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GEOMETRIC FORMULATION OF ELECTRODYNAMICS AND GENERAL
RELATIVITY IN DISCRETE SPACE-TIME

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Geometric Formulation of Electrodynamics and General
Relativity in Discrete Space-Time*

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Abstract

Classical electrodynamics is reconstructed in discrete space-time,
then combined with Regge's discrete version of general relativity.
Quantization using path integrals is considered briefly.

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1. Introduction

By using what might be described as a "geometric interpretation" of Maxwell's equations we will show that electrodynamics can be transferred, in a natural way, from continuous space-time parametrized by coordinates to a discrete version of space-time which requires no coordinates but retains some of the geometric structure of continuous space-time. Then we will combine this result with a formulation of general relativity proposed by Regge¹ and obtain a discrete, coordinate-free model incorporating both electrodynamics and gravitation.

Although part of the motivation for this work is simply that it provides an interesting way of looking at interacting electrodynamics and gravitation, it also has some practical value. The dynamical equations we will obtain are algebraic rather than differential and are therefore convenient for solution by computer.² The solutions gotten in this way should approximate solutions to the differential equations of the continuous theory. But the theory we will construct is not simply the result of replacing space-time by a lattice of points and derivatives by finite differences. Instead the discrete theory retains geometric features of the continuous theory, and we expect that as a consequence it will approximate the qualitative behavior of the continuous theory more efficiently.

Another motivation for this work is the familiar suspicion that a fully satisfactory cure for the ultraviolet divergences of relativistic quantum field theory will eventually require some form of discrete space-time. Thus it seems useful to see what discrete theories suggest themselves as natural extensions of existing continuous theories.

The theory we will describe is classical; the extent to which it might be quantized using Feynman path integration will be discussed briefly toward the end of this paper.

2. Chains, Boundaries and Coboundaries

We will begin by describing an interpretation of some operations which can be performed with arbitrary antisymmetric tensor fields on 4-dimensional curved space-time. Then we will apply these ideas to electrodynamics. Most of the definitions we will use are familiar notions of differential geometry.³ For convenience we will assume space-time is topologically trivial. The extension of our results to nontrivial topologies will be fairly obvious.

A 4-dimensional manifold can support 5 different sorts of covariant tensor fields antisymmetric in all indices: scalar fields, vector fields, and antisymmetric tensor fields with 2, 3 or 4 indices. Antisymmetric tensor fields with 5 or more indices vanish identically. We will call each such field an n -chain, where n is the number of its indices. By exterior differentiation we can map the set of n -chains into the set of $n+1$ chains. That is if we represent an arbitrary n -chain by $f^{(n)}$, exterior differentiation by ∇ , and the components of $f^{(n)}$ in a coordinate basis by $f^{(n)}_{\mu_1 \dots \mu_n}$ we have

$$\begin{aligned} [\nabla f^{(0)}]_{\mu} &= \frac{\partial}{\partial x^{\mu}} f^{(0)} \\ [\nabla f^{(1)}]_{\mu\nu} &= \frac{\partial}{\partial x^{\mu}} f^{(1)}_{\nu} - \frac{\partial}{\partial x^{\nu}} f^{(1)}_{\mu} \\ [\nabla f^{(2)}]_{\mu\nu\lambda} &= \frac{\partial}{\partial x^{\mu}} f^{(2)}_{\nu\lambda} - \frac{\partial}{\partial x^{\nu}} f^{(2)}_{\mu\lambda} - \frac{\partial}{\partial x^{\lambda}} f^{(2)}_{\nu\mu} \\ [\nabla f^{(3)}]_{\mu\nu\lambda\phi} &= \frac{\partial}{\partial x^{\mu}} f^{(3)}_{\nu\lambda\phi} - \frac{\partial}{\partial x^{\nu}} f^{(3)}_{\mu\lambda\phi} - \frac{\partial}{\partial x^{\lambda}} f^{(3)}_{\nu\mu\phi} - \frac{\partial}{\partial x^{\phi}} f^{(3)}_{\nu\lambda\mu} \end{aligned} \quad (2.1)$$

Each $f^{(4)}$ is mapped to 0 by this operation. On the other hand, by taking divergences, each n -chain can be mapped to an $(n-1)$ -chain. Representing the divergence operator by $-\Delta$ and converting it to a form convenient for

use on antisymmetric tensors we have

$$[\Delta f^{(n)}]_{\mu_1 \dots \mu_{n-1}} = -(-g)^{-\frac{1}{2}} \frac{\partial}{\partial x^\nu} (-g)^{\frac{1}{2}} f^{(n)\nu}_{\mu_1 \dots \mu_{n-1}} \quad (2.2)$$

where the quantity g is the determinant of the covariant metric tensor $g_{\mu\nu}$ which we choose to have signature $(+---)$. The result of Δ acting on any 0-chain is defined to be 0. Definitions (2.1) and (2.2) yield the identities

$$\nabla \nabla f^{(n)} = 0 \quad (2.3)$$

$$\Delta \Delta f^{(n)} = 0 \quad (2.4)$$

for all $f^{(n)}$.

