MASTER

IMPACT OF DELAYS IN PLUTONIUM USE
ON THE STATIONARY GROWTH RATE
OF FAST BREEDER REACTOR FUEL

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ABSTRACT

The hierarchy of the four growth rate expressions originally derived from an instantaneous reuse scheme is expanded to account for finite burnup in the core and blanket, β -decay of ^{241}Pu , core and blanket loading schemes, reuse delays due to reprocessing and fabricating fuel and external fuel cycle losses.

The most general growth rate expression, obtained from the asymptotic slope of the accumulating fuel material in an expanding park of breeder reactors, is formally the same in both cases. Formulation of the growth rate based on the condensation of the detailed information of the equilibrium fuel cycle for a single reactor, is more complicated than without delays due to the composition difference between the average residing and excess discharge material. The third growth rate expression results from a slightly more complicated fuel-cycle eigenvalue problem than without delays. The last definition employs isotopic breeding worth factors obtained from the adjoint fuel cycle eigenvalue problem.

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I. Introduction

The isotopic breeding worths and stationary growth rate definition presented by Ott and Borg^{1,2} were introduced to provide a consistently mathematically derived formula for a figure of merit for fast breeder reactors. Most presently used definitions are based on intuition about the fuel cycle and its relationship to growth rate or doubling time. These intuitive formulas are for closed cycle equilibrium operation but they can yield fairly large inaccuracies in predicting the equilibrium growth rate value for plutonium compositions not too close to closed cycle equilibrium compositions. One of the basic differences between these intuitive growth rate formulas and the stationary growth rate is the use of different sets of isotopic weight factors. Two sets are usually used. The standard definition, which is based on an isotope ability to fission at thermal neutron energies, is the most commonly used set in the United States. Substitutional critical mass worths are predominately used by United Kingdom and some other European countries. They describe the reactivity worth of an isotope above that of ^{238}U . On the other hand the isotopic breeding worth factors are obtained from the adjoint fuel cycle eigenvalue problem. The breeding worth factors are analogous to the adjoint neutron flux which eliminates first order perturbations when evaluating the reactivity.

The original work presented by Ott and Borg¹ was performed to lay a solid foundation for evaluating the growth rate, as a figure of merit, which can provide a basis for a universally accepted description of the

breeding capability of fast breeder reactors. A need for such a definition was illustrated in 1974 at an international conference on advance reactor development³. Even more recently during a special session at the American Nuclear Society in New York many papers were presented with a variety of growth rate or doubling time definitions and prescriptions for evaluation⁴⁻¹⁰.

The goal of this paper is to expand the original work performed by Ott and Borg¹ by including the neglected delays in the use of excess bred material. It is practically based on the work presented in Ref. 11. The instantaneous reuse scheme presented in Ref. 1 is expanded to include finite burnup, the effects of fuel loading schemes in the core and blanket, β -decay of ²⁴¹Pu nuclei, delays due to the physical need of an external fuel cycle (out-of-pile material), and the effects of out-of-pile losses. In this paper, all of these additions are incorporated into the growth rate and doubling time expressions of Ref. 1.

A hierarchy of four logically different expressions for the fuel growth rate and thus the doubling time, which yield formally the same value, but have different ranges of application were presented in Ref. 1. Growth rate, γ , and doubling time are related as: $t_D = \frac{\ln 2}{\gamma}$. The first definition, obtained by mathematically expressing the doubling time as a measure of the asymptotic exponential growth of fuel in a system of identical breeder reactors, was the most general. The second definition was a reduction of system quantities to single reactor condensations from the detailed information of the equilibrium fuel cycle analysis. Coefficients of a "continuous integrated fuel cycle model" were obtained from the detailed fuel cycle calculations.

This led to the third expression, which was formulated as an eigenvalue problem with the growth rate as eigenvalue and the equilibrium plutonium composition as eigenvector. A set of isotopic weight factors, which were obtained as a solution of the adjoint of the fuel cycle eigenvalue problem employed in the third procedure, was the basis for the fourth growth rate expression. The resulting "isotopic breeding worth" factors were applied to the production and consumption rates of the four plutonium isotopes. This made the resulting doubling time formula stationary with respect to variations about a reference reactor and practically independent of the fuel composition. All definitions neglected delays in the use of excess bred fuel.

The presentation of the material in this paper follows closely the pattern in Ref. 1. First in Sec. II a growth rate expression is determined from the accumulating material of a growing park of fast breeder reactors. Reduction to "average" park reactor quantities is performed in a straight forward manner, the difference between the park "average" reactor and a reactor in its own closed cycle equilibrium are presented in Sec. III. In Sec. IV. an expression for the growth based on the continuous integrated fuel cycle model and formulation of an eigenvalue problem from the equilibrium solution of the continuous integrated fuel cycle equations is presented. Solution of the adjoint eigenvalue problem and finally the extended formula for the stationary growth rate are described in Sec. V. In Sections VI. typical results are presented.

II. DOUBLING TIME ASSOCIATED WITH THE ACCUMULATING FUEL

The compound doubling time and the growth rate represent measures for the exponential buildup of breeder reactor fuel. Therefore, a quantity, which describes the accumulation of all the fuel present in a growing system of reactors, appears to be the proper basis for the derivation of a general formula for the doubling time or growth rate. This quantity is called the "accumulating fuel" (Ref. 1).

From a three-dimensional burnup calculation and the description of the complete closed fuel cycle operation, including the external cycle and all delays in the fuel reuse, the exact space and time dependence of the buildup of the plutonium isotopes can be determined. For the calculation of the number of accumulating-fuel nuclei, $\hat{N}_i(t)$, these calculations are augmented to consider reuse of the plutonium in an expanding park of reactors and their associated out-of-pile inventories.

During a reloading interval the amount of accumulating fuel does not change, except for a slight loss through nuclei β -decay of ^{241}Pu (e.g., during a down time of 0.1 years, 0.5% of the ^{241}Pu decays, representing a loss of about 0.03% of the fissionable plutonium). Thus, in a plot of the $\hat{N}_i(t)$, with t being the operational time, all curves are continuous at the fuel-charge and discharge times, although the fuel loading in the reactor and its external fuel cycle is changed discontinuously; only ^{241}Pu changes by a minute amount. The exact $\hat{N}_i(t)$ curves exhibit a slight wiggly structure due to a change in reaction rates of the various isotopes during reloading of fuel. But the variations of $\hat{N}_i(t)$ do not normally exceed the standard line thickness

in a plot; one can therefore practically approximate the $\hat{N}_i(t)$ curves as continuous smooth curves. These smooth curves can be calculated directly by approximating charge and discharge of fuel as continuous operations. This yields the "integrated continuous fuel cycle model", (see Refs. 12, 13, and 14). The results presented in Fig. 1, are calculated with this model. It is assumed that the excess plutonium is used, after the reprocessing and refabrication time, to make up a fraction of a new reactor with the same breeding characteristics.

Figure 1 shows as solid lines, the basic time dependence of the accumulating number of plutonium isotopes, $\hat{N}_i(t)$, starting with a base reactor, which is initially fueled with pure ^{239}Pu . The curves with the greatest slope are calculated for core and blanket only, without reuse delays, as in Ref. 1. The other set of $\hat{N}_i(t)$ curves includes in- and out-of-pile plutonium and the reuse delays for each reactor in the park, which obviously leads to a lower growth rate than without these effects.

