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Some Global Problems in Gauge Theories
(Variations on a Theme of Aharonov and Bohm)*

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ABSTRACT

Several situations are discussed, in which the sort of global considerations made famous by Aharonov and Bohm in their discussion of the interaction of charged particles with magnetic flux tubes have important physical implications. It is argued that discrete gauge symmetries in the continuum make sense, and manifest themselves most clearly in Aharonov-Bohm type scattering of charged particles off string singularities. The existence of such discrete symmetries has important implications for the quantum mechanics of topologically non-trivial space-times in general and black holes in particular. It is argued that in the non-abelian case essentially new features arise, most notably that the symmetry group of the homogeneous ground state generally ceases to be globally defined in the presence of a string. When continuous rather than discrete symmetries are involved, a variety of fascinating and as yet poorly understood dynamical effects occur. Perhaps the most striking is a new form of string superconductivity, that exists for purely topological reasons, and is *not* well modeled by regarding the string as a superconducting wire.

In their famous paper, to which we pay tribute today, Aharonov and Bohm emphasized that one may have effects associated with the *global* structure of gauge fields, even in cases where these fields are trivial locally. I would like to describe for you some recent work, which shows that 30 years later we are still far from exhausting the full meaning of this deep concept.

1. Discrete Gauge Theories in the Continuum

1. Upon first hearing the concept of a discrete gauge theory in the continuum might sound rather silly. Since any path can be continuously deformed to a trivial path, the corresponding parallel transports can be continuously deformed to the identity. But for a discrete group this means we must have the identity all along: thus all parallel transport is trivial.

Also, if the discrete gauge group arises as the remnant of an initially continuous group, after all continuous symmetries have been wiped out by the Higgs mechanism, one is accustomed to think that charge is completely screened.

2. Upon further reflection, however, one realizes that each of these arguments has serious limitations.

First, there are situations in which one cannot deform all paths to the identity. This occurs, by definition, on spaces that are not simply connected.

Second, there are situations in which the parallel transport is not a continuous function of the path, or is ill-defined for some paths. An important class of examples, which will be extensively considered below, concerns flux tubes in the Higgs phase of gauge theories.

What about screening? One should appreciate that the screening occurring in the Higgs mechanism is not a mystical process, but essentially a special case of the homely phenomenon of dielectric polarization as occurs in ordinary dielectrics or plasmas. It is true that we are concerned, in the Higgs mechanism, with a particularly effective screening, produced by a condensate of massless charged particles.

Such particles are capable of producing a dipole distribution, to exactly cancel any imposed electric field. The particles in the condensate are singlets under the remnant discrete symmetry, however, and it is unreasonable to expect that any distribution of them can perfectly mimic a non-singlet source.

3. As a definite example, let us consider the case of a $U(1)$ gauge theory spontaneously broken to a discrete Z_p subgroup. That is, we imagine that some charge p field ϕ condenses, but that there are additional unit charge particles, produced by a field ψ , in the theory. The case $p = 2$ is realized in ordinary BCS superconductors, where the doubly charged Cooper pair field condenses, and there are additional singly charged fields to describe the “normal electron” excitations.

Such a theory supports vortex solutions, where the ϕ field behaves asymptotically as a function of the angle θ as

$$\phi(r, \theta) \rightarrow v e^{i\theta}, \quad r \rightarrow \infty \quad (1.1)$$

where v is the value of ϕ in the homogeneous groundstate. Along with this asymptotics for ϕ we must have for the gauge potential

$$A_\theta(r, \theta) \rightarrow \frac{1}{pe} \quad (1.2)$$

so that the covariant derivative $D_\theta \phi = (\partial_\theta \phi - pe A_\theta)$, which appears (squared) in the energy density, vanishes at infinity.