Next let us define an invariant inner product between pairs of n -chains:

$$\langle f^{(n)} | h^{(n)} \rangle = \frac{1}{n!} \int d^4x (-g)^{\frac{1}{2}} f^{(n)\mu_1 \dots \mu_n} h^{(n)}_{\mu_1 \dots \mu_n} \quad (2.5)$$

It follows that if all the chains we consider are required to vanish sufficiently rapidly at ∞ , the operation ∇ is the adjoint of Δ :

$$\langle f^{(n-1)} | \Delta h^{(n)} \rangle = \langle \nabla f^{(n-1)} | h^{(n)} \rangle \quad (2.6)$$

This relation is a version of Stokes theorem.

But at each point in space-time an n -chain with $n \geq 1$ may be considered the tangent to an oriented n -dimensional surface. Thus the full n -chain may be thought of as a formal superposition of n -surface tangents each weighted with an appropriate coefficient. This picture can be extended to include $n = 0$ by considering a point to be a 0-dimensional surface and defining the tangent to a point to be a real number. Consider

the n-chain, $n \geq 1$, giving the tangent to a single oriented n-dimensional surface Σ parametrized by the map $y_i \rightarrow x^\mu(y_i)$, where i runs from 1 to n:

$$f^{(n)\mu_1 \dots \mu_n}(z) = (-g)^{-\frac{n}{2}} \int_{\Sigma} d^n y \delta^{(n)}[z - x(y)] \frac{\partial(x^{\mu_1} \dots x^{\mu_n})}{\partial(y_1 \dots y_n)}$$

The multiplicative factor $(-g)^{\frac{1}{2}}$ is necessary to make $f^{(n)}$ a tensor and

$$\frac{\partial(x^{\mu_1} \dots x^{\mu_n})}{\partial(y_1 \dots y_n)}$$

is the jacobian of the map $y_i \rightarrow x^\mu(y_i)$. For a point x we will choose the corresponding $f^{(0)}$ to be

$$f^{(0)}(z) = (-g)^{-\frac{1}{2}} \delta^{(4)}(z - x)$$

If we now evaluate $[\Delta f^{(n)}]^{\mu_1 \dots \mu_{n-1}}$ for the tensor $f^{(n)}$ of the surface Σ , we find we have obtained the (n-1)-surface tensor of the boundary of Σ . For example, for a tensor $f^{(2)}$ representing a 2-surface parameterized by y_1 and y_2 both running from 0 to 1, we have

$$\begin{aligned} [\Delta f^{(2)}]^\nu(z) &= (-g)^{-\frac{1}{2}} \frac{\partial}{\partial z^\mu} \int_0^1 \int_0^1 dy_1 dy_2 \delta^{(4)}[z - x(y)] \left(\frac{\partial x^\mu}{\partial y_1} \frac{\partial x^\nu}{\partial y_2} - \frac{\partial x^\nu}{\partial y_1} \frac{\partial x^\mu}{\partial y_2} \right) \\ &= (-g)^{-\frac{1}{2}} \int_0^1 \int_0^1 dy_1 dy_2 \left\{ \frac{\partial \delta^{(4)}[z - x(y)]}{\partial y_1} \frac{\partial x^\nu}{\partial y_2} - \frac{\partial \delta^{(4)}[z - x(y)]}{\partial y_2} \frac{\partial x^\nu}{\partial y_1} \right\} \\ &= (-g)^{-\frac{1}{2}} \int_0^1 dy_1 \delta^{(4)}[z - x(y)] \frac{\partial x^\nu}{\partial y_1} \Big|_{y_2=0}^{y_2=1} + (-g)^{-\frac{1}{2}} \int_0^1 dy_2 \delta^{(4)}[z - x(y)] \frac{\partial x^\nu}{\partial y_2} \Big|_{y_1=1}^{y_1=0} \\ &\quad + (-g)^{-\frac{1}{2}} \int_1^0 dy_1 \delta^{(4)}[z - x(y)] \frac{\partial x^\nu}{\partial y_1} \Big|_{y_2=1}^{y_2=0} + (-g)^{-\frac{1}{2}} \int_1^0 dy_2 \delta^{(4)}[z - x(y)] \frac{\partial x^\nu}{\partial y_2} \Big|_{y_1=0}^{y_1=1} \end{aligned}$$

Thus we will refer to Δ as the boundary operator and its adjoint ∇ as the coboundary operator.

