shows that

$$S_{1j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} = 2 R_{0,j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} k_0 + 2 R_{0,j-1} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} k_0 \quad (42a)$$

$$+ 2(\ell+2n-\ell'-2n') R_{1,j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} k_0,$$

and

$$S_{2j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} = R_{0,j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} k_0 + 2 R_{0,j-1} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} k_0 \quad (42b)$$

$$+ R_{0,j-2} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} k_0$$

$$- (\ell+2n-\ell'-2n') (\ell+2n-\ell'-2n'+1) R_{2,j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} k_0$$

$$- 2(n'+1) R_{2j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n'+1 & n'' \end{bmatrix} k_0$$

$$+ 2n R_{2j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n-1 & n' & n'' \end{bmatrix} k_0.$$

C. Large λ Expansions

For this final case we expand $R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} k$ in a power series in $2/(1 + \lambda/2)$, i.e.,

$$R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} k = \frac{1}{(1 + \lambda/2)^{\ell''+3/2-M}} \sum_{j=0}^{\infty} R_{M,j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} k z^j \quad (43)$$

where

$$z = \frac{2}{(1 + \lambda/2)} \quad (44)$$

From (27) we obtain

$$R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & -1 \end{bmatrix}_0 = 0 \quad (45a)$$

for all j ,

and

$$R_{M0} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{bmatrix}_0 = \frac{\pi \Gamma(\ell'' + 3/2 - M)}{\Gamma(\ell + 3/2)}, \quad (45b)$$

for $j = 0$,

with

$$R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{bmatrix}_0 = 0 \quad (45c)$$

for all $1 \leq j$.

Rewriting (28) in terms of z leads to

$$\begin{aligned} R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_0 \\ = -\frac{1}{n''} \left[(\ell'' + 2n'' - \frac{1}{2}) - (\ell'' + n'' + \frac{1}{2} - M)z \right] R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' - 1 \end{bmatrix}_0 \\ - \frac{1}{n''} (\ell'' + n'' - \frac{1}{2}) (1 - z) R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' - 2 \end{bmatrix}_0, \end{aligned}$$

and hence that

$$\begin{aligned} R_{M,j} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_0 \\ = -\frac{1}{n''} (\ell'' + 2n'' - \frac{1}{2}) R_{M,j} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' - 1 \end{bmatrix}_0 \\ + \frac{1}{n''} (\ell'' + n'' + \frac{1}{2} - M) R_{M,j-1} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' - 1 \end{bmatrix}_0 \\ - \frac{1}{n''} (\ell'' + n'' - \frac{1}{2}) R_{M,j} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' - 2 \end{bmatrix}_0 \end{aligned} \quad (46)$$

$$+ \frac{1}{n''} (\ell'' + n'' - \frac{1}{2}) R_{M,j-1} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-2 \end{bmatrix} 0,$$

the recurrence relation needed for $R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix} 0$.

Similarly (26) expressed in terms of z is

$$\begin{aligned} R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix} k \\ = \left[(k - n'' + M - 1) + \frac{1}{2} (\ell'' + k + n'' - M + \frac{1}{2}) z \right] R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix} k-1 \\ - (\ell'' + n'' + \frac{1}{2}) (1 - \frac{1}{2} z) R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-1 \end{bmatrix} k-1 \\ - \frac{1}{2} (k-1) (\ell'' + k - \frac{1}{2}) z R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix} k-2, \end{aligned}$$

which results in the expression

$$\begin{aligned} R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix} k \\ = (k - n'' + M - 1) R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix} k-1 \\ + \frac{1}{2} (\ell'' + k + n'' - M + \frac{1}{2}) R_{Mj-1} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix} k-1 \\ - (\ell'' + n'' + \frac{1}{2}) R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-1 \end{bmatrix} k-1 \\ + \frac{1}{2} (\ell'' + n'' + \frac{1}{2}) R_{Mj-1} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n''-1 \end{bmatrix} k-1 \\ - \frac{1}{2} (k-1) (\ell'' + k - \frac{1}{2}) R_{Mj-1} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix} k-2 \end{aligned} \tag{47}$$

which permits the calculation of all $R_{Mj} \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix} k$.