In this set-up the field strength also vanishes asymptotically. Indeed we note that since under a gauge transformation the fields transform as

$$\begin{aligned} \phi'(x) &= \exp(iQ\Lambda(x))\phi(x) = \exp(ipe\Lambda(x))\phi(x) \\ A'_\mu(x) &= A_\mu(x) + \partial_\mu \Lambda(x) \end{aligned} \quad (1.3)$$

we can formally, by making the choice $\Lambda = -\theta/pe$, remove the space dependence of ϕ in (1.1) and make A_θ in (1.2) vanish altogether – we have transformed back

to the homogeneous groundstate. Of course the gauge transformation function Λ is not quite kosher, since the angle θ is not a legitimate single-valued function. The correct statement is that the vortex asymptotics is trivial and can be gauged away locally, but not globally. Since we can pick a well defined branch of θ in any patch that does not surround the origin, all local gauge invariant quantities must reduce to their groundstate values – this explains, if you like, why $D\phi$ and F vanish. However the line integral of A around a closed loop surrounding the origin, which according to Stokes measures the flux inside, cannot be changed by any legitimate gauge transformation, and it is definitely *not* zero for the vortex. Indeed we find the basic flux unit is $\Phi_p = \frac{2\pi}{pe}$.

Another perspective on the global non-triviality of the vortex, is that our putative gauge transformation $\Lambda = -\theta/pe$ transforms a unit charge field ψ into something that is not single-valued: following (1.3) we find that $\psi'(\theta + 2\pi) = \exp(-\frac{2\pi}{p})\psi'(\theta)$.

What has all this got to do with discrete gauge theories in the continuum? Well, the condensate $\phi = v$ is not invariant under a general gauge transformation, but it is invariant under the discrete subgroup generated by $\Lambda = \frac{1}{pe}$. This discrete subgroup acts trivially both on ϕ and, it would seem, on the gauge field (Λ , since it takes only discrete values, cannot change continuously at all). However, Λ and its various powers are definitely not trivial acting on ψ , which gets multiplied by powers of p^{th} roots of unity. Thus there is a discrete but non-trivial gauge subgroup left. Moreover, *the gauge transformation associated with winding around a vortex is precisely an element of the residual discrete gauge symmetry group* – this is just a restatement, in our new interpretation, of the result of the previous paragraph. Alternatively we could say that the Wilson loop for parallel transport around the vortex defines an element of the residual gauge group:

$$\exp\left(\oint ieA_\mu dx^\mu\right) \in Z_p. \quad (1.4)$$

We anticipated that the discrete gauge group might come into its own in the presence of singularities. Now we see how this is quite simply realized. The required

“singularity” is here supplied by the vortex core. In that core, where the condensate vanishes, the discrete gauge symmetry blows up into a full scale continuous $U(1)$, and the vector potential is unfrozen. The only trace of the vortex visible outside the vortex core is the total flux – which is none other than an element of the residual discrete gauge symmetry group.

4. Now that we have convinced ourselves that the idea of discrete gauge symmetry in the continuum makes sense, the next step is to elucidate its physical implications. The first of these, has to do with the angular momentum of particles in the presence of the vortex.

The asymptotics (1.1) of the scalar order parameter seems not to be rotationally invariant: a scalar field should be unchanged by a rotation, whereas in (1.1) it acquires a phase. However the phase of ϕ is gauge dependent and we cannot infer from (1.1) that any physical property of the vortex violates rotation symmetry. In fact it is easy to see that if we supplement the naive rotation generator J_z with an appropriate gauge transformation:

$$K_z = J_z - \frac{1}{pe} Q \quad (1.5)$$

then K_z leaves both the action, and the asymptotic scalar field configuration (1.1) invariant.

Thus – assuming the core is invariant – a true rotation symmetry of the vortex is generated by K_z . (If the core is not invariant, since the asymptotics is invariant the solution will still have a finite moment of inertia, and upon proper quantization we will get a spectrum of rotational excitations of the vortex similar to the band spectrum of an asymmetric molecule.) By way of comparison, the vortex associated with a global gauge symmetry truly breaks rotational symmetry, and has infinite moment of inertia.

An interesting consequence of the modification of J into K is that the quantization of angular momentum, for quanta orbiting the vortex, is modified. The

angular momentum of quanta with the fundamental charge e , for example, is quantized in units of $-\frac{1}{p} + \text{integer}$. This modification of the quantization condition, and the associated existence of states with fractional angular momentum, is one basic physical consequence of the discrete gauge symmetry.

