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SPECTRUM OF AN ELLIPTIC PLASMA
COLUMN

BY

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NUMERICAL STUDIES OF THE MHD SPECTRUM
OF AN ELLIPTIC PLASMA COLUMN

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ABSTRACT

A numerical procedure is described for determining the MHD spectrum associated with small perturbations about an analytic equilibrium. This configuration has magnetic flux surfaces which are nested similar elliptical cylinders generated by a uniform axial current. Since the system is periodic, it models the essential features of a toroid. The code is used to study the properties of modes in the continuous shear Alfvén and slow acoustic spectra as well as the discrete modes associated with the fast magnetosonic waves and kinks. Modes where the interchange criterion is violated, or nearly violated, are investigated.

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1. INTRODUCTION

There is need for detailed theoretical knowledge about the ideal MHD behavior of tokamak systems, not only to explain and predict experimental results but also to aid the design of future systems. For real configurations, one must consider cross sections and equilibrium profiles which are rarely amenable to direct analytical methods. Because of this, we are undertaking a major computational effort to study the properties of the linear eigenmodes in toroidal systems with arbitrary axisymmetric equilibria. For the sake of gaining a thorough understanding of the physics involved and minimizing the numerical and algebraic uncertainties in such an effort, we chose to proceed in several phases, each using results from previous ones together with other sources, to serve as guidelines in the development of a general numerical code. Thus, we may define phase zero as the description of the diffuse linear pinch; this can be described by a second-order linear differential equation which has been studied extensively [1,4]. We have a highly accurate numerical code for solving this equation as a check of the accuracy for later stages. For phase one, we calculated the eigenvalue spectrum of a straight system, with an elliptical cross section and constant longitudinal current, in the long wavelength, zero pressure limit [5,6]; the numerical results for this case compared well with analytical results which were derived for the constant density case [5,7]. Phase two represents the generalization of phase one to include the

effects of finite wavelength and nonzero pressure. Both phases one and two are constrained to have a constant axial current so that an equilibrium with elliptical magnetic surfaces can be obtained analytically [8]. This avoids problems of inaccuracy with a numerically determined equilibrium and its subsequent interfacing into the normal-mode analysis; this problem is dealt with in phase three which treats the fully toroidal system [9,10].

In this paper, we present some of the results obtained from phase two. We generate the whole MHD spectrum in various regions of the parameter space of the equilibrium; even with a constant axial current we have sufficient flexibility that the dependence of the stability properties on the equilibrium parameters can be studied for numerous configurations. In section 2, we describe the equilibrium and our numerical approach to the stability problem, showing the coordinates used, the decomposition of the Lagrangian displacement, and the solution of the eigenvalue problem. We discuss the results in the next three sections where we study, respectively, the properties of the modes which lie in the continuous Alfvén and sound spectra, modes which violate or nearly violate the interchange stability criterion [11-13], and the discrete spectra such as the magnetosonic waves and the kinks. Since our code is suitable for studies of systems with either circular or noncircular cross sections, we intersperse results obtained with one or two dimensional models whenever this is useful for understanding the spectrum.

2. EQUILIBRIUM AND NUMERICAL PROCEDURE

For a straight configuration with periodicity length L , we write the equilibrium magnetic field in the form

$$\vec{B} = B_0 \left(\frac{2\pi}{L} f(\psi) \vec{\nabla} z \times \vec{\nabla} \psi + g(\psi) \vec{\nabla} z \right). \quad (1)$$

Here, ψ , which labels a magnetic surface, and z , the distance along the axis, represent two of our coordinates. We choose the third coordinate such that the field lines are straight. We have

$$\frac{d\vec{r} \times \vec{B}}{B_0} = \left(\frac{2\pi}{L} f d\psi \right) \vec{\nabla} z - (J g d\psi) \vec{\nabla} \theta + (J g d\theta - \frac{2\pi f}{L} dz) \vec{\nabla} \psi \quad (2)$$

with the Jacobian $J \equiv (\vec{\nabla} \psi \cdot \vec{\nabla} \theta \times \vec{\nabla} z)^{-1}$. Since this vanishes when $d\vec{r}$ is tangent to the magnetic field, the field lines are described by

$$d\psi = 0 \text{ and } \frac{d\theta}{dz} = \frac{2\pi f(\psi)}{LJ g(\psi)}. \quad (3)$$