Again expressions (27) and (28) are valid with $R_{Mj} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right]_k$ replacing $R_M \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right]_k$, since there is no explicit λ dependence in them.

The substitution of (37) into (15) and (16) results in

$$S_1 \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right] = \frac{2}{(1 + \lambda/2)^{\ell''+3/2} (2-z)} \sum_{j=0}^{\infty} \left\{ (2-z) R_{0,j} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right]_{k_0} \right. \\ \left. + (\ell+2n-\ell'-2n') R_{1,j} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right]_{k_0} \right\} z^j$$

and

$$S_2 \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right] = \frac{1}{(1 + \lambda/2)^{\ell''+3/2} (2-z)^2} \sum_{j=0}^{\infty} \left\{ (2-z)^2 R_{0,j} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right]_{k_0} \right. \\ - (\ell+2n-\ell'-2n') (\ell+2n-\ell'-2n'+1) R_{2,j} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right]_{k_0} \\ - 2(n'+1) R_{2,j} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n'+1 & n'' \end{smallmatrix} \right]_{k_0} \\ \left. + 2n R_{2,j} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n-1 & n' & n'' \end{smallmatrix} \right]_{k_0} \right\} z^2.$$

If we define

$$S_{1j} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right] = 4 R_{0j} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right]_{k_0} - 2 R_{0j-1} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right]_{k_0} \quad (48a) \\ + 2(\ell+2n-\ell'-2n') R_{1j} \left[\begin{smallmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{smallmatrix} \right]_{k_0}$$

and

$$\begin{aligned}
 S_{2j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} &= 4 R_{0j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix}_{k_0} - 4 R_{0j-1} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix}_{k_0} \\
 &+ R_{0j-2} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix}_{k_0} \\
 &- (\ell+2n-\ell'-2n') (\ell+2n-\ell'-2n'+1) R_{2j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix}_{k_0} \\
 &- 2(n'+1) R_{2j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n'+1 & n'' \end{bmatrix}_{k_0} \\
 &+ 2n R_{2j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n-1 & n' & n'' \end{bmatrix}_{k_0} ,
 \end{aligned} \tag{48b}$$

it follows that

$$S_1 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} = \frac{1}{(1 + \lambda/2)^{\ell''+3/2} (2 - z)} \sum_{j=0}^{\infty} S_{1j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} z^j, \tag{49a}$$

and

$$S_2 \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} = \frac{1}{(1 + \lambda/2)^{\ell''+3/2} (2 - z)^2} \sum_{j=0}^{\infty} S_{2j} \begin{bmatrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{bmatrix} z^j. \tag{49b}$$

REFERENCES

1. H. K. Meier, W. I. van Rij, C. O. Beasley, Jr., and J. E. McCune,
"The Collisional Plasma Model: A Velocity-Space Orthogonal-Function
Representation for the Distribution Function of a Collisional
Plasma," ORNL/TM-5314, (1976).
2. Bateman Manuscript Project, Higher Transcendental Functions and Tables
of Integral Transforms, McGraw-Hill, New York (1953).

THIS PAGE
WAS INTENTIONALLY
LEFT BLANK

APPENDIX

Derivation of the Recursion Relations

Since the generalized Laguerre polynomials have the property

$$L_0^{\ell+1/2}(x) = 1$$

it follows from expression (19) that

$$Q \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & j \end{bmatrix}_k = \int_0^\infty dx x^{\ell''+k+1/2} e^{-(1+\lambda/2)x} \Phi(-j, \ell'' + \frac{3}{2}; \frac{\lambda}{2}x), \quad (A1)$$

$$= \frac{\Gamma(\ell'' + k + \frac{3}{2})}{(1 + \frac{\lambda}{2})^{\ell''+k+3/2}} F \left(-j, \ell'' + k + \frac{3}{2}; \ell'' + \frac{3}{2}; \frac{\lambda}{2 + \lambda} \right),$$

the above integral being a standard Laplace transform. F is a hypergeometric function² of type ${}_2F_1(a, b; c; z)$.