Finally let us examine the action of the inner product on the tensor $f^{(n)}$ of a single surface Σ for which the number, s , of linearly independent space-like tangent vectors is the same at all points. Let $y_1, \dots, y_n$ parametrize Σ and let $y_{n+1}, \dots, y_4$ parametrize a family of nearly identical copies of Σ displaced from each other slightly in the direction orthogonal to Σ . Define the proper $4 - n$ dimensional measure of the region Σ' swept out by $y_{n+1}, \dots, y_4$ at fixed $y_1, \dots, y_n$ to be $m^{(4-n)}(\Sigma')$

$$m^{(4-n)}(\Sigma') = \int_{\Sigma'} d y_{n+1} \dots d y_4 \sqrt{\frac{1}{(4-n)!} \left| \frac{\partial(x^{\mu_{n+1}} \dots x^{\mu_4})}{\partial(y_{n+1} \dots y_4)} \frac{\partial(x^{\nu_{n+1}} \dots x^{\nu_4})}{\partial(y_{n+1} \dots y_4)} g_{\mu_{n+1} \nu_{n+1}} \dots g_{\mu_4 \nu_4} \right| } , \quad (2.7)$$

and require $m^{(4-n)}(\Sigma')$ to be the same for all $y_1, \dots, y_n$. For $n=4$, $m^{(4-n)}(\Sigma')$ is defined to be 1. Then the normalized n -surface tensor

$$f^{(n)\mu_1 \dots \mu_n}(z) = [-m^{(4-n)}(\Sigma) g]^{-\frac{1}{2}} \int d^4 y \delta^{(4)}[z - x(y)] \frac{\partial(x^{\mu_1} \dots x^{\mu_n})}{\partial(y_1 \dots y_n)}$$

yields the value

$$\langle f^{(n)} | f^{(n)} \rangle = (-1)^s m^{(n)}(\Sigma) ,$$

where, for $n \geq 1$, $m^{(n)}(\Sigma)$ is the proper n -measure of Σ given by an expression of the same form as (2.7), and the proper 0-measure of a point is defined to be 1.

The preceding ideas can be applied to electrodynamics immediately. The electromagnetic field F is a 2-chain and therefore corresponds to a weighted superposition of 2-dimensional surfaces; the vector potential A is a 1-chain and therefore a superposition of lines. Maxwell's equations are

$$\begin{aligned}\nabla F &= 0 \\ \Delta F &= 0\end{aligned}\tag{2.8}$$

and state that both the boundary and coboundary of the electromagnetic field vanish, and the relation between F and A

$$F = \nabla A\tag{2.9}$$

states that F is the coboundary of A . An action integral which yields (2.8) is

$$Q = \langle \nabla A | \nabla A \rangle\tag{2.10}$$

if (2.9) is taken as the definition of F .

3. Electrodynamics in a Simplicial Complex

We will now replace continuous space-time with a form of discrete space-time which can be equipped with a natural set of n -chains, a boundary operator, an inner product, and a coboundary operator, and these in turn will allow us to reconstruct electrodynamics.⁴ The mathematical designation for the discrete space-time we will describe is a "simplicial complex".⁵

Let S_0 be a set containing at most a countably infinite number of elements. For convenience we will take S_0 to be either the integers or a proper subset of the integers. For each $n=1, \dots, 4$ let S_n be a set of $(n+1)$ -tuples of elements of S_0 . We require that if $\{i_1, \dots, i_{n+1}\}$, $n \geq 1$,

occurs in S_n then each subset of $\{i_1, \dots, i_{n+1}\}$ is an element of some S_m , $m \leq n$. From every element of S_n form all possible distinct ordered $(n+1)$ -tuples. These will be written $[i_1, \dots, i_{n+1}]$ and may be thought of as oriented n -surfaces. Each will be called an n -simplex. By allowing n -simplexes to be formally multiplied by arbitrary real numbers and then forming finite sums of such quantities, the collection of n -simplexes can be turned into a vector space. A vector in this space will be called an n -chain and written $c^{(n)}$. If $[j_1, \dots, j_{n+1}]$ is a permutation of $[i_1, \dots, i_{n+1}]$ we assume