5. Since the spin and statistics of particles are deeply related, we might be led to ask whether there is peculiar statistics of the excitations with peculiar angular momentum. There is indeed. The emergence of fractional statistics for the sort of vortex-particle composite that carries fractional angular momentum may be seen as a consequence of the same kind of Aharonov-Bohm phase that is responsible for the fractional angular momentum. Indeed as one such composite particle is wound around another, it acquires phase as the charge is transported through the vector potential, according to the action $\int A \cdot j dt = q \int A \cdot dr$.

The physics of fractional statistics particles, or anyons, has grown from a curiosity into quite a substantial subject with solid applications to the fractional quantized Hall effect and many more conjectural applications. Since I have extensively reviewed this rich and rapidly developing subject elsewhere recently, I will not further expand on it here.

6. A direct physical consequence of the modification of the angular momentum quantization condition is the existence of a deeply geometric scattering process, which is in fact basically the one studied by Aharonov and Bohm in their original paper. I will merely quote the famous result:

$$\frac{d\sigma}{d\theta} = \frac{1}{2\pi k} \sin^2\left(\frac{q\Phi}{2}\right) \frac{1}{\sin^2\left(\frac{\theta}{2}\right)}, \quad (1.6)$$

where $q\Phi$ is the product of charge times flux, and make a few comments on the different factors. The cross-section is directly proportional to the DeBroglie wavelength of the incoming particle. This is very reasonable: all those wave packets, and only those, that actually pass through the core of the vortex have their internal phase relationships reshuffled. Secondly, there is a factor which depends on

the *fractional part* of the charge-flux product. This is also the fractional part of the angular momentum, or the fractional part of the statistics, according to our other interpretations. Third, there is an angular factor, that again has a purely geometric origin. Note that this factor is singular in the forward direction (it goes as the square root of the Rutherford factor in Coulomb scattering). This singularity reflects the long-range character of the interaction responsible for the scattering, and insures that *there is a characteristic part of the scattering cross section, separable in principle from the contribution of any short-range interaction, that directly reflects the discrete gauge charge, fractional angular momentum, or fractional statistics*. This criterion can be used as an *operational definition* of these properties for the fully dressed physical particles, or quasiparticles in a condensed system, at the level of the physical S -matrix. It is similar to the strict definition of electric charge as the appropriate coefficient in low-energy Thomson scattering.

The modification of the angular momentum quantization condition in the presence of a vortex, of course modifies and may in favorable cases remove altogether the centrifugal barrier. This is very important for cosmic strings, where the core may contain condensates of particles which mediate baryon number violation, for example. We might expect generally that the cross-section for reaching the core and tapping the potential symmetry-breaking fields lurking there must depend sensitively, at low energy, on the form of the centrifugal barrier. And detailed study does indeed reveal striking dependence of this sort; cross-sections for baryon-number violating processes may come close to saturating unitarity limits (similar to the Callan-Rubakov effect for magnetic monopoles), and vary by many orders of magnitude depending on the precise value of the charge-flux product.

7. The existence of observable discrete gauge quantum numbers has significant implications for fundamental physics.

One implication is a modification of, or at least a significant addendum to, the no hair theorems of black hole physics. The quantum numbers associated with discrete gauge symmetries are well-defined, conserved, and measurable at

arbitrarily large distances. Therefore a necessary part of specifying the quantum state of a black hole, is to specify its discrete gauge quantum numbers. On the other hand, these quantum numbers leave no *classical* residue at large distances. The usual no hair theorems, stating that the only properties of a black hole are its mass, charge, and angular momentum, are based on an analysis of the classical field equations. (Even at this level there is an additional possibility, axion charge, as was discussed in Giddings' talk here.) It appears, when we come to consider the quantum mechanics of black holes, that there are additional states, distinguishable by their quantum-mechanical properties: specifically, they differ in their Aharonov-Bohm scattering from cosmic strings.