An exponential increase which is identical for all plutonium isotopes is asymptotically exhibited by the accumulating fuel, $\hat{N}_i(t)$, (comp. Refs. 1, 15, 16 and 17). The logarithmic slope of this asymptotic behavior, indicated in Figure 1 by the dashed lines, defines the asymptotic "growth rate" (γ^∞), and the associated doubling time.

$$\gamma^\infty = \frac{d \ln \hat{N}_i^{\text{as}}(t)}{dt} = \frac{1}{\hat{N}_i^{\text{as}}(t)} \frac{d \hat{N}_i^{\text{as}}(t)}{dt} = \frac{\ln 2}{t_D} \quad (1)$$

The asymptotic time dependence, denoted by $\hat{N}_i^{\text{as}}(t)$, being the same for all plutonium isotopes implies that any combination of isotopes gives the same growth rate. A general definition of the figure-of-merit, γ^∞ ,

is obtained by applying isotopic weight factors, w_i , (first definition¹):

$$\gamma^\infty = \frac{\ln 2}{t_D} = \frac{\sum_i w_i \frac{d\hat{N}_i^{as}(t)}{dt}}{\sum_i w_i \hat{N}_i^{as}(t)} \quad (2)$$

Equation (2) describes the asymptotic growth of critical mass material, of fissile isotopes, of plutonium or any other combination. The growth rate, Eq. (2), is the same for all w_i .

The difference between Eq. (2) and the corresponding expression in Ref. 1 is in the definition of the accumulating fuel. In- and out-of-pile delays in the use of plutonium including the corresponding inventories are explicitly considered; in the earlier development, instantaneous reuse, which automatically eliminates the external fuel cycle, was assumed. Therefore, $\hat{N}_i^{as}(t)$, in Eq. (2), includes the core plus blanket material as in Ref. 1 and in addition the external fuel cycle material.

Integrating Eq. (1), with appropriate limits, shows that all $\hat{N}_i^{as}(t)$ are proportional to the same exponential function:

$$\hat{N}_i^{as}(t) = \hat{N}_i^{as}(0) e^{\gamma^\infty t} \quad (3)$$

$\hat{N}_i^{as}(0)$ represents the amplitude of the asymptotic curve, obtained by extrapolating $\hat{N}_i(t)$, in Figure 1, back to $t = 0$. Inserting Eq. (3) into Eq. (2) illustrates that the doubling time, as defined by Eq. (2), is independent of the isotopic weight factors.

The asymptotically accumulating fuel, as described by Eq. (3), is located in more than one system. Actually, the "number" of

systems must increase exponentially to accommodate the increasing $\hat{N}_i^{as}(t)$:

$$\hat{N}_i^{as}(t) = \bar{N}_{pi}^{\infty} n(t) \quad , \quad (4)$$

with

$$n(t) = n_0 e^{\gamma^{\infty} t} \quad . \quad (5)$$

The quantities $\bar{N}_{pi}^{\infty}$ represent the average cycle and park number of plutonium nuclei in and out-of-pile for all $n(t)$ reactors of the park. Newly fueled reactors have not attained their individual equilibrium state. Therefore, the "park average" equilibrium plutonium (composition and amount per reactor) is slightly different from that of the base reactor system in its "individual equilibrium" state.

The quantity n_0 in Eq. (5) is equal to the number of "initial" base systems if operation would have started with equilibrium plutonium in- and out-of-pile. The value of n_0 cancels in the doubling time formula.

III. THE GROWTH RATE FORMULA OF A SINGLE REACTOR

III.A. GENERAL DERIVATION

The general definition of the growth rate, Eq. (2) describes the accumulation of fuel in a growing system of reactors with their associated out-of-pile inventories. This general definition can be reduced to a formula containing only quantities derived from the

base reactor, which has per definition the same breeding characteristics as the reactor, which uses the excess bred plutonium. The reduction to base system quantities has to employ the balance of the out-of-pile inventory and reaction rates in the reactor.

The out-of-pile inventory increases with an increasing number of reactors. Therefore, the growth rate of the complete system, reactor plus out-of-pile inventory, is smaller than it would be without the consideration of the out-of-pile fuel. For the purpose of a growth rate definition one includes only the minimum amount of out-of-pile fuel. Any excess is to be removed from the system. It is convenient to assume that the removal of excess fuel occurs at the time of reloading, i.e., when a batch of fuel is discharged from the reactor. Then, a breeder reactor continuously builds up excess fuel during the operational phase; the excess is removed at the time of reloading (t_c). This leads to a saw-tooth type of curve for all $N_i(t)$, if - as it is commonly done - the variable t describes the operation time.

Also the rates of buildup and burnup of actinide isotopes change discontinuously at the reloading times. This leads to a discontinuous change in the slopes of the $N_i(t)$ - curves at the t_c values (comp. e.g., Ref. 17). Of practical importance for the growth rate determination are only the cycle average reaction rates which are constant for equilibrium operation.

The reduction of the general growth rate formula, Eq. (2), to an expression for a single reactor is performed in two steps. First, Eq. (2) is reduced to a formula for a reactor, representing an equilibrium "park-average" system, a procedure which is quite straight forward. In contrast to this, the derivation of an expression for a

single reactor, being in its "individual-equilibrium" mode of operation, requires additional considerations. Although both results should not differ in any practically significant way, the distinction is made to enhance the logical clarity of the derivation.

The rate of increase of accumulating fuel, $\frac{dN_i^{as}}{dt}$, is directly related to the average surplus plutonium production, i.e., excess discharge, in all $n(t)$, reactors in the park, Eq. (6).

The spectrum of reactors in the park is asymptotically in an equilibrium; i.e., the ratio of reactors of various ages approaches constant values when $N(t)$ is in its asymptotic phase. Therefore, the average equilibrium park-reactor has a constant cycle-averaged composition, $\bar{N}_{Pi}^{\infty}$; it also has a constant cycle-averaged fuel surplus production rate given by:

$$\frac{dN_i^{as}}{dt} = (\bar{P}_{Pi}^{\infty} - \bar{L}_{Pi}^{\infty})n(t) = \begin{cases} \text{equilibrium park and cycle} \\ \text{average surplus} \\ \text{production rate} \\ \text{of fuel isotope } i. \end{cases} \quad (6)$$

$\bar{P}_{Pi}^{\infty}$ and $\bar{L}_{Pi}^{\infty}$ represent the average production rate and loss rate, respectively, for isotope i at the park equilibrium, and the bar indicates the cycle average for each reactor in the park.

If Eqs. (4) to (6) are inserted into Eq. (2) and $n(t)$ canceled, a general doubling time definition expressed in park average reaction rates is obtained:

$$\gamma^{\infty} = \frac{\ln 2}{t_D} = \frac{\sum_i w_i (\bar{P}_{Pi}^{\infty} - \bar{L}_{Pi}^{\infty})}{\sum_i w_i \bar{N}_{Pi}^{\infty}} \quad (7)$$

The growth rate of Eq. (7) is the same as Eq. (2) and therefore also independent of the weight factors w_1 .

Equation (7) defines the growth rate for an expanding reactor park, expressed in park averaged reaction rates and fuel composition. To find these quantities requires additional effort, beyond the calculation of the base system (which started the park). Although all reactors in the park are identical in their design, they have a somewhat different fuel composition. This difference is caused by a difference in the composition of the plutonium surplus and the residing composition of a given reactor. Since the surplus is used to start a new reactor, the daughter reactor will at first have a different composition than the parent (or base) reactor.

Normally, one calculates the doubling time or the growth rate for a single reactor in its closed cycle equilibrium. This growth rate cannot be quite identical to the one of Eqs. (2) or (7), since the park-average aspect does not appear in the single-reactor growth rate calculation. Furthermore, the derivation of this growth rate definition involves a different concept, the amount and composition of the excess fuel instead of the residing composition in a park as in Eq. (7).

The balance equation for the plutonium isotopes, averaged over a complete interval of the closed fuel cycle in its asymptotic phase is given by

$$\overline{\left(\frac{dn_i}{dt}\right)} = 0 = \overline{P_i^\infty} - \overline{L_i^\infty} - \overline{S_{Si}^\infty} \quad (8)$$

The derivative term on the LHS vanishes since there are no changes

in the N_i between corresponding times in the equilibrium cycle [see Ref. 1, Eqs. (24) and (34)]. The production term P_i^∞ is the same as in Ref. 1, but the loss term is generalized as compared to Ref. 1 by including β -decay of ^{241}Pu and out-of-pile losses. Furthermore, all delays in the reuse of the excess fuel are considered in the derivation of Eq. (3) (see below). As a consequence, the excess fuel, discharged from the "system" (core, blanket and out-of-pile cycle), has a different composition than the fuel residing in the system.