From expression (20) and (A1) it is seen that

$$R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_k = \frac{(-1)^{n''} \pi \Gamma(\ell'' + n'' + \frac{3}{2}) \Gamma(\ell'' + k + \frac{3}{2})}{\Gamma(\ell + \frac{3}{2}) \Gamma(\ell'' + \frac{3}{2}) (1 + \frac{\lambda}{2})^{\ell''+k+3/2}} \quad (A2)$$

$$\times \sum_{j=0}^{n''} \frac{(-1)^j 2^j \Gamma(\ell'' + j + \frac{3}{2} - M)}{j! (n'' - j)! \Gamma(\ell'' + j + \frac{3}{2})}$$

$$\times F(j + M, \ell'' + k + \frac{3}{2}; \ell'' + \frac{3}{2}; \frac{\lambda}{2 + \lambda}).$$

The property²

$$F(a, b; b; z) = \frac{1}{(1 - z)^a} \quad (A3)$$

can be applied to (A2) to give

$$\begin{aligned}
 R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix} 0 &= \frac{(-1)^{n''} \pi \Gamma(\ell'' + n'' + \frac{3}{2})}{\Gamma(\ell + \frac{3}{2}) (1 + \frac{\lambda}{2})^{\ell'' + 3/2 - M}} \\
 &\times \sum_{j=0}^{n''} \frac{(-1)^j 2^j \Gamma(\ell'' + j + \frac{3}{2} - M)}{j! (n'' - j)! \Gamma(\ell'' + j + \frac{3}{2})} \left(\frac{1}{1 + \frac{\lambda}{2}} \right)^j \\
 &= \frac{(-1)^{n''} \pi \Gamma(\ell'' + n'' + \frac{3}{2}) \Gamma(\ell'' + \frac{3}{2} - M)}{n''! \Gamma(\ell + \frac{3}{2}) \Gamma(\ell'' + \frac{3}{2}) (1 + \frac{\lambda}{2})^{\ell'' + 3/2 - M}} \\
 &\times F \left(-n'', \ell'' + \frac{3}{2} - M; \ell'' + \frac{3}{2}; \frac{2}{1 + \frac{\lambda}{2}} \right). \tag{A4}
 \end{aligned}$$

In particular, (A4) gives

$$R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & -1 \end{bmatrix} 0 = 0,$$

and

$$R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{bmatrix} 0 = \frac{\pi \Gamma(\ell'' + \frac{3}{2} - M)}{\Gamma(\ell + \frac{3}{2}) (1 + \frac{\lambda}{2})^{\ell'' + 3/2 - M}},$$

which is expression (24). The identity

$$\begin{aligned}
 (2a - c - az + bz) F(a, b; c; z) \\
 + (c - a) F(a - 1, b; c; z) + a(z-1) F(a + 1, b; c; z) = 0, \tag{A5}
 \end{aligned}$$

applied to (A4) with $a = -n'' + 1$, $b = \ell'' + 3/2 - M$, $c = \ell'' + 3/2$,

and $z = 2/(1 + \lambda/2)$, gives the recurrence relation (25). Equation (A5),

with a and b interchanged, and with $a = -j + M$, $b = \ell'' + k + 1/2$,

$c = \ell'' + 3/2$, and $z = \lambda/(2 + \lambda)$ gives

$$\begin{aligned}
& - (k - 1) F\left(-j + M, \ell'' + k - \frac{1}{2}; \ell'' + \frac{3}{2}; \frac{\lambda}{2 + \lambda}\right) \\
& + \left[(\ell'' + 2k - \frac{1}{2}) - (\ell'' + k + j - M + \frac{1}{2}) \frac{\lambda}{2 + \lambda} \right] \\
& \quad \cdot F\left(-j + M, \ell'' + k + \frac{1}{2}, \ell'' + \frac{3}{2}, \frac{\lambda}{2 + \lambda}\right) \\
& - \frac{(\ell'' + k + \frac{1}{2})}{(1 + \frac{\lambda}{2})} F\left(-j + M, \ell'' + k + \frac{3}{2}; \ell'' + \frac{3}{2}; \frac{\lambda}{2 + \lambda}\right) = 0
\end{aligned}$$