Choosing the Jacobian $J = 1/2\pi$ ensures that the field lines are straight, makes the safety factor, $q(\psi) = g(\psi)/2\pi f(\psi)$, and simplifies the expression for the operator $\vec{B} \cdot \vec{\nabla}$. The magnetic surfaces are obtained by setting

$$\hat{e}_z \cdot \vec{\nabla} \times \vec{B} = \frac{2\pi B_0}{L} \vec{\nabla} \cdot [f(\psi) \vec{\nabla} \psi] = J_z. \quad (4)$$

We choose f and J_z to be constants so that [8]

$$\psi = \psi_0 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right) \quad (5)$$

where

$$\psi_0 \equiv \frac{a^2 b^2 L J_z}{4\pi B_0 f (a^2 + b^2)} \quad (6)$$

We also find $\psi_0 = \pi ab$ by integrating over the plasma volume, thus relating f to J_z . With these choices for $\psi(x,y)$ and f , we can determine θ from the relation $J \partial \psi(x,y) / \partial x = \partial y(\psi, \theta) / \partial \theta$, obtaining

$$y = b \left(\frac{\psi}{\psi_0} \right)^{1/2} \sin \theta, \quad (7)$$

$$x = a \left(\frac{\psi}{\psi_0} \right)^{1/2} \cos \theta. \quad (8)$$

The coordinate system has the properties

$$|\vec{\nabla} \psi|^2 = \frac{4\pi\psi}{ab} \left(\frac{b^2 + a^2}{2} + \frac{b^2 - a^2}{2} \cos 2\theta \right), \quad (9)$$

$$|\vec{\nabla} \theta|^2 = \frac{\pi}{ab\psi} \left(\frac{b^2 + a^2}{2} - \frac{b^2 - a^2}{2} \cos 2\theta \right), \quad (10)$$

$$\vec{\nabla} \psi \cdot \vec{\nabla} \theta = - \frac{\pi}{ab} (b^2 - a^2) \sin 2\theta, \quad (11)$$

$$\vec{\nabla} \psi \cdot \vec{\nabla} z = \vec{\nabla} \theta \cdot \vec{\nabla} z = 0. \quad (12)$$

We deduce immediately several features from these relations. First, for $a \neq b$ we have a nonorthogonal system; this may introduce some complications but it conforms to our philosophy that the physics of the situation should dominate mathematical considerations. Second, since the metric elements are equilibrium quantities, their dependence on 2θ indicates the existence of coupling among different Fourier modes in θ for noncircular cross sections but the odd modes always decouple from the even ones; this is a consequence of the symmetry property of the ellipse. Not all equilibrium quantities are functions of θ however; for example, the safety factor $q = g(\psi)/2\pi f$ depends only on ψ . Also, from pressure balance, $\vec{J} \times \vec{B} = \vec{\nabla} p$, we find $p = p(\psi)$ and

$$q^2(\psi) = q^2(\psi_0) - 2 \left[\frac{p(\psi)}{B_0^2} q^2(\psi_0) - \frac{2\pi^2}{L^2} (a^2 + b^2) \left(1 - \frac{\psi}{\psi_0}\right) \right] \quad (13)$$

where we have used $p(\psi_0) = 0$, $g(\psi_0) = 1$.

The current-carrying conductors in the vacuum necessary to maintain the elliptical plasma equilibrium are assumed to have sufficiently high impedance that they do not affect the stability calculations. We allow, however, for the presence of a wall surrounding the vacuum region, with conductivity σ such that magnetic fields can penetrate the wall on the slow equilibrium time scale τ_{eq} , but not on the fast perturbation time scale τ_p , i.e., we require $\tau_{eq} \gg \sigma L_{eq}^2 \gg \tau_p$, L_{eq} being some equilibrium scale length. Thus the conducting wall need not coincide with a magnetic surface in the vacuum.