Let $\vec{N}_S^\infty$ denote the asymptotic (cycle averaged) residing fuel and $\vec{N}_{Sd}^\infty$ the composition vector of the discharge current, $\vec{g}_S^\infty$. The normalization of $\vec{N}_{Sd}^\infty$ is yet to be determined. The difference in the composition of $\vec{N}_S^\infty$ and $\vec{N}_{Sd}^\infty$ can be expressed in terms of a diagonal matrix H_S :

$$\vec{N}_{Sd}^\infty = H_S \vec{N}_S^\infty \quad (9)$$

In Ref. 1, H_S^∞ could be approximated by a unit matrix, since reuse delays were neglected. Here, the H-matrix needs to be considered explicitly.

The operation of a breeder reactor or the computational solution of the closed fuel cycle problem yields the discharge current $\vec{g}_S^\infty$. A growth rate is conceptually defined by factoring the discharge current into a scalar factor and a vector having the discharge fuel composition:

$$\vec{g}_S^\infty = \gamma \vec{N}_{Sd}^\infty \quad (10)$$

The disposition over the magnitude of $\vec{N}_{Sd}^\infty$ determines the meaning of γ . As discussed above, the logarithmic rate of change of the $\hat{W}_i^{as}(t)$

represents the growth of complete systems, core plus blanket plus out-of-pile fuel. Therefore, the amount of fuel, $\vec{N}_{Sd}^{\infty}$, in the factorization defining $\vec{\gamma}^{\infty}$ must be sufficient to make up a complete system:

First, discharged fuel is placed in the core to build a critical reactor. The density-constrained reactivity worth (comp. Refs. 1, 18 and 19) is the proper* isotopic weight for the determination of the needed amount of discharged plutonium:

$$\alpha_C \sum_i w_i^{CM} N_{Sdi} = \sum_i w_i^{CM} N_{Ci} \quad , \quad (11a)$$

with α_C giving the fraction of $\vec{N}_{Sd}$ needed for this purpose and $\vec{N}_C$ being the equilibrium amount of plutonium in the core.

A second and third portion of fuel discharged from the parent reactor is needed to fuel the daughter reactor while it is building up its blanket plutonium and its out-of-pile inventory; α_B , α_X and $\vec{N}_B$, $\vec{N}_X$ denote the respective fractions and numbers:

$$\alpha_B \sum_i w_i^{CM} N_{Sdi} = \sum_i w_i^{CM} N_{Bi} \quad , \quad (11b)$$

and

$$\alpha_X \sum_i w_i^{CM} N_{Sdi} = \sum_i w_i^{CM} N_{Xi} \quad . \quad (11c)$$

Combining the three contributions of Eqs. (11) gives:

$$\sum_i w_i^{CM} N_{Sdi} = \sum_i w_i^{CM} N_{Si} \quad , \quad (12)$$

*The reactivity effect of the blanket plutonium requires special considerations which are not subject of this paper.

with

$$\alpha_C + \alpha_B + \alpha_X = 1 \quad , \quad (13a)$$

and

$$\vec{N}_C + \vec{N}_B + \vec{N}_X = \vec{N}_S^\infty \quad . \quad (13b)$$

Multiplying Eq. (10) scalarly with $\vec{w}^{CM}$ and inserting Eqs. (8) and (12) yields for the growth rate:

$$\gamma^\infty = \frac{\sum_i w_i^{CM} g_{Si}}{\sum_i w_i^{CM} N_{Sdi}} \quad , \quad (14)$$

and

$$\gamma^\infty = \frac{\sum_i w_i^{CM} \{P_i^\infty - L_i^\infty\}}{\sum_i w_i^{CM} N_{Si}^\infty} \quad . \quad (15)$$

The same notation, γ^∞ , is used for the growth rates defined by Eqs. (7) and (15), since the numerical difference is expected to be negligibly small. The growth rate for an individual reactor, Eq. (15), is not completely independent of weight factors as is γ^∞ from Eq. (7) and the formula derived in Ref. 1 for an individual reactor. The reason is that numerator and demoninator in Eq. (15) involve different fuel compositions, as indicated by Eq. (9). Splitting $\underline{H}_S$ in Eq. (9) in to a unit matrix and a difference $\delta \underline{H}$ and inserting it in Eq. (15) can

be used to explore the weight factor dependence of Eq. (15):

$$\underline{H}_S = \underline{1} + \underline{\delta H} \quad , \quad (16)$$

which gives for N_S^∞

$$\underline{N}_S^\infty = \underline{N}_{Sd}^\infty - \underline{\delta H} \underline{N}_S^\infty \quad , \quad (17)$$

and for γ^∞

$$\gamma^\infty = \frac{\sum_i^{CM} w_i [P_i^\infty - L_i^\infty]}{\sum_i^{CM} w_i N_{Sdi}^\infty - \sum_i^{CM} w_i \delta H_i N_{Si}^\infty} \quad . \quad (18)$$

The second term in the denominator is small, since the δH_i are small and have both signs. Thus, a two term Taylor expansion provides sufficient accuracy:

$$\gamma^\infty \approx \frac{\sum_i w_i [P_i^\infty - L_i^\infty]}{\sum_i w_i N_{Sdi}^\infty} \left\{ 1 + \frac{\sum_i^{CM} w_i \delta H_i N_{Si}^\infty}{\sum_i^{CM} w_i N_{Sdi}^\infty} \right\} \quad . \quad (19)$$

The factor in front of the braces is independent of the weight factors as before, since the composition of the numerator and denominator are the same, only the small correction term in the braces needs to be evaluated with the substitutional critical mass worths.

III. B. DERIVATION FROM THE SPACE AND TIME DEPENDENT ACTINIDE BALANCE EQUATIONS

The growth rate is formulated as a condensed quantity of an integrated fuel cycle model derived from the space and time dependent actinide balance equations. The schematic structure of the model consists of a core, blanket and external fuel cycle which, as single units, communicate with each other and the outside by charge and discharge currents, such as a ^{238}U feed current and an excess bred plutonium discharge current. Fission products and ^{243}Am , which are produced at the end of the plutonium chain, are also discharged. There is, however, no need to explicitly treat these latter two currents.

The space and time dependent balance equations for ^{238}U and the plutonium isotopes are given by:

$$\frac{\partial N_8(\vec{r}, t)}{\partial t} = -\lambda_8^8(\vec{r}, t)N_8(\vec{r}, t) + \xi_8(\vec{r}, t) - \kappa_8(\vec{r}, t) \quad (20a)$$

and

$$\begin{aligned} \frac{\partial N_i(\vec{r}, t)}{\partial t} = & -\lambda_a^i(\vec{r}, t)N_i(\vec{r}, t) + \lambda_c^{i-1}(\vec{r}, t)N_{i-1}(\vec{r}, t) \\ & + \xi_i(\vec{r}, t) - \kappa_i(\vec{r}, t) \end{aligned} \quad (20b)$$

The index i denotes the actinide isotopes ($i = 9, 0, 1$ and 2 for ^{239}Pu , ^{240}Pu , ^{241}Pu and ^{242}Pu). The space and time dependent microscopic reaction rates are given by:

$$\lambda_a^i(\vec{r}, t) = \int_0^\infty \sigma_a^i(E)\phi(\vec{r}, E, t)dE + \lambda_{ra}^i \quad (21a)$$

and

$$\lambda_c^i(\vec{r}, t) = \int_0^\infty \sigma_1^c(E) \phi(\vec{r}, E, t) dE \quad (21b)$$

$\phi(\vec{r}, E, t)$ is the space, energy and time dependent neutron flux; $\sigma_1^c(E)$ is the energy dependent microscopic cross section for absorption or capture; λ_{ra}^i represents the radioactive decay constant for isotope i . The discontinuous charge and discharge operations of ^{238}U and the Pu isotopes are described by $\xi_i(\vec{r}, t)$ and $\kappa_i(\vec{r}, t)$.