or

$$\begin{aligned}
& F\left(-j + M, \ell'' + k + \frac{3}{2}; \ell'' + \frac{3}{2}; \frac{\lambda}{2 + \lambda}\right) \\
& = \frac{(1 + \lambda/2)}{(\ell'' + k + \frac{1}{2})} \left\{ \left[\left(\ell'' + 2k - \frac{1}{2} \right) - \left(\ell'' + k + n'' - M + \frac{1}{2} \right) \frac{\lambda}{2 + \lambda} \right. \right. \\
& \quad \left. \left. + (n'' - j) \frac{\lambda}{2 + \lambda} \right] F\left(-j + M, \ell'' + k + \frac{1}{2}; \ell'' + \frac{3}{2}, \frac{\lambda}{2 + \lambda}\right) \right. \\
& \quad \left. - (k - 1) F\left(-j + M, \ell'' + k - \frac{1}{2}; \ell'' + \frac{3}{2}; \frac{\lambda}{2 + \lambda}\right) \right\}
\end{aligned}$$

This substituted into (A2) yields

$$\begin{aligned}
R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix} k &= \frac{(-1)^{n''} \pi \Gamma(\ell'' + n'' + \frac{3}{2}) \Gamma(\ell'' + k + \frac{1}{2})}{\Gamma(\ell + \frac{3}{2}) \Gamma(\ell'' + \frac{3}{2}) (1 + \frac{\lambda}{2})^{\ell'' + k + 1/2}} \\
& \cdot \sum_{j=0}^{n''} \frac{(-1)^j 2^j}{j! (n'' - j)!} \frac{\Gamma(\ell'' + j + \frac{3}{2} - M)}{\Gamma(\ell'' + j + \frac{3}{2})} \left\{ \left[(\ell'' + 2k - \frac{1}{2}) - (\ell'' + k + n'' - M + \frac{1}{2}) \frac{\lambda}{2 + \lambda} \right. \right. \\
& \quad \left. \left. + (n'' - j) \frac{\lambda}{2 + \lambda} \right] F\left(-j + M, \ell'' + k + \frac{1}{2}; \ell'' + \frac{3}{2}, \frac{\lambda}{2 + \lambda}\right) \right\}
\end{aligned}$$

$$\begin{aligned}
& - (k - 1) F\left(-j + M, \ell'' + k - \frac{1}{2}; \ell'' + \frac{3}{2}; \frac{\lambda}{2 + \lambda}\right) \Bigg\} \\
& = \left[(\ell'' + 2k - \frac{1}{2}) - (\ell'' + k + n'' - M + \frac{1}{2}) \frac{\lambda}{2 + \lambda} \right] R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_{k-1} \\
& - (\ell'' + n'' + \frac{1}{2}) \frac{\lambda}{2 + \lambda} R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' - 1 \end{bmatrix}_{k-1} \\
& - \frac{(k - 1) (\ell'' + k - \frac{1}{2})}{(1 + \frac{\lambda}{2})} R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & n'' \end{bmatrix}_{k-2} \tag{A6}
\end{aligned}$$

which is expression (26).

Starting with the identity

$(n + 1) L_{n+1}^\alpha(x) - (2n + \alpha + 1 - x) L_n^\alpha(x) + (n + \alpha) L_{n-1}^\alpha(x) = 0$ which is valid for the generalized Laguerre polynomials, and taking $\alpha = \ell' + \frac{1}{2}$, and replacing n by $n' - 1$ one obtains

$$\begin{aligned}
L_{n'}^{\ell' + 1/2}(x) &= \frac{1}{n'} \left\{ \left(\ell' + 2n' - \frac{1}{2} - x \right) L_{n'-1}^{\ell' + 1/2}(x) \right. \\
&\quad \left. - \left(\ell' + n' - \frac{1}{2} \right) L_{n'-2}^{\ell' + 1/2}(x) \right\} \tag{A7}
\end{aligned}$$