We now describe the stability analysis. This is carried out by finding the stationary points of the Lagrangian, $L = T - \delta W$,

of our system for perturbations about the equilibrium. Here [14],

$$2T = \omega^2 \int_P d\tau \rho |\vec{\xi}|^2 \quad (14)$$

and

$$\delta W = \delta W_P + \delta W_V, \quad (15)$$

where

$$2\delta W_P = \int_P d\tau \left(|\vec{Q}|^2 - \vec{J} \cdot \vec{Q}^* \times \vec{\xi} + (\gamma_P \vec{\nabla} \cdot \vec{\xi} + \vec{\xi} \cdot \vec{\nabla}_P) \vec{\nabla} \cdot \vec{\xi}^* \right), \quad (16)$$

$$2\delta W_V = \int_V d\tau |\vec{\nabla} \chi|^2. \quad (17)$$

We have assumed that the Lagrangian displacement vector depends on time as $\vec{\xi}(\vec{r}, t) = \vec{\xi}(\vec{r}) \exp(-i\omega t)$; $\vec{Q} \equiv \vec{\nabla} \times (\vec{\xi} \times \vec{B})$ is the perturbed magnetic field in the plasma, and χ is the perturbed magnetic scalar potential in the vacuum. Admissible functions $\vec{\xi}$ are those for which the kinetic energy is finite, and which satisfy the boundary conditions that the normal component of $\vec{\nabla} \chi$ vanishes at the outer wall and $\vec{Q} \cdot \vec{\nabla} \psi = \vec{\nabla} \chi \cdot \vec{\nabla} \psi$ at the plasma-vacuum interface.

In order to utilize the energy principle as efficiently as possible, we let ourselves be guided by the physical situation. Thus, we decompose the displacement $\vec{\xi}$ as

$$\vec{\xi} = \frac{\vec{\nabla} \theta \times \vec{B}}{B_0 g} \frac{L}{4\pi^2} \xi_\psi + \frac{\vec{B} \times \vec{\nabla} \psi}{B_0 g L} \xi_s + i \frac{\vec{B}}{B_0} \tau; \quad (18)$$

this isolates the sound mode which is polarized mainly along the magnetic field. In addition, we express the components of $\vec{\xi}_\perp$ in terms of scalars ζ and δ with

$$\xi_\psi = \delta - \frac{4\pi^2 i}{L} \frac{\partial \zeta}{\partial \theta} , \quad (19)$$

$$\xi_s = iL \frac{\partial \zeta}{\partial \psi} . \quad (20)$$

Then, since

$$\vec{\nabla} \cdot \vec{\xi}_\perp = \frac{L}{2\pi} \left[\frac{\partial}{\partial \psi} \delta + \frac{\vec{\nabla}_\psi \cdot \vec{\nabla}_\theta}{2\pi L g} \frac{\partial}{\partial z} \left(\delta - \frac{4\pi^2 i}{L} \frac{\partial \zeta}{\partial \theta} \right) - \frac{2\pi i}{L^2 g} |\vec{\nabla}_\psi|^2 \frac{\partial^2}{\partial z \partial \psi} \zeta \right] \quad (21)$$

and $\partial/\partial z \sim L^{-1}$, this transformation tends to decouple the various modes in long thin systems in the sense that for the slow sound branch, where the motion is mainly along the field, $\zeta \sim \delta \sim 0$, for the intermediate Alfvén branch, where $\vec{\nabla} \cdot \vec{\xi}_\perp \sim 0$, $\tau \sim \delta \sim 0$, and for the fast magnetosonic branch $\tau \sim 0$.

The situation where $\tau \sim \delta \sim 0$ and $L \rightarrow \infty$ corresponds to the case studied by Dewar et al. [5], and Laval et al. [7].