The space, energy and time dependent neutron flux is evaluated from the neutronics of a critical reactor i.e.,

$$[\underline{F} - \underline{M}] \phi = 0 \quad (22)$$

$\underline{F}$ is the source operator and $\underline{M}$ is the migration and loss operator.

Equation (22) describes the difference between neutron production and losses. For a critical reactor this difference is identical to zero and also specifies that the reactivity is zero. When a reactor is refueled, spent fuel material is discharged and replaced by fresh fuel. The make-up density of the charged material is maintained constant, which specifies that

$$\sum_{i=8}^2 \zeta_i(\vec{r}, t) = \text{Constant} \quad (23)$$

Equations (22) and (23) are necessary to determine the solution of Eqs. (20) and therefore must be considered as an integral part of the actinide balance equations.

Equations (20) are applicable for all fuel residing locations and are integrated with respect to space over each region i.e., core, axial and radial blankets and the external fuel cycle. The resulting balance equations for the regions, core and blanket, are given by:

$$\frac{dN_{R8}(t)}{dt} = -\lambda_{Ra}^i(t)N_{R8}(t) + \xi_{R8}^d(t) - \kappa_{R8}^d(t) \quad (24a)$$

and

$$\frac{dN_{Ri}(t)}{dt} = -\lambda_{Ra}^i(t)N_{Ri}(t) + \lambda_{Rc}^{i-1}(t)N_{Ri-1}(t) + \xi_{Ri}^d(t) - \kappa_{Ri}^d(t) \quad (24b)$$

The region of integration is denoted by R, (R = C, B and X for core, blanket and external fuel cycle), and discontinuous values are indicated by the superscript d. The integration with respect to space, of Eq. (22) and (23), yields the following integrated number densities, microscopic reaction rates, and charge and discharge rates:

$$N_{Ri}(t) = \int_R N_i(\vec{r}, t) d\vec{r} \quad (25)$$

$$\lambda_{Ra,c}^i(t) = \frac{1}{N_{Ri}(t)} \int_R \lambda_{a,c}^i(\vec{r}, t) N_i(\vec{r}, t) d\vec{r} \quad (26)$$

$$\xi_{Ri}^d(t) = \int_R \xi_i^d(\vec{r}, t) d\vec{r} \quad (27)$$

and

$$\kappa_{Ri}^d(t) = \int_R \kappa_i^d(\vec{r}, t) d\vec{r} \quad (28)$$

Since there is only β -decay of ^{241}Pu and no burnup or buildup of actinides in the external fuel cycle, the integrated balance equations for this region are obtained basically from material flow considerations. The integrated external fuel cycle balance equations are given by: =.

$$\frac{dN_{X8}(t)}{dt} = \xi_{X8}^d(t) - \kappa_{X8}^d(t) \quad (29a)$$

and

$$\frac{dN_{Xi}(t)}{dt} = -\lambda_{rs}^i N_{Xi}(t) + \xi_{Xi}^d(t) - \kappa_{Xi}^d(t) \quad (29b)$$

Equations (24) and Eqs. (29) are still exact if the solution of Eqs. (20), (22) and (23) are used when calculating the integral quantities. The N_{Ri} variation described by Eqs. (24) and (29) accounts for changes that occur in the core, blanket and external fuel cycle over a loading interval.

Cycle averaged quantities for the core, blanket and external fuel cycle are obtained by integrating Eqs. (24) and Eqs. (29) with respect to time over a "closed" equilibrium loading interval, Δt , and dividing by Δt . Then the derivative terms disappear, since there is no net charge across equilibrium cycle intervals

$$0 = -\overline{L}_{R8}^{\infty} + \overline{\xi}_{R8}^{\infty} - \overline{\kappa}_{R8}^{\infty} \quad (30a)$$

and

$$0 = -\overline{L}_{Ri}^{\infty} + \overline{P}_{Ri}^{\infty} + \overline{\xi}_{Ri}^{\infty} - \overline{\kappa}_{Ri}^{\infty} \quad (30b)$$

for the core and blanket. Similarly, the cycle averaged external fuel cycle balance is described by:

$$0 = -\lambda_{ra}^i \bar{N}_{Xi}^{\infty} + 0 + \bar{\xi}_{Xi}^{\infty} - \bar{\kappa}_{Ri}^{\infty} \quad (30c)$$

Cycle averaged loss rates are defined by:

$$\bar{L}_{Ri}^{\infty} = \frac{1}{\Delta t} \left[\int_{\Delta t}^{\Delta t + \Delta t} \lambda_{Ra}^i(t) N_{Ri}(t) dt \right]^{eq} \quad (31)$$

$\bar{P}_{Ri}^{\infty}$, $\bar{\xi}_{Ri}^{\infty}$, $\bar{\kappa}_{Ri}^{\infty}$ and $\bar{N}_{Ri}^{\infty}$ denote the cycle averaged production, charge, discharge, and residing actinides and are evaluated in the same way as the loss, [comp. Eq. (31)].

The discontinuous charge to the external fuel cycle is identical to the sum of the core and blanket discharge. Similarly, the discharge from the external fuel cycle is equal to the core charge plus the discharge of excess plutonium from the system. Since the reactor is a breeder, only ^{238}U , is charged to the system. There are also material losses during reprocessing and fabrication which must be considered. Usually these losses are accounted for by reducing the core and blanket charge to the external fuel cycle. The cycle averaged external fuel cycle balance given by Eq. (30c) is modified to account for the relationships discussed above:

$$0 = (1 - \epsilon_{X8}) (\bar{\kappa}_{C8}^{\infty} + \bar{\kappa}_{B8}^{\infty}) + \bar{U}_8^{\infty} \quad (32a)$$

and

$$0 = (1 - \epsilon_{Xi}) (\bar{\kappa}_{Ci}^{\infty} + \bar{\kappa}_{Bi}^{\infty}) - \lambda_{ra}^i \bar{N}_{Xi}^{\infty} - \bar{\xi}_{Ci}^{\infty} - \bar{g}_{Si}^{\infty} \quad (32b)$$

The terms $\bar{U}_8^{\infty}$ and $\bar{g}_{Si}^{\infty}$ describe the needed uranium make up charge, and the system excess plutonium discharge. The factor $(1 - \epsilon_{Xi})$ represents the reduction of the external fuel cycle charge due to reprocessing and fabrication losses.

An expression for the system cycle averaged plutonium balance is obtained by summing the contributions from each of the material residing locations, Eqs. (30b) and (32b):

$$0 = \bar{P}_{Ci}^{\infty} - \bar{L}_{Ci}^{\infty} - \epsilon_{Xi} \bar{\kappa}_{Ci}^{\infty} + \bar{P}_{Bi}^{\infty} - \bar{L}_{Bi}^{\infty} - \epsilon_{Xi} \bar{\kappa}_{Bi}^{\infty} - \lambda_{ra}^i \bar{N}_{Xi}^{\infty} - \bar{S}_{Si}^{\infty} \quad (33)$$

Equation (33) is the detailed version of Eq. (8), with all terms derived from the space and time dependent actinide balance equations:

$$\bar{P}_i^{\infty} = \bar{P}_{Ci}^{\infty} + \bar{P}_{Bi}^{\infty} \quad (34a)$$

and

$$\bar{L}_i^{\infty} = \bar{L}_{Ci}^{\infty} + \bar{L}_{Bi}^{\infty} + \epsilon_{Xi} (\bar{\kappa}_{Ci}^{\infty} + \bar{\kappa}_{Bi}^{\infty}) + \lambda_{ra}^i \bar{N}_{Xi}^{\infty} \quad (34b)$$

Inserting Eqs. (34) in Eq. (15) gives the fuel growth rate derived for an individual reactor.