$$[j_1, \dots, j_{n+1}] = (-1)^P [i_1, \dots, i_{n+1}],$$

where P is 1 for an odd permutation and 0 otherwise. Each $c^{(n)}$ can be written in the form

$$c^{(n)} = \sum_{[i_1, \dots, i_{n+1}] \in U_n} c^{(n)}(i_1, \dots, i_{n+1}) [i_1, \dots, i_{n+1}], \quad (3.1)$$

where $c^{(n)}(i_1, \dots, i_{n+1})$ is a real number and the set of oriented simplexes U_n contains exactly one (arbitrarily chosen) ordering $[j_1, \dots, j_{n+1}]$ for each $\{i_1, \dots, i_{n+1}\} \in S_n$.

A boundary operator Δ taking any $c^{(n)}$ into some $c^{(n-1)}$ can be defined as follows. For an individual n -simplex assume

$$\Delta[i_1, \dots, i_{n+1}] = \sum_{j=1}^{n+1} (-1)^{j+1} [i_1, \dots, \hat{i}_j, \dots, i_{n+1}], \quad (3.2)$$

where the symbol $\hat{i}_j$ indicates i_j has been omitted and the result of omitting i from the 0-simplex $[i]$ is 0. The action of Δ on an arbitrary n -chain is gotten from (3.2) by requiring Δ to be linear. It follows that

$$\Delta \Delta c^{(n)} = 0 \quad (3.3)$$

for all $c^{(n)}$.

Next we will construct an inner product $\langle c^{(n)} | d^{(n)} \rangle$ on n-chains. For any individual n-simplex assume

$$\langle [i_1, \dots, i_{n+1}] | [j_1, \dots, j_{n+1}] \rangle = \mu(i_1, \dots, i_{n+1}) \quad (3.4)$$

where $\mu(i_1, \dots, i_{n+1})$ is a real number; for each pair of simplexes differing in the set $\{i_1, \dots, i_{n+1}\}$ from which they are formed assume

$$\langle [i_1, \dots, i_{n+1}] | [j_1, \dots, j_{n+1}] \rangle = 0 \quad (3.5)$$

For any 0-simplex $[i]$, $\mu(i)$ is chosen to be 1. For $n \geq 1$, $\mu(i_1, \dots, i_{n+1})$ can be either positive or negative, but for convenience we exclude $\mu(i_1, \dots, i_{n+1}) = 0$. Those n-simplexes with negative measure may be thought of as containing an odd number of linearly independent space-like vectors. The inner product between arbitrary n-chains is gotten from (3.4) and (3.5) by requiring $\langle c^{(n)} | d^{(n)} \rangle$ to be linear in each of its arguments. The coboundary operator ∇ can now be defined as the adjoint of Δ with respect to the inner product. Equation (3.3) combined with the restriction $\mu(i_1, \dots, i_{n+1}) \neq 0$ for all $\{i_1, \dots, i_{n+1}\} \in S_n$ implies

$$\nabla \nabla c^{(n)} = 0 \quad (3.6)$$

for all $c^{(n)}$.

Finally we must impose a number of constraints on the structure of the simplicial complex to insure that it adequately approximates a curved 4-dimensional space-time governed by a metric with signature (+---). First we need a pair of "topological" requirements: each (n-1)-simplex must occur in the boundary of at least one n-simplex, for $n = 1, \dots, 4$, and each

3-simplex must occur in the boundary of exactly two distinct 4-simplexes. These conditions imply, roughly speaking, that the complex will not have any isolated regions of dimension lower than 4, any boundaries, or any "joints" which would require a space of dimension higher than 4 for a local embedding. The significance of the two conditions we have just given can be pictured more clearly by considering the effect of corresponding requirements on a complex composed only of 0,1,2 and 3 simplexes. It can be proved⁵ that if these conditions are fulfilled then there is necessarily some curved, continuous 4-dimensional space-time which can be covered with the simplicial complex in such a way that each 4-simplex is assigned to a unique singly connected 4-volume, each 3-simplex assigned to a unique singly connected 3-volume and so on.