To proceed with the extremization process we use Galerkin's method of expanding the perturbations in a finite subset of a complete set of expansion functions, i.e.,

$$\vec{\xi} = \sum \xi_{\ell m n} \vec{\phi}_m(\psi) \exp \left[i \left(\ell \theta - 2\pi n z / L \right) \right] . \quad (22)$$

This enables us to cast the variational problem into a matrix eigenvalue equation

$$\left(\omega^2 \langle \vec{T} \rangle - \langle \delta \vec{W} \rangle \right) \cdot \vec{\xi} = 0 \quad (23)$$

where $\langle \vec{T} \rangle$ and $\langle \delta \vec{W} \rangle$ are, respectively, the Heisenberg matrices formed from the kinetic and potential energies in the space of our expansion functions. For the radial functions $\vec{\phi}(\psi)$ we use finite elements. It is necessary that the elements chosen for ζ and δ have bounded derivatives since these quantities appear in the Lagrangian. This can be accomplished by choosing piecewise linear functions, normalized to unity at $\psi = \psi_i$ and vanishing at and outside of ψ_{i+1} . An alternate choice is based on the observation [15] that it is necessary to accurately represent $\vec{\nabla} \cdot \vec{\xi}_1$ when it is small, in order to reasonably calculate small eigenvalues ω^2 . From Eqs. (9), (11), and (21), evaluated for the circular case $a = b$, terms in δ can cancel those in ζ to arbitrary accuracy if the elements for ζ are chosen as above while the elements for δ are given by

$$\begin{aligned} \phi_i &= \int_{\psi_{i-1}}^{\psi} d\psi \left[\psi/g(\psi) \right] / \int_{\psi_{i-1}}^{\psi_i} d\psi \left[\psi/g(\psi) \right], \quad \psi_{i-1} < \psi < \psi_i \\ \phi_i &= \int_{\psi}^{\psi_{i+1}} d\psi \left[\psi/g(\psi) \right] / \int_{\psi_i}^{\psi_{i+1}} d\psi \left[\psi/g(\psi) \right], \quad \psi_i < \psi < \psi_{i+1} \end{aligned} \quad (24)$$

For τ we choose piecewise constant functions normalized to unity for $\psi_{i-1} < \psi < \psi_i$ and vanishing elsewhere. The grid points ψ_i can be chosen so that we can accurately represent modes which tend to be localized. Note that we have, at most, coupling only between nearest neighbors so that a tridiagonal system of equations in ψ is obtained. The merits of using finite elements rather than global expansion functions have been reported elsewhere [6].

The details of the calculation of the vacuum contribution to the Lagrangian are given in Appendix A.

It should be emphasized here that the calculation of the matrix elements, which is carried out analytically up to the point where the integration over ψ is performed, entails a vast amount of algebra so that extreme care had to be taken; in fact, we have made use of symbolic programming [16] to check the algebraic input to the computer code.

3. THE CONTINUOUS SPECTRUM

Systems with circular cross sections can be described by the ordinary differential equations in ψ [4]:

$$D\xi'_\psi = \hat{C}_1 \xi_\psi + \hat{C}_2 P, \quad (25)$$

$$DP' = \hat{C}_3 \xi_\psi - \hat{C}_1 P, \quad (26)$$

where

$$P \equiv -(\gamma p + B^2) \vec{\nabla} \cdot \vec{\xi}_1 - i \frac{\gamma p}{B_0} \vec{B} \cdot \vec{\nabla} \tau + \frac{LB_\theta^2}{2\pi\psi} \xi_\psi.$$