IV. THE GROWTH RATE FORMULA BASED ON THE INTEGRATED CONTINUOUS FUEL CYCLE MODEL

IV.A. GROWTH RATE CALCULATED FROM THE SOLUTION OF THE EQUILIBRIUM CONTINUOUS INTEGRATED FUEL CYCLE EQUATIONS

Derivation of the growth rate formula, based on an instantaneous reuse model presented in Ref. 1, is expanded to include both in- and out-of-pile delays. Basis for the extended approach is a continuous integrated fuel cycle model^{11,12,13,14}.

The expanded continuous integrated model gives a realistic treatment of the fuel cycle by including all delays and losses. Since the model is based on a two- or-three-dimensional static equilibrium space and time dependent calculation, the theoretical accuracy of the determined γ^{∞} and t_D can be of an accuracy comparable¹⁷ to that of the two- or- three-dimensional burnup model.

Schematic structure of the integrated continuous fuel cycle model consists of a core, blanket and external fuel cycle, which act as single units. The individual residing locations communicate with each other and the outside through continuous charge and discharge currents. Derivation of the continuous integrated fuel cycle equations begins with the space and time dependent actinide balance equations [comp. Sec. III.B, Eqs. (20), (22) and (23)]. The derivation follows that outlined for the discontinuous fuel cycle description (comp. Sec. III), and slightly diverges after Eq. (29). Only the balance equations for equilibrium operation are necessary for a growth rate expression; therefore, only these terms are presented, (see also Ref. 1):

Time dependent microscopic reaction rates, $\lambda_{Ra,c}^i(t)$, [comp. Eq. (26)] are replaced by constant average values and continuous currents replace the discontinuous charge and discharge operations, [comp. Eq. (27) and Eq. (28)]. These approximations for the $\lambda_{Ra,c}^i(t)$, $\xi_{Ri}^d(t)$ and $\kappa_{Ri}^d(t)$, are formally based on the equilibrium solution Eqs. (20), (22) and (23) and therefore, validity of the equilibrium solution is retained.

Continuous currents, ξ_{Ri}^∞ and κ_{Ri}^∞ , which correctly describe the integral charge and discharge during the equilibrium cycle, are introduced through the following equation:

$$\xi_{Ri}^\infty = \frac{\xi_{Ri}^d}{\Delta t}, \quad (35a)$$

and

$$\kappa_{Ri}^\infty = \frac{\kappa_{Ri}^d}{\Delta t}. \quad (35b)$$

where Δt represents the loading interval. The external fuel cycle equations are automatically continuized along with the core and blanket.

The core, blanket and external fuel cycle continuous integrated actinide equilibrium balance equations are in analogy to Eq. (30), given by:

$$0 = -\lambda_{Ra}^8 N_{R8}^\infty + \xi_{R8}^\infty - \kappa_{R8}^\infty, \quad (36a)$$

$$0 = -\lambda_{Ra}^1 N_{R1}^\infty + \lambda_{Rc}^{i-1} N_{Ri-1}^\infty + \xi_{Ri}^\infty - \kappa_{Ri}^\infty, \quad (R = C \text{ and } B) \quad (36b)$$

$$0 = (1 + \epsilon_{X8})(\kappa_{C8}^{\infty} + \kappa_{B8}^{\infty}) - \xi_{C8}^{\infty} - \xi_{B8}^{\infty} + U_8^{\infty} \quad (37a)$$

and

$$0 = -\lambda_{ra}^i N_{X1}^{\infty} + (1 - \epsilon_{X1})(\kappa_{C1}^{\infty} + \kappa_{B1}^{\infty}) - \xi_{C1}^{\infty} - \xi_{S1}^{\infty} \quad (37b)$$

The equilibrium discharge currents from the core are approximated by the following:

$$\kappa_{C1}^{\infty} = \left(\frac{1}{n\Delta t} - \frac{1}{2} \lambda_{Ca}^i \right) N_{C1}^{\infty} + \frac{1}{2} \lambda_{Cc}^{i-1} N_{Ci-1}^{\infty} \quad (38)$$

with n denoting the number of batches.

The equilibrium discharge from the blanket is calculated along with the core and blanket microscopic reaction rates. In order to calculate the time independent λ 's a static depletion analysis at least to loading equilibrium, is necessary. Since the blanket is a once thru process, i.e., only ^{238}U is used as feed, the equilibrium plutonium discharge currents and ^{238}U charge for the axial and radial blankets will not be too different from the loading equilibrium values. Therefore the κ_{AB1}^{∞} and κ_{RB1}^{∞} are time independent quantities available from the same calculation which determines the λ 's.

The excess system discharge g_{S1} may be described by the discharge coming from the blankets and core. A portion of this core and blanket discharge must be refueled to the core thus the excess is given by:

$$g_{S1}^{\infty} = \alpha^{\infty} (1 - \epsilon_{X1}^{\infty}) \left(\Delta t (\kappa_{C1}^{\infty} + \kappa_{AB1}^{\infty}) + m \Delta t \kappa_{RB1}^{\infty} \right) \quad (39)$$

with α^∞ describing that portion of the core and blanket discharge which is discharged from the system and the subscripts AB and RB represent axial and radial blanket quantities.

The reloading interval for the radial blanket may be different from the core and axial blanket, this difference is denoted by m , which is the ratio of residing time in the radial blanket compared to the core. The ϵ'_{Xi} accounts for both radioactive decay losses and reprocessing losses which occur during residence in the external fuel cycle.

Since the core feed and excess discharge are identical in composition, ξ_{Ci}^∞ is given by:

$$\xi_{Ci}^\infty = (1-\alpha^\infty)(1-\epsilon'_{Xi})(\Delta t(\kappa_{Ci}^\infty + \kappa_{ABi}^\infty) + m\Delta t\kappa_{RBi}^\infty) \quad , \quad (40)$$

Two constraint conditions for the core are applied based on Eq. (22) and (23). They are the criticality constraint:

$$\sum_i w_{Ci}^0 N_{Ci}^\infty + \delta\rho_p = 0 \quad , \quad (41)$$

and the fuel density constraint condition for core material:

$$\sum_i N_{Ci}^\infty = K_N \quad . \quad (42)$$

The adjoint weighted reactivity worth of actinide isotope i , ($i=8,9,0,1$ and 2) is denoted by w_{Ci}^0 and $\delta\rho_p$ defines the contributions from all other components of the reactivity. These two constraints were originally introduced in Refs. 12 and 18.

Use of the constraint conditions and Eqs. (36) to (40) yields an expression for α^∞ in terms of the N_{Ci}^∞ , (comp. Ref. 12, 13, 14 and the Appendix of Ref. 11). Similarly an expression for the needed makeup of ^{238}U from the outside maybe expressed in terms of the N_{Ci}^∞ and α^∞ . See Refs. 10, 11, 12, 13, 14 and 17 for more specific derivation of each of the terms presented in Eq. (36) to Eq. (42).

Equations (36) to (42) are thirty equations for thirty unknowns five N_{Ri}^∞ 's for each region, five κ_{Ci}^∞ , four plutonium ξ_{Ci}^∞ 's, four g_{Si}^∞ , α^∞ and U_8^∞ . The solution can be readily obtained.

The quantities described above can be interpreted as condensations of a detailed space and time dependent depletion analysis. These types of calculations are regularly preformed with computer codes such as 2DB²¹ and REBUS²².

Summing Eqs. (36b) and (37b) gives an expression for the system balance equations in a form identical to Eqs. (34), which were based on calculations of the type preformed by 2DB and REBUS:

$$\bar{P}_i^\infty = \lambda_{Cc}^{i-1} N_{Ci-1}^\infty + \lambda_{Bc}^{i-1} N_{Bi-1}^\infty \quad (43a)$$

and

$$\bar{L}_i^\infty = -\lambda_{Ca}^i N_{Ci}^\infty - \lambda_{Ba}^i N_{Bi}^\infty - \epsilon_{Xi} (\kappa_{Ci}^\infty - \kappa_{Bi}^\infty) - \lambda_{ra}^i N_{Xi}^\infty \quad (43b)$$

where the coefficients N_{Ci}^∞ , N_{Bi}^∞ , N_{Xi}^∞ and κ_{Ci}^∞ are evaluated from the solution of Eqs. (36) thru Eqs. (42). These expressions describe the production and loss terms in Eq. (8) based on the equilibrium solution of the continuous integrated fuel cycle balance equations.