This expression is valid for $1 \leq n'$ provided one defines $L_{-1}^{\ell' + 1/2}(x)$

to be zero. Now substitute this into expression (19) and

$$\begin{aligned}
Q \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & n' & j \end{bmatrix}_k &= \int_0^\infty dx x^{\ell'' + k + 1/2} e^{-(1 + \lambda/2)x} \\
&\cdot \frac{1}{n'} \left[\left(\ell' + 2n' - \frac{1}{2} - x \right) L_{n'-1}^{\ell' + 1/2}(x) \right. \\
&\quad \left. - \left(\ell' + n' - \frac{1}{2} \right) L_{n'-2}^{\ell' + 1/2}(x) \right] \Phi(-j, \ell'' + \frac{3}{2}; \frac{\lambda}{2} x)
\end{aligned}$$

$$= \frac{1}{n'} \left\{ (\ell' + 2n' - \frac{1}{2}) Q \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & n'-1 & j \end{bmatrix}_k \right. \\ \left. - Q \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & n'-1 & j \end{bmatrix}_{k+1} - (\ell' + n' - \frac{1}{2}) Q \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & n'-2 & j \end{bmatrix}_k \right\}$$

Which again is valid for $1 \leq n'$, and

$$Q \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & -1 & j \end{bmatrix}_k = 0$$

since

$$L_{-1}^{\ell'+1/2} = 0.$$

The substitution of the above into (20) results in

$$R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & n' & n'' \end{bmatrix}_k = \frac{1}{n'} \left\{ (\ell' + 2n' - \frac{1}{2}) R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & n'-1 & n'' \end{bmatrix}_k \right. \\ \left. - R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & n'-1 & n'' \end{bmatrix}_{k+1} \right. \\ \left. - (\ell' + n' - \frac{1}{2}) R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & n'-2 & n'' \end{bmatrix}_k \right\}, \quad (A8)$$

with

$$R_M \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & -1 & n'' \end{bmatrix}_k = 0, \quad (A9)$$

since

$$Q \begin{bmatrix} \ell & \ell' & \ell'' \\ 0 & -1 & j \end{bmatrix}_k = 0.$$

Expression (A9) is (27a) and (A8) is (27b).

In a similar manner using (A7) and (19) it is seen that

$$\begin{aligned}
Q \left[\begin{matrix} \ell & \ell' & \ell'' \\ n & n' & j \end{matrix} \right]_k &= \int_0^\infty dx x^{\ell'' + k + 1/2} e^{-(1 + \lambda/2)x} \\
&\cdot \frac{1}{n} \left[\left(\ell + 2n - \frac{1}{2} - x \right) L_{n-1}^{\ell + 1/2}(x) \right. \\
&- \left. \left(\ell + n - \frac{1}{2} \right) L_{n-2}^{\ell + 1/2}(x) \right] L_{n'}^{\ell' + 1/2}(x) \Phi(-j, \ell'' + \frac{3}{2}; \frac{\lambda}{2} x) \\
&= \frac{1}{n} \left\{ \left(\ell + 2n - \frac{1}{2} \right) Q \left[\begin{matrix} \ell & \ell' & \ell'' \\ n-1 & n' & j \end{matrix} \right]_k \right. \\
&- Q \left[\begin{matrix} \ell & \ell' & \ell'' \\ n-1 & n' & j \end{matrix} \right]_{k+1} \\
&- \left. \left(\ell + n - \frac{1}{2} \right) Q \left[\begin{matrix} \ell & \ell' & \ell'' \\ n-2 & n' & j \end{matrix} \right]_k \right\}
\end{aligned}$$

and this substituted into (20) yields

$$\begin{aligned}
R_M \left[\begin{matrix} \ell & \ell' & \ell'' \\ n & n' & n'' \end{matrix} \right]_k &= \frac{1}{(\ell + n + \frac{1}{2})} \left\{ \left(\ell + 2n - \frac{1}{2} \right) R_M \left[\begin{matrix} \ell & \ell' & \ell'' \\ n-1 & n' & n'' \end{matrix} \right]_k \right. \\
&- R_M \left[\begin{matrix} \ell & \ell' & \ell'' \\ n-1 & n' & n'' \end{matrix} \right]_{k+1} \\
&- \left. (n - 1) R_M \left[\begin{matrix} \ell & \ell' & \ell'' \\ n-2 & n' & n'' \end{matrix} \right]_k \right\}
\end{aligned} \tag{A10}$$

This is expression (28) and the derivations are complete.