Another implication concerns discrete symmetries. There are several "phenomenological" reasons why one might like to have discrete symmetries in fundamental models of particle physics. Such symmetries seem to be necessary to forbid catastrophically large rates for proton decay in models of low-energy supersymmetry. We might also like to use them in understanding the regularities of the quark and lepton mass matrices. On the other hand, it has been argued that all global symmetries, whether continuous or discrete, are strongly violated by wormhole effects. If the discrete symmetries in question are gauge symmetries, however, there is no cause for alarm. Discrete gauge symmetries cannot be violated by wormholes or by anything else.

8. The results of the previous paragraphs, and in particular the very strong statement in the previous sentence, are best understood as consequences of a simple yet profound distinction between gauge symmetries and global symmetries. Whereas global symmetries express that the laws of physics look the same from different perspectives, or when certain variables are substituted for others, gauge symmetries express a *redundancy* in the variables: physical results can only depend on gauge invariant quantities, even though the laws may be most conveniently expressed using gauge variant quantities in intermediate expressions. Thus gauge symmetries are tautologies; they can no more be violated than can $1 + 1 = 2$. Even such exotic processes as particles disappearing past event horizons, or baby

universes pinching off from and merging with our own, cannot alter tautologies.

The statement that gauge symmetries cannot be broken may sound peculiar to someone who has studied the standard gauge theory of weak interactions, or even the BCS theory of superconductivity, since the usual exposition of these theories makes heavy use of the concept of spontaneous breakdown of gauge symmetries. However a more accurate statement is that in the “broken” phases the gauge symmetries, rather than being violated, merely become useless. Whereas in the unbroken phase there are superselection sectors characterized by different values of the charge, in the unbroken phase only the $Q = 0$ sector exists. Actually, as we have seen, one may have a residual discrete gauge symmetry; in this case, what has happened is that there are a finite number of different superselection sectors, corresponding to the different possible values of the discrete gauge charge. Indeed the number and structure of the superselection sectors can be used to distinguish between possible phases of (totally or partially) “broken” gauge theories.

(A penetrating discussion of these issues, including nonabelian aspects that are at this moment not fully understood, is contained in the a recent paper by Preskill and Krauss, cited below.)

2. Nonabelian Aspects of Discrete Gauge Symmetries

1. Essentially new features appear when we generalize the considerations above to the case when the unbroken symmetry group H is nonabelian. Perhaps the most striking one is that in the presence of a flux tube the symmetry group H of the homogeneous vacuum, which is also the symmetry group observed locally by observers far from the tube, generally *ceases to be well defined* globally.

To be specific, let the flux through the tube, which is parametrized by an element of H that specifies the result of parallel transport around the tube, be h . Then, as we shall see in a moment, only those elements of H which commute with h are legitimate, globally defined symmetries. The other elements of H are

not so much broken, as obliterated: they are not even defined on the physical Hilbert space. This despite the fact that off the tube, and in any region that does not actually encircle the tube, the physics is rigorously the same as $-$ is a gauge transform of $-$ the physics of the homogeneous vacuum groundstate.

2. The physical reason for this phenomenon, is closely tied up with the Aharonov-Bohm effect. Consider charged particles transforming according to some representation R of H . As such particles wind around the flux tube, their internal quantum numbers are transformed according to the matrix $R(h)$. Let us imagine diagonalizing this matrix, by taking appropriate combinations of the charge states. In this basis, the action of h is simply to generate a phase proportional to the eigenvalue and to the number of windings of the superposition of particles. The physics is identical to that in the abelian case, where it leads to Aharonov-Bohm scattering as in (1.6) , with a coefficient governed by the eigenvalue.

Now according to Schur's lemma particles lying in an irreducible representation of the centralizer $C(h)$ of h will all be characterized by the same eigenvalue. This is as it must be, if $C(h)$ is to be a legitimate symmetry group. The physical properties of particles in an irreducible representation of a legitimate symmetry group, including specifically their scattering cross-sections, must be the same.