IV.B. GROWTH RATE AS EIGENVALUE OF THE CONTINUOUS INTEGRATED FUEL CYCLE PROBLEM

In Sec. IV.B. the growth rate is evaluated after the solution of the equilibrium continuous integrated fuel cycle equations. Previously, in the instantaneous reuse scheme the growth rate appeared directly in the model because the average residing core composition and the excess discharge were assumed to be the same. This made it possible to transform the integrated model directly into an eigenvalue problem with the core growth rate as eigenvalue. When the entire system (core, blanket and external cycle) is considered, an eigenvalue problem can also be formulated for the entire system; but the parameters appear to depend sensitively on the distribution of fuel within the system. This sensitivity is drastically reduced, when the eigenvalue problem is formulated for the core plutonium only. Therefore this approach is preferred here. If the delays in the use of excess bred material are eliminated, the formulation, presented here, becomes identical to that of Ref. 1.

The system equilibrium excess discharge is expressed as a product of the core growth rate and discharge vector $\vec{N}_{Cd}^{\infty}$, in analogy to Eq. (9) and (10). Then, $\vec{N}_{Cd}^{\infty}$ is expressed in terms of the core average residing material:

$$\vec{g}_S^{\infty} = \gamma_C^{\infty} \vec{N}_{Cd}^{\infty} \quad , \quad (44)$$

and

$$\vec{N}_{Cd}^{\infty} = \frac{\vec{H}}{-\vec{C}} \vec{N}_C^{\infty} \quad , \quad (45)$$

i.e.,

$$g_{Si}^{\infty} = \gamma_C^{\infty} N_{Cdi}^{\infty} = \gamma_C^{\infty} H_{Ci}^{\infty} N_{Ci}^{\infty} \quad (46a)$$

The precise split between γ_C^{∞} and N_{Cdi}^{∞} is defined such that:

$$\sum_{i=9}^2 w_i^{CM} N_{Cdi}^{\infty} = \sum_{i=9}^2 w_i^{CM} N_{Ci}^{\infty} \quad (46b)$$

Substitutional critical mass weight factors must be used for the normalization given in Eq. (46b), (comp. Sec. II).

The transformation coefficients, H_{Ci}^{∞} , are determined from Eq. (46a), with a normalization defined by Eq. (46b). Rearranging Eq. (46a) gives:

$$H_{Ci}^{\infty} = g_{Si}^{\infty} / (\gamma_C^{\infty} N_{Ci}^{\infty}) \quad (47a)$$

Substituting the core growth rate expression analogous to Eq. (14) gives:

$$H_{Ci}^{\infty} = \frac{g_{Si}^{\infty}}{\sum_{i=9}^2 w_i^{CM} g_{Si}^{\infty}} \cdot \frac{\sum_{i=9}^2 w_i^{CM} N_{Ci}^{\infty}}{N_{Ci}^{\infty}} \quad (47b)$$

Since the H_{Ci}^{∞} are close to 1, approximate values can readily be found, and suffice in most cases. But formally, the H_C -matrix is defined on the basis of the complete solution, in analogy to group constants e.g., for the thermal group.

Formulation of an eigenvalue problem is identical to the presentation given in Ref. 1, (comp. Sec. IV.F. of Ref. 1). The primary difference is the explicit use of the H-matrix. The equation analogous to Eq. (50) in Ref. 1 is given by

$$\underline{C}^{\infty} \underline{N}_C^{\infty} = \gamma \underline{C}_C^{\infty} \underline{H}_C^{\infty} \underline{N}_C^{\infty} \quad (48)$$

with

$$\underline{C}^{\infty} = (C_{ij}^{\infty}) \quad , \quad (49)$$

$$\underline{N}_C^{\infty} = \begin{pmatrix} N_{C9}^{\infty} \\ N_{C0}^{\infty} \\ N_{C1}^{\infty} \\ N_{C2}^{\infty} \end{pmatrix} \quad , \quad (50)$$

and

$$\underline{H}_C^{\infty} = \begin{pmatrix} H_{99}^{\infty} & 0 & 0 & 0 \\ 0 & H_{00}^{\infty} & 0 & 0 \\ 0 & 0 & H_{11}^{\infty} & 0 \\ 0 & 0 & 0 & H_{22}^{\infty} \end{pmatrix} \quad , \quad (51)$$

The elements of $\underline{C}^{\infty}$ are:

$$\begin{aligned} C_{99}^{\infty} &= -\lambda_{Ca}^9 - \lambda_{Cc}^8 \frac{b_9}{b_8} + I_9 d_9 \quad , \\ C_{91}^{\infty} &= -\lambda_{Cc}^8 \frac{b_i}{b_8} + I_9 d_i \quad , \end{aligned} \quad (52)$$

$$C_{ii-1}^{\infty} = \lambda_{Cc}^{i-1} + I_i^{\infty} d_{i-1}$$

$$C_{ii} = -\lambda_{Ca}^i + I_i^{\infty} d_i \quad i = 0, 1, 2,$$

and

$$C_{ij}^{\infty} = I_i^{\infty} d_j \quad (i, j) = (0, 1), (0, 2), (1, 9), (1, 2), (2, 9), (2, 0),$$

with

$$d_i = \frac{(b_8 - b_i)}{\kappa_N b_8} \quad (53a)$$

$$b_i = w_{Ci}^{\rho} + \frac{\delta_{\rho} p}{\kappa_N} \quad (53b)$$

and

$$I_i^{\infty} = -\lambda_{Ba}^i N_{Bi}^{\infty} + \lambda_{Bc}^{i-1} N_{Bi-1}^{\infty}$$

$$- \epsilon_{Xi} (\kappa_{Ci}^{\infty} + \kappa_{Bi}^{\infty}) - \lambda_{ra}^i N_{Xi} \quad (53c)$$

The I_i^{∞} replaces g_{Bi}^{∞} in Ref. 1; otherwise the formulations are essentially identical. If all delays are neglected Eq. (48) to Eq. (53) are the same as Eqs. (49) to (52) in Ref. 1.

Equation (48) is an eigenvalue problem with the core equilibrium composition as the corresponding eigenvector. It is generalized compared to the eigenvalue problem described in Ref. 1 since the difference between the discharge and average residing material is explicitly introduced through the H_C^{∞} matrix. Equation (48) yields

four quantities since the normalization of $\vec{N}_C^\infty$ is arbitrary.

It is convenient to normalize the plutonium vector to 1:

$$\alpha_N \vec{N}_C^\infty = \vec{n}_C^\infty \quad (54)$$

$$\sum_9^2 \vec{n}_{Ci}^\infty = 1 \quad (55)$$

The generalized eigenvalue problem:

$$\underline{C} \vec{n}_C^\infty = \gamma_C^\infty \underline{H} \vec{n}_C^\infty \quad (56)$$

and Eq. (55) yield then the core composition of Pu^∞ and γ_C^∞ for the normalization described in $\underline{H}_C^\infty$.

From the derivation of Eq. (56) it is apparent that the application of the matrix $\underline{C}^\infty$ to the vector $\vec{n}_C^\infty$ gives the difference of production and absorption rates for each isotope; e.g., for the equilibrium composition one obtains:

$$\underline{C}^\infty \vec{n}_C^\infty = \frac{1}{\alpha_N} \begin{pmatrix} P_9^\infty - L_9^\infty \\ P_0^\infty - L_0^\infty \\ P_1^\infty - L_1^\infty \\ P_2^\infty - L_2^\infty \end{pmatrix} \quad (57)$$

P_i^∞ and L_i^∞ contain equilibrium reaction rate information from the core and all other fuel residing locations.