For the existence of continuous spectra, it is necessary that this set of equations be singular. The spectrum is given by solving the local dispersion relation at the singular point ψ_s , which is obtained by setting the coefficient of the highest derivative equal to zero, i.e.,

$$D(\omega^2, \psi_s) \equiv [\Omega^2 - (\ell - nq)^2] [\Omega^2(\gamma p + B^2) - \gamma p(\ell - nq)^2] = 0 \quad (27)$$

where

$$\Omega^2 \equiv \omega^2 \frac{\rho}{\pi B_\theta^2} \psi = \rho \left(\frac{L}{4\pi^2 f} \right)^2 \omega^2 \quad (28)$$

This gives the Alfvén and sound continua

$$\Omega_A^2 = (\ell - nq)^2 \Big|_{\psi_s} \quad (29)$$

$$\Omega_s^2 = \frac{\gamma p (\ell - nq)^2}{(\gamma p + B^2)} \Big|_{\psi_s} \quad (30)$$

Then any value of ω^2 for which Eq. (29) or (30) is satisfied inside the plasma region, lies in the continuum. The associated eigenfunction is singular at that surface ψ_s .

In the elliptical system, θ is not ignorable so that instead of Eqs. (25) and (26) we have a partial differential equation in ψ and θ . The coefficient of the highest ψ -derivative is then an operator that is differential in θ .

The vanishing of this operator in the space of periodic functions is an eigenvalue problem for the frequencies ω^2 that lie in the various continua. This solution can be found either by direct integration or, alternatively, by using Fourier analysis in θ to construct a matrix eigenvalue equation. We have carried out both procedures, but the latter approach parallels that of our δW analysis and is followed here. The set of equations is a matrix eigenvalue problem reduced from that described in Eq. (23):

$$\omega^2 \rho(\psi_s) \begin{bmatrix} T_{\zeta'\zeta'} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T_{\tau\tau} \end{bmatrix} \begin{bmatrix} \zeta' \\ \delta' \\ \tau \end{bmatrix} = \begin{bmatrix} \delta W_{\zeta'\zeta'} & \delta W_{\zeta'\delta'} & \delta W_{\zeta'\tau} \\ \delta W_{\delta'\zeta'} & \delta W_{\delta'\delta'} & \delta W_{\delta'\tau} \\ \delta W_{\tau\zeta'} & \delta W_{\tau\delta'} & \delta W_{\tau\tau} \end{bmatrix} \begin{bmatrix} \zeta' \\ \delta' \\ \tau \end{bmatrix}, \quad (31)$$

where the elements T and δW are matrices in ℓ associated with $d\zeta/d\psi$, $d\delta/d\psi$, and τ . The second equation in this set can be used to eliminate δ' so that

$$\omega^2 \rho(\psi_s) \hat{T} - \hat{\delta W} = 0, \quad (32)$$

which determines the eigenvalues ω^2 for each ψ_s . Here

$$\hat{\delta W} = \begin{pmatrix} \delta W_{\zeta'\zeta'} - \delta W_{\zeta'\delta} \delta W_{\delta'\delta}^{-1} \delta W_{\delta'\zeta'} & \delta W_{\zeta'\tau} - \delta W_{\zeta'\delta} \delta W_{\delta'\delta}^{-1} \delta W_{\delta'\tau} \\ \delta W_{\tau\zeta'} - \delta W_{\tau\delta} \delta W_{\delta'\delta}^{-1} \delta W_{\delta'\zeta'} & \delta W_{\tau\tau} - \delta W_{\tau\delta} \delta W_{\delta'\delta}^{-1} \delta W_{\delta'\tau} \end{pmatrix} \quad (33)$$

and

$$\hat{T} = \begin{pmatrix} T_{\zeta'\zeta'} & 0 \\ 0 & T_{\tau\tau} \end{pmatrix} \quad (34)$$