On the other hand particles which do not lie in the same irreducible representation of $C(h)$ have no particular reason to have the same eigenvalue. Indeed one can show that given any subgroup U of H properly larger than $C(h)$, then for some irreducible representation S of U , $S(h)$ will not be proportional to the identity. This will imply that the action of U definitely alters the properties of particles, specifically their cross-sections for scattering off the flux tube, and therefore that U is not a legitimate symmetry group.

3. The basis in which the scattering off a single flux tube diagonalizes, is generally different from the basis in which the scattering off a flux tube characterized by a different flux diagonalizes. Thus considering arrays of several flux tubes, widely separated, one can construct analyzers that separate different components

of a beam of non-abelian charged particles. Is it possible to determine the representation of H to which a given particle belongs? I suspect that this is possible, and Alford, March-Russell and I showed it in detail for a special example; but a general proof is lacking. It should also be possible to distinguish further quantum numbers, corresponding to any complete set of commuting observables – i.e. to any maximal abelian subalgebra of the group algebra of H . These questions are of intrinsic mathematical interest; they are, of course, also relevant to the question of possible non-abelian black hole hair.

4. Another essentially non-abelian effect, is that in the nonabelian case not only may a charge scatter off a flux tube, but also one pure flux tube can scatter off another.

There is an essential subtlety in specifying the flux associated with a non-abelian flux tube, i.e. the parallel transport around it. That is, that this flux is not gauge invariant. If we perform a gauge transformation by $h \in H$, then the parallel transport g around the tube is modified to hgh^{-1} . This would seem to indicate that the flux is defined only to within a conjugacy class. However, when we have two or more flux tubes the ambiguity is still only one overall conjugation. Thus it seems appropriate, and it is certainly convenient, to fix a gauge and to speak as if each flux tube is characterized by a definite group element. Of course all physical results must then be invariant against an overall conjugation.

Now we come to the heart of the matter: why do such flux tubes scatter at all? The reason is most easily grasped visually. We refer to Figure 1, where the passage of one flux-tube-charge composite (a, ξ) over another (b, η) is depicted. From this Figure, it is apparent that the effect of this passage is to make the change:

$$\begin{aligned} (a', \xi') &= (b, \eta) \\ (b', \eta') &= (bab^{-1}, R(b)\xi) \end{aligned} \tag{2.1}$$

where the charges transform according to the representation R . Iterating this once,

we find the result for a complete winding of one composite particle around another:

$$\begin{aligned}
(a^{(1)}, \xi^{(1)}) &= (bab^{-1}, R(b)\xi) \\
&= ((ba)a(ba)^{-1}, R(ba)R(a)^{-1}\xi) \\
(b^{(1)}, \eta^{(1)}) &= ((bab^{-1})^{-1}b(bab^{-1}), R(bab^{-1})\xi) \\
&= ((ba)b(ba)^{-1}, R(ba)R(b)^{-1}\eta).
\end{aligned} \tag{2.2}$$

In the second step of each of these equations we have written them in such a way that the result of further iterations can be read off easily. Noting that $b^{(1)}a^{(1)} = ba$, we readily find that after p windings

$$\begin{aligned}
(a^{(p)}, \xi^{(p)}) &= ((ba)^p a (ba)^{-p}, R(ba)^p R(a)^{-p} \xi) \\
(b^{(p)}, \eta^{(p)}) &= ((ba)^p b (ba)^{-p}, R(ba)^p R(b)^{-p} \eta).
\end{aligned} \tag{2.3}$$

For pure flux tubes, a result essentially equivalent to (2.2) was given by Carlip in the context of 2+1 dimensional gravity. He phrased it in terms of a geometric construction known as the Dehn twist. It may be appropriate to add a word concerning the relationship between these approaches.