The $\vec{N}_{Ci}^\infty$ which are obtained simultaneously with γ_C^∞ can be used to find $\vec{N}_{C8}^\infty$, P_1^∞ , A_1^∞ and thus the core growth rate:

$$\gamma_C^\infty = \frac{\sum_{i=1}^2 w_i (P_i^\infty - L_i^\infty)}{\sum_{i=9}^9 w_i H_{Ci}^\infty N_{Ci}^\infty} = \frac{\ln 2}{t_{CD}} \quad (58)$$

The weight factors in Eqs. (58) are arbitrary.

IV.C. RELATIONSHIP BETWEEN CORE AND SYSTEM GROWTH RATE

To determine the system or core growth rate, the excess system discharge is split into two quantities, a magnitude and a vector, [Comp. Eq. (10) and Eq. (44)]. The precise split between the magnitude and the vector are defined by Eq. (12) and (46b). A relationship between the core and system growth rates is established by assuming that the excess plutonium, $\vec{g}_S$, is used to make up only cores:

$$\vec{g}_S^\infty = \gamma^\infty \vec{N}_{Sd} = \gamma_C^\infty \vec{N}_{Cd} \quad (59)$$

Multiplying Eq. (59) scalarly with the substitutional critical mass weights gives:

$$\sum_{i=9}^2 w_i^{CM} g_{Si}^\infty = \gamma^\infty \sum_{i=9}^2 w_i^{CM} N_{Sdi}^\infty = \gamma_C^\infty \sum_{i=9}^2 w_i^{CM} N_{Cdi}^\infty \quad (60a)$$

Substitution of the normalization conditions Eqs. (12) and (46b) yields instead of Eq. (60a)

$$\sum_i w_i^{CM} B_{Si}^{\infty} = \gamma^{\infty} \sum_i w_i^{CM} N_{Si} = \gamma_C \sum_i w_i^{CM} N_{Ci} \quad (60b)$$

Rearranging of Eq. (60b) gives the following relationship between the system and core growth rate and doubling times:

$$\gamma^{\infty} = \frac{\ln 2}{t_D} = \gamma_C \cdot R_{CS} = \frac{\ln 2}{t_{CD}} \cdot R_{CS} \quad , \quad (61)$$

where

$$R_{CS}^{\infty} = \frac{\sum_{i=9}^2 w_i^{CM} N_{Ci}^{\infty}}{\sum_{i=9}^2 w_i^{CM} N_{Si}^{\infty}} \quad (62)$$

The value for R_{CS}^{∞} is composition independent for the specified split of critical mass material defined by Eqs. (13).

V. ISOTOPIC BREEDING WORTHS AND THE GROWTH RATE

Values for the isotopic breeding worth weight factors can be obtained from the adjoint fuel cycle problem¹, given by:

$$\underline{C}^{\infty} \vec{n}_C^* = \gamma_C^{\infty} \underline{H}_C^* \vec{n}_C^* \quad , \quad (63)$$

with the normalization $n_9^* = 1$. The eigenvector of the adjoint problem defines the isotopic breeding worths (relative to ^{239}Pu):

$$n_i^* = w_i^* \quad . \quad (64)$$

The extension of the fourth growth rate formula of Ref. 1 which employs the isotopic breeding worth values together with the contributions from each material residing location of the system and includes losses and delays is given by:

$$\gamma_C^{\infty} = \frac{\sum_9^2 w_i^* \left[\lambda_{Cc}^{i-1} N_{Ci-1}(t) - \lambda_{Ca}^{i-1} N_{Ci}(t) + P_{Bi}^{\infty} - L_{Bi}^{\infty} - \epsilon_{Xi} (\kappa_{Ci}^{\infty} + \kappa_{Bi}^{\infty}) - \lambda_{ra}^i N_{Xi}^{\infty} \right]}{\sum_9^2 w_i^* H_{Ci}^{\infty} N_{Ci}^{\infty}} \quad (65)$$

Recalling Eqs. (61) and (62) yields the following expression for the system growth rate:

$$\gamma^{\infty} = \frac{\ln 2}{t_D} = \gamma_C^{\infty} \cdot \frac{\sum_9^2 w_i^{CM} N_{Ci}}{\sum_9^2 w_i^{CM} N_{Si}} \quad , \quad (66)$$

where λ_{Cc}^i and λ_{Ca}^i represent the associated microscopic reaction rates of isotope i for the core. P_{Bi}^∞ and L_{Bi}^∞ are the equilibrium production and loss rate of isotope i for the blanket; κ_i^∞ is the blanket or core discharge rate; N_i is the average number of type i atoms. The w_i^* are the "isotopic breeding worths" defined in the previous section.

Equation (65) and (66) are independent of the core composition for the specifically defined system.

For practical application, Eq. (65) is slightly modified to eliminate the calculation of the equilibrium non-core quantities and the H -matrix. This modified expression for the core growth rate is given by:

$$\gamma_C^\infty = \frac{\sum_9^2 w_i^* \left[(P_{Ci} - L_{Ci} + P_{Bi} - L_{Bi} - \epsilon_{Xi} (\kappa_{Ci} + \kappa_{Bi}) - \lambda_{ra}^i N_{Xi}) \right]}{\sum_9^2 w_i^* H_{Ci} N_{Ci}} \quad (67)$$

The core growth rate given by Eq. (67) is slightly dependent upon the core composition due to a deviation of the neutron spectrum and H -matrix from corresponding equilibrium quantities used in Eq. (65). Since all parameters are to be calculated from loading equilibrium quantities, the prediction of γ_C^∞ is still very good. The term in the numerator which accounts for losses in the external fuel cycle is very small in magnitude compared to the core and blanket contribution. Since the values for P_{Bi} and L_{Bi} at loading equilibrium are not too different from the corresponding equilibrium value, there is a minimal impact on the numerator. Similarly, for loading equilibrium compositions, the

diagonals of the $\underline{H}^{\infty}$ and $\underline{H}$ are also not too different in value. The system growth rate, R_{CS} times Eq. (67), is also slightly composition dependent but only through the dependence of the γ_C^{∞} . The ratios of the critical masses in system and core described by the R_{CS} are very insensitive to compositions, thus this term is essentially the same for loading equilibrium and closed cycle equilibrium.

VI. INVESTIGATIONS

VI.A. DELAYS AND THE ACCUMULATING FUEL

The effect of delays on the growth of the accumulating fuel may be demonstrated with the aid of Figs. 2 and 3, which illustrate the time dependence of the accumulating number of ^{240}Pu and ^{241}Pu atoms. Any of the plutonium isotopes could have been chosen since all have identical asymptotic exponential growth rates (comp. Sec. II).

The first case assumes instantaneous reuse of the excess bred plutonium. Case two considers only finite burnup in-pile and does not include the out-of-pile delays. The growth rate and doubling time for these first two cases are essentially undistinguishable but the value for the growth rate in the second case is slightly smaller which reflects a larger doubling time. Values for the growth rate and doubling time are about 0.076/year and 9.12 years, respectively. The third and fourth cases illustrate the accumulating material with all delays considered. The defined fuel management program specified in- and out-of-pile residing times of 912 days and 304 days, respectively. Reprocessing losses were held at two percent. Case three neglects the effect of ^{241}Pu β -decay. The associated asymptotic growth rate for this case is 0.057/year which gives a doubling time of 12.16 years. Case four illustrates the accumulating ^{240}Pu and ^{241}Pu atoms for the realistic situation, and includes β -decay of ^{241}Pu . A growth rate and doubling time of 0.056/year and 12.38 years is realized. As indicated by the above discussion, it is highly advantageous to reduce the delay in the use of excess bred material to increase the growth rate and decrease the doubling time.