INTERNAL DISTRIBUTION

- 1-3. Lab Records
- 4. Lab Records - RC
- 5. Tech. Support Div., ORO, ERDA
- 6. Y-12 Document Reference
- 7-8. Central Research Library
- 9. Thermonuclear Library
- 10. ORNL Patent Office
- 11. J. D. Callen
- 12. J. F. Clarke
- 13. R. A. Dory
- 14. G. G. Kelley
- 15. D. G. McAlees
- 16. O. B. Morgan
- 17. M. W. Rosenthal
- 18-43. H. K. Meier

EXTERNAL DISTRIBUTION

- 44. Plasma Physics Library, Plasma Physics Laboratory, Princeton Univ., Forrestal Campus, P.O. Box 451, Princeton, NJ 08540
- 45. Controlled Thermonuclear Research Library, Lawrence Livermore Laboratory, P.O. Box 808, Livermore, CA 94550
- 46. Q Division Library, c/o F. L. Ribe, Los Alamos Scientific Laboratory, P.O. Box 1663, Los Alamos, NM 87544
- 47. Controlled Thermonuclear Research Library, c/o Weston M. Stacey, Jr., Argonne National Laboratory, 9700 S. Cass Ave., Argonne, IL 60439
- 48. CTR Computer Center, c/o Dr. John Killeen, Lawrence Livermore Laboratory, P.O. Box 808, Livermore, CA 94550
- 49. Librarian, Culham Laboratory, U.K. Atomic Energy Authority, Abingdon, Oxon, OX14, 3DB, United Kingdom
- 50. Ruth Lengye, Bibliothek, Max Planck Institut für Plasmaphysik, 8046 Garching bei München, Federal Republic of Germany
- 51. Library, Centre de Recherches en Physique des Plasma, 21 Avenue des Bains, 1007, Lausanne, Switzerland
- 52. A. M. Dupas, Documentation S.I.G.N., Department de La Physique du Plasma, Et de La Fusion Controlee, Association EURATOM-CEA, Sur La Fusion, Centre d'Etudes Nucleaires, BP 85 Centre Du TRI 38041 Grenoble Cedex (France)
- 53. Bibliothèque, Service du Confinement des Plasmas, C.E.A., B.P. No. 6, 92, Fontenay-aux-Roses (Seine) France

54. Library, International Centre for Theoretical Physics, Trieste, Italy
55. Library, Laboratorio Gas Ionizati, Frascati, Italy
56. V. E. Ivanov, Physical-Technical Institute of the Ukranian Academy of Sciences, Sukhumi, U.S.S.R.
57. M. S. Rabinovich, Lebedev Institute of Physics, Academy of Sciences of the U.S.S.R., Leninsky Prospect 53, Moscow, U.S.S.R.
58. Thermonuclear Laboratory, Kurchatov Institute of Atomic Energy, 46 Ulitsa Kurchatova, P.O. Box 3402, Moscow, U.S.S.R.
59. Library, Institute for Plasma Physics, Nagoya University, Nagoya, Japan 464
60. Library, FOM-Institut voor Plasma-Fysica, Rijnhuizen, Jutphaas, Netherlands
61. Plasma Physics Group, Department of Engineering Physics, Australian National University, P.O. Box 4, Canberra A.C.T. 2600 Australia
62. Thermonuclear Library, Japan Atomic Energy Research Institute, Tokai, Naka, Ibaraki, Japan
63. CTR Reading Room, c/o Prof. Dieter J. Sigmar, Room 37-391, M.I.T., Cambridge, MA 02139
64. CTR Reading Room, c/o Prof. D. W. Kerst, Dept. of Physics, Sterling Hall, Univ. of Wisconsin, Madison, WI 53706
65. CTR Reading Room, c/o Prof. I. B. Bernstein, Yale Univ., New Haven, CT 06510
66. Center for Plasma Physics and Thermonuclear Research, Univ. of Texas, Physics Building 330, Austin, TX 78712
67. CTR Reading Room, c/o Prof. B. D. Fried, Physics Dept., Univ. of California, Los Angeles, CA 90024
68. CTR Reading Room, c/o Prof. David C. Montgomery, Physics & Astronomy Dept., Univ. of Iowa, Iowa City, IA 52240
69. Magneto-Fluid-Dynamics Library, Courant Inst. of Math. Sci., New York Univ., 251 Mercer St., New York, NY 10012
70. CTR Reading Room, c/o Prof. Allan N. Kaufman, Physics Dept., Univ. of California, Berkeley, CA 94720
71. CTR Reading Room, c/o Prof. W. B. Thompson, Univ. of California, San Diego, Physics Dept., La Jolla, CA 92037