To investigate the dynamics of our flux tubes, we must consider how to weight the amplitudes contributed by different possible space-time trajectories, in a Feynman path integral. Since the gauge field outside the tubes is locally trivial, the very most naive reaction is to ignore it; but of course Aharonov and Bohm taught us long ago that this is wrong. The next level of sophistication is to realize that the global residue of flux outside the tubes may be localized, by appropriate gauge transformations, onto small lines emanating from the tubes (i.e. it is only within these regions that the vector potential and parallel transport are non-trivial). Now as we consider a trajectory where flux tubes wind around one another we can, by appropriate gauge transformations, change the direction of the lines emanating from them in such away that they never cross one another. In this way, it seems that once again we have defined away the interaction. However the result of shunting the singular lines around this way, is that they get tangled up, as shown in

Figure 2. A test charge circling just one or the other of our composite particles in the final state will, in this description, have to cross several singular lines. In general, therefore, it will be parallel transported through a different transformation than for the original unwound configuration, which involved just crossing a single line. Similarly, a test flux winding around one of our composite particles will be affected by its passage through the surrounding spaghetti. Thus the effective fluxes and charges in the final state are definitely different from their original values in this (Carlip's) description as well. In fact it is not difficult to convince oneself that both descriptions of the process lead to the same result, although in my opinion the first is more transparent.

From this point the analysis proceeds along channels similar to the analysis of nonabelian charge-flux tube scattering: one diagonalizes the conjugation operation, and in this basis the calculation of the amplitude reduces to the original calculation of Aharonov and Bohm. One may then, if desired, reassemble the group theoretic factors to produce the amplitudes for generally inelastic scattering (i.e. with change of quantum numbers) in the basis where the flux tubes have definite H -flux.

In my opinion nonabelian charge-flux-tube composites are the proper generalization of anyons to the nonabelian case. They arise as quasiparticles in a simple generalization of the flux phase construction, as follows. The essence of the flux phase construction is the postulation of a correlation

$$\Gamma_{ij} = \langle c_i^\dagger c_j \rangle \quad (2.4)$$

for hopping between sites i and j , such that the total amplitude for hopping around a closed loop contains a complex phase depending on the area and the orientation (and no other properties) of the loop. With such correlations, the system has manufactured an effective magnetic field. If there are several types of fermions – e.g. with different spin, or coming from different components of a disconnected Fermi surface – then, in this construction, the same phase is assigned to each. Generically quasiparticles around such an ordered state will be associated with

defects in the ordering – vortices – and will carry fractional statistics. Now if we generalize (2.4) by allowing Γ to depend on indices labeling fermion types, then plausibly the quasiparticles acquire non-abelian statistics. Though we certainly cannot claim at present that ordering of this type occurs in the ground state of any realistic 2 dimensional Hamiltonian, such a possibility does not seem absurd and deserves further investigation.

3. Alice Electrodynamics

1. Now I'd like to mention some of the additional effects that occur when a nonabelian residual gauge symmetry group H contains a continuous part. In the discrete case the global non-existence of a locally valid symmetry, although perhaps surprising, does not offend our physical sensibilities – after all, the physical consequences of discrete gauge symmetries are pretty esoteric in the first place. It is quite a different matter when we are informed, say, that because of a flux tube somewhere in the Andromeda galaxy, electromagnetic gauge symmetry does not exist. *That* takes some explaining.

2. The phenomena that may arise for the general case of continuous, non-abelian H are rich and only very partially explored. To be concrete I'd like to discuss a comparatively simple case, that is however sufficient to display several striking phenomena, which I believe are rather generic. My example is the Alice string.

This string arises in an $SO(3)$ gauge theory which is broken down to $O(2)$ in the homogeneous ground state by the condensation of a Higgs field ϕ in the 5-dimensional symmetric traceless tensor, or isospin 2, representation. Suppose that $\phi_{11} = \phi_{22} = -\frac{1}{2}\phi_{33}$ are the non-vanishing components of the vacuum expectation value of ϕ in the homogeneous ground state. Then the residual $O(2)$ is generated by rotations around the 3-axis together with a parity transformation, say reflection in the 1-axis. Note that the residual symmetry is nonabelian: conjugation by the parity operation reverses the sense of a rotation.

An experimentalist in such a world might identify the residual gauge rotation as (his or her version of) electromagnetism.

Now consider a string whose flux is the parity reflection in $H = O(2)$. A charged particle, after winding around such a string, comes back with the sign of its electric charge reversed – a sort of looking-glass-world behavior. This is the Alice string, and now you can appreciate the reason for the name.