IV.B. ISOTOPIC BREEDING WORTH FACTORS

Table 1 illustrates typical values for the isotopic breeding worth, standard and substitutional critical mass definitions of isotopic weight factors, relative to ^{239}Pu . The isotopic breeding worth for ^{241}Pu is much lower than was reported in Ref. 1. Reference 1 treated an idealized instantaneous reuse scheme and neglected ^{241}Pu decay. Here with a more specific treatment, ^{241}Pu is lost through decay in-and-out of pile. These decayed nuclei do not contribute to breeding; thus, the breeding worth of ^{241}Pu relative to ^{239}Pu decreases. Investigations have shown that the isotopic breeding worth of ^{241}Pu relative to ^{239}Pu increases to a value close to that presented in Ref. 1 when decay is not considered. The isotopic breeding worth of ^{242}Pu is about half that of ^{240}Pu ; but from a reactivity worth standpoint, shown by the substitutional critical mass weights, they are essentially the same. Difference between the isotopic breeding worth values for these two isotopes can be understood: a neutron absorbed by ^{240}Pu breeds ^{241}Pu or fissions but neutron capture by ^{242}Pu does not lead to a fuel isotope.

IV.C. COMPOSITION DEPENDENCE

The dependence of growth rate definitions and the fuel composition is investigated for a sequence of compositions as they appear as function of time during a transition from a certain initial condition to equilibrium.

Isotopic dependence of the doubling time formula, given by Eq. (65) and Eq. (66), based on the isotopic breeding worths, standard weights and substitutional critical mass weights, is investigated.

For the calculation of the γ 's in Fig. 4 the following quantities have been used with their reference reactor values:

$$\lambda_{Ca,c}, H_{Si}^{\infty}, H_{Ci}^{\infty}, \kappa_{Bi}^{\infty}, \kappa_{Ci}^{\infty}, \text{ and } N_{Xi}^{\infty}$$

Figure 4 shows the time dependence of the core doubling time for the three sets of isotopic weight factors given in Table 1, and two initial compositions, ^{238}U with ^{239}Pu and ^{240}Pu . The time dependent integrated fuel cycle model equations were solved to obtain time dependent core composition and reaction rates. The solid lines indicate core doubling time, t_{CD} , calculated for an initial composition of ^{238}U and ^{239}Pu while the dashed lines indicate an initial composition of ^{238}U and ^{240}Pu .

Doubling time based on the standard weight factors, t_{CD}^{ST} , in both cases is initially the farthest away from the equilibrium values, t_{CD}^{eq} . The substitutional critical mass weights, t_{CD}^{CM} , also yield high and low initial values when compared to t_{CD}^{eq} . As the solution of the time dependent reaction rates approaches equilibrium, t_{CD}^{CM} , and t_{CD}^{ST} converge to the equilibrium core doubling time. Doubling time evaluated with the isotopic breeding worths, t_{CD}^* , yields the exact equilibrium value for all compositions.

The composition dependence of the system doubling time, for the three sets of isotopic weight factors, is proportional to the curves of Fig. 4. The proportionality constant is given by R_{CS} , (comp. Sec. IV.C.). R_{CS} for the system is, 1.72. All points in Fig. 4 are multiplied by this factor to obtain the system growth rate.

As seen from the discussion in this section, use of standard and

substitutional critical mass weight factors produce significant error in predicting the equilibrium core and system doubling time for compositions much different from equilibrium. Only the isotopic breeding worth weight factors yield accurate estimates of the core and system doubling time for off-equilibrium compositions.

VI.D. STATIONARY CHARACTER OF GROWTH RATE FORMULAS

The stationary character of the growth rate based on the three sets of isotopic weight factors was investigated. Equation (65) was modified by considering instantaneous reuse in the same way as described in Ref. 1. Figure 5 illustrates the variational characteristics of the three growth rate definitions (based on an instantaneous reuse scheme) for changes in the reactor power. The radius of the reactor is modified so that the power is increased or decreased by 40%. Thus, the range is from 0.6 to 1.4 times 1000 MWe of the reference system.

Note that what is plotted is in a sense the error of 10^{-6} . The smallest dashed line gives the quantity calculated with isotopic breeding worths and is stationary. At either end of this curve, there is an error of about 1.5×10^{-6} which is less than 0.001% for the same given set of breeding worth factors. The other two curves are multiplied by 10^{-3} in order to illustrate some dependences on the same figure. Figure 5 illustrates a tremendous sensitivity of the growth rate when using w^{+ST} and w^{+CM} to perturbations in reactor size.

The same trends illustrated in Fig. 5 would be seen if perturbations in fuel and coolant volume fractions were investigated. By increasing and decreasing the power and size of the reactor, there is a corresponding softening and hardening of the neutron spectrum. This is the same behavior which is observed when modifying the fuel and coolant volume fractions.

Figure 6 illustrates the character of growth rates for cross section uncertainties, specifically the capture fission ratio in ^{239}Pu . The scale of variation ranges from plus or minus 5% in α . Again, the same stationary character of the growth rate based on isotopic breeding worths is observed. Likewise, the other two types of weighting do not exhibit a stationary behavior.

SUMMARY

The hierarchy of the four growth rate expression originally derived from an instantaneous reuse scheme have been expanded to account for finite burnup in the core and blanket, for β -decay of ^{241}Pu , for core and blanket loading schemes, for reuse delays due to reprocessing and fabricating fuel, and for external fuel cycle losses.

The general growth rate definition, obtained from the asymptotic slope of the accumulating material in a growing system of identical breeder reactors, is formally the same in both cases. The general definition is independent of weighting factors. The major differences in the definitions for individual reactors, result from including in- and out-of-pile plutonium and from the difference between residing and discharge plutonium. In equilibrium operation, the difference of these two fuel compositions is small; the impact of this difference needs to be evaluated with substitutional critical mass worth weights.

The bulk of the growth rate is independent of the isotopic weight factors, so that computational ease, such as stationarity of the bulk of γ , can be adopted as a criterion for the determination of a special weight factor set. This leads to the definition of isotopic breeding worth, calculated from an adjoint fuel cycle problem. The isotopic breeding worth factors calculated from the expanded model differ from the previously calculated set (Ref. 1) primarily in the ^{241}Pu worth; inclusion of the β -decay of ^{241}Pu leads to a significant reduction of the worth of these nuclei for breeding.

The quantitative investigation of the stationarity character of

the growth rate calculated with the breeding worths, w_1^* leads to a stationary definition: i.e., the application of a reference set of w_1^* to reactors that are increasingly different from the reference system leads to an error which increases only quadratically. The standard and w_1^{CM} based definitions lead to a linear deviation that is, in the practically important domain, thousands of times larger than the small quadratic error of the stationary definition.

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Figure 1: Time dependence of the accumulating fuel with ^{239}Pu as initial fuel (solid line). Asymptotic time dependence extrapolated to time equal zero (dashed line).

Figure 2: Time dependence of the accumulating ^{240}Pu atoms, with ^{239}Pu as initial fuel (solid lines), for different delays. Asymptotic time dependence extrapolated to time equal zero (dashed line).

Figure 3: Time dependence of the accumulating ^{241}Pu atoms, with ^{239}Pu as initial fuel (solid lines), for different delays. Asymptotic time dependence extrapolated to time equal zero (dashed line).

Figure 4: Time dependence of the core doubling time definitions with ^{239}Pu and ^{238}U as initial fuel (solid lines) and ^{240}Pu and ^{238}U as initial fuel (dashed lines).

Figure 5: Stationary character of growth rate formulas for change in reactor power

Figure 6: Stationary character of growth rate formulas for change in ^{239}Pu capture to fission ratio, α

Table 1

Isotopic Weight Factors Relative to ^{239}Pu

Weight Definition

Isotope	Isotopic Breeding Worths		Standard	Crit. Mass
	$\delta + \beta$	*.Ref. 1		
^{239}Pu	1.00	1.00	1.0	1.00
^{240}Pu	0.53	0.50	0.0	0.22
^{241}Pu	1.05	1.20	1.0	1.32
^{242}Pu	0.26	0.11	0.0	0.19

* Isotopic breeding worths calculated with delays and ^{241}Pu β -decay.