We will also need a second set of constraints which are "metric". Consider a single 4-simplex $\{i_1, \dots, i_5\}$ and all the 3, 2, 1 and 0-simplexes which can be formed from subsets of $\{i_1, \dots, i_5\}$. Let us embed this collection of simplexes in flat Minkowski space by assigning each 0-simplex $\{j\}$ to a point $p(j)$, each 1-simplex $\{j_1, j_2\}$ to a straight line $p(j_1, j_2)$ running from $p(j_1)$ to $p(j_2)$, and so on. If $p^\mu(j)$ are the Minkowski space components of $p(j)$, each n-simplex can be assigned to an n-indexed anti-symmetric surface tensor by the relations

$$M^{\mu}(j_1, j_2) = p^{\mu}(j_2) - p^{\mu}(j_1)$$

$$\begin{aligned} M^{\mu_1 \dots \mu_n}(j_1, \dots, j_{n+1}) &= M^{\mu_1 \dots \mu_{n-1}}(j_1, j_2, \dots, j_{n-1}, j_n) M^{\mu_n}(j_1, j_{n+1}) \\ &\quad - M^{\mu_1 \dots \mu_{n-1}}(j_1, j_2, \dots, j_{n-1}, j_{n+1}) M^{\mu_n}(j_1, j_n) \\ &\quad \dots - M^{\mu_1 \dots \mu_{n-1}}(j_1, j_{n+1}, \dots, j_{n-1}, j_n) M^{\mu_n}(j_1, j_2) \end{aligned}$$

Then let the function $N(j_1, \dots, j_{n+1})$ be given by

$$N(j_1, \dots, j_{n+1}) = \frac{1}{n!} M^{\mu_1 \dots \mu_n}(j_1, \dots, j_{n+1}) M^{\nu_1 \dots \nu_n}(j_1, \dots, j_{n+1}) \eta_{\mu_1 \nu_1} \dots \eta_{\mu_n \nu_n}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. From $N(j_1, \dots, j_{n+1})$ define a signed proper n-measure for each n-simplex by

$$\mu(j_1, \dots, j_{n+1}) = \frac{N(j_1, \dots, j_{n+1})}{\sqrt{|N(j_1, \dots, j_{n+1})|}} \quad (3.7)$$

It follows that for each n-simplex, $[\mu(j_1, \dots, j_{n+1})]^2$ can be expressed as the absolute value of a homogeneous polynomial of degree $2n$ in the set of 1-measures $\mu(j_1, j_2)$. Let us abstract this set of equations and use it to determine each $[\mu(j_1, \dots, j_{n+1})]^2$, $n \geq 2$, in (3.4) from the set of $\mu(j_1, j_2)$ in (3.4). In addition, since the embedding we have made is in flat Minkowski space with signature $(+---)$, the quantities $\mu(j_1, \dots, j_{n+1})$ in (3.7) fulfill a set of inequalities. For example, the 4-measure $\mu(i_1, \dots, i_5)$ must be negative, and if three 1-simplexes with negative measure form the boundary of a 2-simplex, then the measure of the 2-simplex must be positive. Let us also abstract this set of inequalities and impose them on the measures in (3.4). Once the various metric constraints we have just described are imposed, the simplicial complex's inner product $\langle \dots | \dots \rangle$ acting on 1-chains becomes the discrete version of the metric tensor $g_{\mu\nu}$ of curved continuous space-time.

We can now construct electrodynamics on the simplicial complex simply by reinterpreting each of the quantities in Eqs. (2.8), (2.9) and (2.10). If F is a 2-chain on the simplicial complex, Δ the complex's boundary operator and ∇ the coboundary operator, then Eqs. (2.8) become the simplicial complex's version of Maxwell's equations. If A is now a 1-chain on the complex, (2.9) is again the relation between the electromagnetic field

and potential. Using the complex's inner product (2.10) becomes a satisfactory action integral. We have

$$\begin{aligned} \delta Q &= \langle \nabla \delta A | \nabla A \rangle + \langle \nabla A | \nabla \delta A \rangle \\ &= 2 \langle \nabla \delta A | \nabla A \rangle \\ &= 2 \langle \delta A | \Delta \nabla A \rangle = 0 . \end{aligned} \quad (3.8)$$

But since we assumed none of the $\mu(i_1, \dots, i_n)$ in (3.4) are 0, (3.8) implies

$$\Delta \nabla A = 0 . \quad (3.9)$$

Using (3.9), (3.6) and (2.9) we obtain (2.8).