3. As our first example of a physical phenomenon associated with the Alice string, let us consider electrostatics. How is the electric field of a charge affected by the presence of an Alice string?

To find the field, it is simplest to work in a singular gauge, such that the vector potential is zero outside of a singular line (or, in three dimensions, a singular sheet) starting at the string and pointing directly away from the charge. Then the boundary condition on this line, is that the electric field must *change sign* across it. It is not difficult to solve the electrostatic problem with this boundary condition: one finds that the electric field is that of a charge in the presence of a conducting plate – with the singular line, which of course has no direct physical significance, playing the role of the plate!

The little paradox, that a fictitious line should be apparently the seat of a physical effect, is easily resolved. The point is that the sign of the electric field has no gauge-invariant meaning, being altered by the unbroken gauge transformation of parity in the internal $O(2)$. Only its magnitude is physically significant, and the magnitude of the electric field is perfectly continuous across the singular line.

Note that the charge is *attracted* to the string, by a long-range force.

4. A second striking phenomenon concerns the scattering of particles from the string.

The photon field changes sign as one circles the string: thus the *photon* is maximally Aharonov-Bohm scattered off an Alice string!

Consider the gauge bosons of the original $SO(3)$, which acquire mass from the

condensate. We have every right to call the charge eigenstates W^+ and W^- , since the model is closely similar to the standard model of weak interactions. Now after parallel transport around the Alice string, one finds that W^+ has transformed into W^- while W^- has transformed into W^+ . Thus the combination $W^+ + W^-$ is left unchanged, while the combination $W^+ - W^-$ changes sign after parallel transport around the string. Thus the first combination does not scatter at all, while the second combination maximally Aharonov-Bohm scatters. The net result is that when, say a W^+ beam impinges on the string, half the flux winds up going in the forward direction, but appears as a 50-50 mixture of W^+ and W^- , while the rest appears with the familiar $\frac{1}{\sin^2(\frac{\theta}{2})}$ cross-section, and at any angle again as a 50-50 mixture of both charge eigenstates.

5. A third striking phenomenon is the possible existence of nonlocalized charge in the presence of string-antistring pairs – what we like to call *Cheshire charge*.

Cheshire charge is perhaps best introduced through a thought experiment. Let there be given an Alice string and an antistring at some finite distance in two dimensions; the discussion also applies, with trivial modifications, to a large loop of Alice string in three dimensions. In this geometry the full $O(2)$ symmetry group is well-defined; the sorts of global problems which obstructed the consistent definition of some generators for a single string cancel out. In particular electric charge can be consistently defined and is rigorously conserved. Still, if we wind a charged particle slowly around a closed path enclosing the string, but not the antistring, it will return with the opposite charge. Where is the rest of the charge? Evidently, it must be left behind in a form somehow associated with the string-antistring system.

Now the winding can be done arbitrarily slowly. If the antistring were not there, one could take a circular orbit, and then the symmetry of the process would guarantee that no work is done. We should expect therefore that the energy necessary to wind around goes smoothly to zero as the string-antistring separation increases. The energy associated with the charge left behind, must become arbitrarily small

as the distance between string and antistring increases. Since some of this energy is Coulomb energy (more precisely electric field energy), clearly the charge distribution (more precisely the electric field distribution) must become more and more spread out as the strings separate. While Gauss' law is not mocked, and the integrated surface integral of the electric field far from the string holds firm, the field at any fixed point goes to zero as the strings separate. It calls to mind the smile of the Cheshire cat.

The profound point is that Gauss' law can only be used to define the charge inside surfaces containing both the string and the antistring. (Or, in three dimensions, the entire loop.) Thus it cannot be used to identify the local charge density.