72. CTR Reading Room, c/o Prof. Alvin W. Trivelpiece, Dept. of Physics & Astronomy, Univ. of Maryland, College Park, MD 20742
73. CTR Reading Room, c/o Prof. T. Kammash, 103 Research Admin. Bldg., N. Campus, Univ. of Michigan, Ann Arbor, MI 48105
74. CTR Reading Room, c/o Dr. Ravi N. Sudan, Phillips Hall, Cornell Univ., Ithaca, NY 14850
75. Prof. Marshall N. Rosenbluth, Institute for Advanced Study, Princeton, NJ 08540
76. CTR Reading Room, c/o Prof. R. Gross, Plasma Research Lab., Columbia Univ., New York, NY 10027
77. CTR Reading Room, c/o Prof. Roy Gould, California Inst. of Tech., Pasadena, CA 91103
78. Dr. Nicholas A. Krall, Science Applications, Inc., P.O. Box 2354, 1200 Prospect St., La Jolla, CA 92037
79. CTR Reading Room, c/o Dr. Jay P. Boris, Plasma Physics, Naval Research Laboratory, Washington, DC 20390
80. CTR Library, General Atomic Co., P.O. Box 81608, San Diego, CA 92138
81. CTR Library, c/o Dr. Alan F. Haught, United Technologies Research Labs, East Hartford, CT 06108
82. Dr. Robert E. Price, Division of Controlled Thermonuclear Research, Energy Research and Development Administration, Washington, DC 20545
83. Dr. Bennett Miller, Division of Controlled Thermonuclear Research, Energy Research and Development Administration, Washington, DC 20545
84. Dr. Arthur Sleeper, Division of Controlled Thermonuclear Research, Energy Research and Development Administration, Washington, DC 20545
85. Dr. Walter Sadowski, Division of Controlled Thermonuclear Research, Energy Research and Development Administration, Washington, DC 20545
86. Dr. D. H. Priestler, Division of Controlled Thermonuclear Research, Energy Research and Development Administration, Washington, DC 20545
87. Dr. Paul H. Rutherford, Princeton Plasma Physics Lab., Princeton Univ., Princeton, NJ 08540

88. Dr. L. D. Pearlstein, L-388, Lawrence Livermore Laboratory, P.O. Box 808, Livermore, CA 94550
89. Dr. J. P. Friedberg, Los Alamos Scientific Laboratory, Los Alamos NM 87544
90. Dr. David W. Ross, Center for Plasma Physics & Thermonuclear Research, Dept. of Physics, Univ. of Texas, Austin, TX 78712
91. Dr. Gareth E. Guest, General Atomic Co., P.O. Box 81608, San Diego, CA 92138
92. Dr. Claude Mercier, Service du Theorie des Plasmas, Centre d'Etudes Nucleaires, Fontenay-aux-Roses (Seine) France
93. Dr. J. B. Taylor, Culham Laboratory, UKAEA Abingdon, Oxon, OX14 3DB, United Kingdom
94. Dr. D. Pfirsch, Institute for Plasma Physics, 8046 Garching bei München, Federal Republic of Germany
95. Dr. V. D. Shafranov, I. V. Kurchatov Inst. of Atomic Energy, 46 Ulitsa Kurchatova, P.O. Box 3402, Moscow, U.S.S.R.
96. Dr. Harold Grad, Courant Inst. of Math. Sci., New York Univ., 251 Mercer St., New York, NY 10012
97. Dr. J. G. Cordey, Culham Laboratory, UKAEA, Abingdon, Oxon, OX14, 3DB, United Kingdom
98. Dr. David Baldwin, L-388, Lawrence Livermore Lab., P.O. Box 808, Livermore, CA 94550
99. Prof. Bruno Coppi, Dept. of Physics, M.I.T., Cambridge, MA 02139
100. Dr. Harold P. Furth, Princeton Plasma Physics Lab., Princeton Univ., P.O. Box 451, Princeton, NJ 08540
- 101-127. Tech. Information Center, ERDA.