4. Electrodynamics and General Relativity

A formulation of general relativity on a simplicial complex developed in Ref. [1] can be described as follows. The metric tensor of continuous space-time $g_{\mu\nu}$ is replaced by the inner product of the simplicial complex acting on the set of 1-chains, while the curvature tensor $R_{\mu\nu\alpha\beta}$ is replaced by a symmetric bilinear form $R[c^{(2)}, c^{(2)'}]$ on the set of 2-chains of the simplicial complex. Using the inner product on the set of 2-chains, the bilinear form $R[c^{(2)}, c^{(2)'}]$ can be related to a linear operator R on 2-chains:

$$R[c^{(2)}, c^{(2)'}] = \langle c^{(2)} | R c^{(2)'} \rangle$$

If $[i, j, k]^*$ is the dual to the 2-simplex $[i, j, k]$ given by the linear map

$$[i, j, k]^* : c^{(2)} \longrightarrow \langle [i, j, k] | c^{(2)} \rangle ,$$

then, in an obvious notation, R has the diagonal form

$$R = \sum_{\{i,j,k\} \in U_2} R(i,j,k) \frac{[i,j,k][1,j,k]^*}{\mu(i,j,k)},$$

where $R(i,j,k)$ is a real number which can be expressed as a function of the values of the inner product on the set of 1-simplexes, $\mu(i,j)$. Using R or $R(i,j,k)$ the action integral of general relativity becomes

$$\begin{aligned} T_r(R) &= \sum_{\{i,j,k\} \in U_2} \langle [i,j,k] | R[i,j,k] \rangle \\ &= \sum_{\{i,j,k\} \in U_2} R(i,j,k) \mu(i,j,k). \end{aligned} \quad (4.1)$$

Einstein's equations are gotten by requiring

$$\frac{\delta T_r(R)}{\delta \mu(i,j)} = 0$$

for all $\{i,j\} \in U_1$.

An action for interacting gravity and electrodynamics can now be obtained simply by adding (4.1) to the action for electrodynamics in Sect. 3:

$$Q = T_r(R) + \langle \nabla A | \nabla A \rangle. \quad (4.2)$$

If we vary this with respect to A , Maxwell's equations (2.8) are reproduced again. Varying with respect to $\mu(i,j)$ for all $\{i,j\} \in U_1$ we obtain the interacting version of Einstein's equations. The first term on the right side of (4.2) yields

$$\begin{aligned} \frac{\delta T_r(R)}{\delta \mu(i,j)} = & \sum_{[i',j',k'] \in U_2} \frac{\delta R(i',j',k')}{\delta \mu(i,j)} \mu(i',j',k') \\ & + \sum_{[i',j',k'] \in U_2} R(i',j',k') \frac{\delta \mu(i',j',k')}{\delta \mu(i,j)} \end{aligned} \quad (4.3)$$

But it is shown in [1] that the first term on the right side of (4.3) vanishes. In addition, the "metric" constraints imposed on the complex in Sect. 3 give

$$\begin{aligned} 16 \mu(i,j,k)^2 = & |2 \mu(i,j)^2 \mu(j,k)^2 + 2 \mu(i,k)^2 \mu(j,k)^2 + 2 \mu(i,k)^2 \mu(i,j)^2 \\ & - \mu(i,j)^4 - \mu(j,k)^4 - \mu(i,k)^4| \end{aligned} \quad (4.4)$$

and $\mu(i,k)$ has the same sign as the argument of the absolute value. The second term on the right of (4.3) becomes

$$\frac{\delta T_r(R)}{\delta \mu(i,j)} = \sum_{[i,j,k'] \in U_2} R(i,j,k') \mu(i,j) \frac{[\mu(j,k')^2 + \mu(i,k')^2 - \mu(i,j)^2]}{|\mu(i,j,k')|} \quad (4.5)$$

Varying $\langle \nabla A | A \nabla \rangle$ with respect to $\mu(i,j)$ yields

$$\begin{aligned}
 \frac{\delta \langle \nabla A | \nabla A \rangle}{\delta \mu(i,j)} &= \frac{\delta}{\delta \mu(i,j)} \sum_{[i',j',k'] \in U_2} \frac{\langle A | \Delta[i',j',k'] \rangle \langle \Delta[i',j',k'] | A \rangle}{\mu(i',j',k')} \\
 &= - \sum_{[i,j,k'] \in U_2} \left\{ \frac{\langle \nabla A | [i,j,k'] \rangle \langle [i,j,k'] | \nabla A \rangle}{|8 \mu(i,j,k')^3|} \mu(i,j) \right. \\
 &\quad \times [\mu(j,k')^2 + \mu(i,k')^2 - \mu(i,j)^2] \Big\} \\
 &\quad + 2 \frac{\langle A | [i,j] \rangle \langle [i,j] | \Delta \nabla A \rangle}{\mu(i,j)^2} \quad (4.6)
 \end{aligned}$$