6. Since the broken $SO(3)$ symmetry supports not only Alice strings but also magnetic monopoles, one can conduct similar thought experiments with magnetic monopoles. The conclusion is, that a loop of Alice string can appear, viewed from afar, as a magnetic monopole. Perhaps surprisingly, it turns out that it is much easier to understand how the loop carries magnetic charge, than how it carries electric charge. I will merely state the result, leaving the proof as an exercise for the reader. (It has a pretty solution, requiring not calculation but rather that one visualize the correct picture.) Consider a loop of loops around the loop of string: i.e. draw a little circle linking the loop of string somewhere, and imagine pulling this circle around in a closed path, always linking the string loop, until it comes all the way back to its initial location. Mark off a particular point on the circle, and a smooth path of this point back to itself following the circle's orbit. Now for each fixed position of the circle, we may compute the parallel transport for the path from the marked point around the circle and back to itself. It gives us an element of the residual symmetry group $O(2)$. Now as we follow the circle's orbit, these elements give us a closed path within $O(2)$. Since this group itself has essentially the topology of a circle, we can define the winding number of our path in $O(2)$. The exercise is, to show that this winding number is the magnetic monopole charge of the Alice string loop.

4. Zero modes and topological superconductivity

1. The various peculiarities we found in the electrodynamics of Alice strings, are quite generic and robust for flux tubes when the residual symmetry H is non-abelian.

2. One might expect that there should be a dimensionally reduced analog of the Nambu-Goldstone mechanism associated with the reduction of symmetry by a string. That is, we have found that H is reduced to some subgroup by the presence of the strings, and one might expect there to be arbitrarily low energy modes associated with the slow modulation of the string by the H generators which are no longer symmetries. More precisely, since we are concerned with gauge theories, one might expect that there is a dimensionally reduced version of the Higgs phenomenon, with the strings acting as superconducting wires.

Something close to this is true, but there is an interesting surprise. There are indeed essentially massless modes that carry charge and current, and string loops do form superconductors of a sort. However, as our discussion of Cheshire charge brought out, there is no reason to expect that the charge and current are localized on the loop. A detailed analysis of the relevant modes shows that, in fact, they are not. We are faced with a new form of superconducting electrodynamics, for which a loop of superconducting wire is not an adequate metaphor.

3. The superconductivity associated with breaking of a symmetry by a non-abelian flux tube is very robust, since it occurs for essentially group theoretic and topological reasons. It should be taken into account, in considering the possible behavior of cosmic strings produced in the early universe.

5. References

The material discussed in this lecture is contained in the following recent papers, which contain many further references:

L. Krauss, F. Wilczek *Discrete Gauge Symmetry in Continuum Theories*, Phys. Rev. Lett. **62**, 1221 (1989).

M. Alford, J. March-Russell, F. Wilczek *Discrete Quantum Hair on Black Holes and the Non-abelian Aharonov-Bohm Effect*, Harvard preprint HUTP 89/A040.

J. Preskill, L. Krauss *Local Discrete Symmetry and Quantum Mechanical Hair* Caltech preprint CALT-68-1601.

M. Alford, J. March-Russell, F. Wilczek *Enhanced Baryon Number Violation Around Cosmic Strings*, Nucl. Phys. **B328**, 140 (1989).

M. Alford, K. Benson, S. Coleman, J. March-Russell, F. Wilczek *Interactions and Excitations of Non-abelian Vortices*, Harvard preprint HUTP 89/A051

M. Alford, K. Benson, S. Coleman, J. March-Russell, F. Wilczek Harvard preprint HUTP 89/A052

F. Wilczek, Y.-S. Wu *Space-Time Approach to Holonomy Scattering*, Institute for Advanced Study preprint IASSNS 90/02

FIGURE CAPTIONS

- 1) The modification of charge-flux composites by passage around one another. The composites are represented by a charge, located at the cross or double-cross, and a singular line across which there is non-trivial parallel transport. Note that these two structural elements must be slightly separated, to give an unambiguous specification of the quantum numbers. As one composite winds counterclockwise through another – operation α – it hits the singular line; to pull it through the singular line, we add a tube and antitube to the right and perform operation β . The resulting object on the left is not quite of the form of a charge split from a flux tube; to put it in that form we perform operation γ .
- 2) An alternative procedure to Figure 1, where the singular lines are twisted to avoid one another as one charge-flux composite winds around the other.

(α)

(β)

(γ)

FIGURE 1.

FIGURE 2.