Using Maxwell's equations and combining (4.5) and (4.6), the interacting version of Einstein's equations for the simplicial complex can be written

$$\begin{aligned}
 \sum_{[i,j,k'] \in U_2} \left\{ \frac{\langle [i,j,k'] | R[i,j,k'] \rangle}{|\mu(i,j,k')|} - \frac{\langle [i,j,k'] | \nabla A \rangle \langle \nabla A | [i,j,k'] \rangle}{|\mu(i,j,k')^3|} \right\} \\
 \times [\mu(j,k')^2 + \mu(i,k')^2 - \mu(i,j)^2] = 0 \quad (4.7)
 \end{aligned}$$

for each $[i,j] \in U_1$ where the sum is carried out over k' .

Solutions to (4.7) can be interpreted as approximate solutions to the differential equations of continuous interacting gravity and electrodynamics by a straightforward extension of methods described in [1]. Roughly speaking,

each 4-simplex is mapped into flat Minkowski space, then continuous fields are constructed from corresponding discrete quantities by averaging over collections of many neighboring simplexes. The details will not be given here.

5. Quantization

The theory described so far is classical. We will now consider quantization using a Feynman path integral.

If the simplicial complex and its inner product are assumed given, the electromagnetic field by itself can be quantized without great difficulty. For convenience let us restrict the set of points S_0 on which the complex is based to be finite. The various other requirements on the complex, given previously, remain unchanged.

The first candidate for an n-point function is the integral

$$T[A(i_1, j_1) \dots A(i_n, j_n)] = \prod_{[i, j] \in U_1} \int dA(i, j) \exp(i \langle \nabla A | \nabla A \rangle) \prod_{l=1}^n A(i_l, j_l), \quad (5.1)$$

where $A(i, j)$ is defined by (3.1). But if λ is an arbitrary 0-chain,

Eq. (3.6) implies the action $\langle \nabla A | \nabla A \rangle$ is invariant under the gauge transform

$$A' = A + \nabla \lambda \quad (5.2)$$

The orbit obtained by applying all such transformations to a fixed A has infinite volume. Therefore (5.1) diverges. This difficulty can be corrected by an adaptation of the work of Fadeev and Popov.⁶

Since the complex is finite the subspace of 1-chains, Q , which can be expressed in the form $\nabla \lambda$ is finite dimensional. Let the set of 1-chains $a_1, a_2, \dots$ form a linearly independent basis for Q and let $a_1, a_2, \dots, A_1, A_2, \dots$

form a linearly independent basis for the entire space of 1-chains. There are many different ways $A_1, A_2, \dots$ can be chosen. Each possible choice is equally good for our purposes. An arbitrary 1-chain $c^{(1)}$ can be written in two distinct forms

$$c^{(1)} = \sum_{[i,j] \in U_1} c^{(1)}(i,j) [i,j] \quad , \quad (5.3)$$

$$c^{(1)} = \sum_k c^{(1)}(k) A_k + \sum_k c^{(1)'}(k) a_k \quad (5.4)$$

The relation between the two sets of coefficients is a linear transformation

$$c^{(1)}(i,j) = \sum_k L(i,j|k) c^{(1)}(k) + \sum_k L'(i,j|k) c^{(1)'}(k) \quad , \quad (5.5)$$

where

$$L(i,j|k) = \mu(i,j)^{-1} \langle [i,j] | A_k \rangle$$

$$L'(i,j|k) = \mu(i,j)^{-1} \langle [i,j] | a_k \rangle.$$

If D is the determinant of the linear transformation (5.5), the integral in (5.1) can be rewritten

$$D \prod_{k_1} \int dA(k_1) \prod_{k_2} \int dA'(k_2) \exp \left[i \sum_{l_1, l_2} A(l_1) A(l_2) \langle \nabla A_{l_1} | \nabla A_{l_2} \rangle \right] \prod_{m=1}^n \left[\sum_p L(i_m, j_m | p) A(p) + \sum_p L'(i_m, j_m | p) A'(p) \right] \quad (5.6)$$

The coefficients $A'(k)$ do not appear in the argument of the exponential in (5.6) since each a_k fulfills $\nabla a_k = 0$. Therefore the integrals over $dA'(k)$ in (5.6) diverge. We now simply omit these integrals and set each $A'(k)$ in the remainder of (5.6) to 0. This procedure is equivalent to a

choice of gauge in canonical quantization of continuous electrodynamics. The n-point functions we obtain will then depend on precisely how the set $A_1, A_2, \dots$ was chosen. But if the n-point functions for A are used to construct n-point functions for quantities such as ∇A , which are invariant with respect to the gauge transform (5.2), the result can easily be shown independent of the choice of $A_1, A_2, \dots$. Our replacement for (5.6) is now

$$\prod_k \int dA(k) \exp \left[i \sum_{l_1, l_2} A(l_1) A(l_2) \langle \nabla A_{l_1} | \nabla A_{l_2} \rangle \right] \prod_{m=1}^n \left[\sum_p L(i_m, j_m | p) A(p) \right].$$

This expression is still not finite, however, since the equation

$$\sum_l A(l) \Delta \nabla A_l = 0$$

has nonzero solutions and therefore the matrix $\langle \nabla A_{l_1} | \nabla A_{l_2} \rangle$ has 0 as an eigenvalue. This further difficulty can be circumvented by introducing a small "photon mass" ϵ extremely close to 0.

Our final result becomes

$$T[A(i_1, j_1) \dots A(i_n, j_n)] = \prod_k \int dA(k) \exp \left\{ i \sum_{l_1, l_2} A(l_1) A(l_2) \left[\langle \nabla A_{l_1} | \nabla A_{l_2} \rangle - \epsilon^2 \langle A_{l_1} | A_{l_2} \rangle \right] \right\} \times \prod_{m=1}^n \left[\sum_p L(i_m, j_m | p) A(p) \right].$$

This is a gaussian integral and can be expressed using the inverse matrix

$M_{k_1 k_2}$ defined by

$$\sum_{k_2} M_{k_1, k_2} \left[\langle \nabla A_{k_2} | \nabla A_{k_3} \rangle - \varepsilon^2 \langle A_{k_2} | A_{k_3} \rangle \right] = \delta_{k_1, k_3}.$$

For example, if the vacuum to vacuum amplitude is

$$T = \prod_k \int dA(k) \exp \left\{ i \sum_{l_1, l_2} A(l_1) A(l_2) \left[\langle \nabla A_{l_1} | \nabla A_{l_2} \rangle - \varepsilon^2 \langle A_{l_1} | A_{l_2} \rangle \right] \right\},$$

then the normalized 2-point function becomes

$$T[A(i_1, j_1) A(i_2, j_2)] / T = \sum_{k_1, k_2} L(i_1, j_1 | k_1) L(i_2, j_2 | k_2) M_{k_1, k_2}.$$

The question which remains is whether path integration can also be used to quantize the simplicial complex's metric, given by the collection of 1-measures $\mu(i, j)$, or the complex's topology, described by the sets $S_0, \dots, S_4$. If the complex's topology is fixed, we would expect that the set of $\mu(i, j)$ could be quantized since path integral quantization has been carried through, at least formally, for the metric of the continuous theory^{6,7}. Quantization of the discrete theory may actually be simplified to some extent by the absence of frame dependent quantities. Once a path integral has been put together its behavior could perhaps be determined using methods similar to those Wilson⁸ has applied to conventional field theories on a lattice. Quantization of the complex's topology, however, is a more difficult question. I hope to return to this subject elsewhere.

Footnotes and References

1. T. Regge, *Nuovo Cimento* 19, 558 (1961).
2. Computer solutions of Regge's theory by itself have been done by P. A. Collins and R. M. Williams, *Phys. Rev.* D7, 965 (1973).
See also R. Sorkin, *Phys. Rev.* D12, 385 (1975).
3. S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Interscience Publishers, New York, 1963.
4. This was suggested in part by D. Bohm, B. J. Hiley and A.E.G. Stuart, *International Journal of Theoretical Physics* 3, 171 (1970).
5. A discussion of simplicial complexes is given by P. S. Alexandroff, Combinatorial Topology, Graylock Press, Rochester, 1957.
6. V. N. Popov and L. D. Faddeev, *Perturbation Theory for Gauge Invariant Fields*, Kiev ITP report (unpublished), L. D. Faddeev and V.N. Popov, *Phys. Lett.* 25B, 29 (1967).
7. L. D. Faddeev and V. N. Popov, *Sov. Phys. - Usp.* 16, 777 (1974).
8. K. G. Wilson and J. Kogut, *Phys. Reports* 12, 75 (1974).