is nonsingular. If the elements of $\hat{T}$ and $\hat{\delta W}$ are independent of ψ , then all branches of $\omega^2(\psi_s)$ are constants. This singular frequency is then either an infinite degeneracy or an accumulation point of the full spectrum. In general, however, equilibrium quantities are nonconstant functions of ψ so that the continuum occupies a finite band. As an illustration, we show in Fig. 1 the results obtained numerically from Eq. (32) for a varying density profile, $\rho = (1.0 - \psi/3.0\psi_0)$, circular cross section, zero pressure and length $L/a=40\pi$. Note the characteristic parabolic-like curves as dictated by the $(\ell - nq)^2$ behavior of Eq. (29). In this case q is nearly constant, and the limits on the continua are determined by the highest and lowest values of the density profile. Since the pressure is zero the sound continuum has an infinite degeneracy on the axis. In Fig. 2, we introduce ellipticity with $b = 2a$ and we see the existence of coupling among the different bands; note also that this coupling generates gaps in the spectrum, a point which may be of some importance if

the continuous spectrum is utilized as a mechanism for plasma heating.

We have stated that the presence of singularities in the differential equation is necessary for the existence of a continuous spectrum. Such a spectrum exists if the equations possess "weak solutions" [17-19] which are then continuum eigenfunctions. These can be characterized in terms of an indicial expansion about the singular point of the differential equation [6,20]:

$$\xi_\psi = A(\ln|\psi - \psi_s| + \dots) + B(1 + \dots) , \quad (35)$$

where A and B are constants. Functions for which the values of A on either side of the singularity are equal satisfy the conditions for the existence of weak solutions, regardless of the difference in the values of B on the two sides. While these eigenfunctions do not have a finite kinetic energy norm, they can be approximated arbitrarily closely by a sequence of functions of finite norm, e.g., the finite elements used in the present computation.

In most cases the regular solution, $A = 0$, does not satisfy the boundary conditions at the magnetic axis and the plasma-vacuum interface. Thus, the singular solution must be included. Although this singular behavior is characteristic of the continuous spectrum, there can exist isolated modes where the singular solution is, in fact, not present. This is so since the

constant A in Eq. (35) depends on the equilibrium parameters so that with the right choice of these parameters A can vanish. It can be shown that as parameters are adjusted to make kinks, or other discrete modes, approach the edge of the continuum, the constant A of nearby continuum modes must approach zero. This demonstration requires the construction of approximate eigenfunctions near the plasma-vacuum interface for kink modes lying just outside the continuum, and has been carried through for the special case of circular cross-section plasma columns. The boundary condition at the plasma-vacuum interface has been given by Bernstein et al. [14], and, for Eqs. (25) and (26), can be expressed as $\xi_\psi(\psi_0)/P(\psi_0) = C_V$, where C_V depends on the configuration but is independent of the eigenvalue Ω^2 . Using Eq. (35), together with the corresponding expression for P, the boundary condition can be put in the form

$$\ln(\psi_s - \psi_0) = \frac{C_V \partial D / \partial \psi}{C_2 + C_1 C_V} - \frac{B}{A} . \quad (36)$$

Here ψ_s is the value ψ at which the analytic continuation of D into the vacuum vanishes for the eigenfrequency of the kink. From this we see that ψ_s can approach ψ_0 , and the kink merge with the continuum, only when either A or $C_V + \hat{C}_2/\hat{C}_1$ approaches zero. The latter condition can be shown by similar arguments to be equivalent to the condition that A, evaluated for the eigenfunction in the region $\psi_s \leq \psi \leq \psi_0$, approaches zero. Thus, in either case, discrete modes join to continuum

eigenfunctions with bounded ξ_ψ . It is thus reasonable to identify these as kink modes in the continuum, even though they have unbounded kinetic energy.

It follows from Eq. (36) that the eigenvalues of kink modes are not analytic functions of the plasma parameters for eigenvalues just outside the continuum since

$$\Omega_k^2 - \Omega_0^2 \approx - \frac{\partial D / \partial \psi}{\partial D / \partial \Omega^2} \exp(-B/A) \quad (37)$$

and A and B are expected to be analytic. On the other hand, eigenvalues of kink modes in the continuum, as identified above, are analytic functions of the parameters.

In order to illustrate this point, we use the full matrix eigenvalue equation to generate the spectrum. The continuum is then approximated by a band of discrete eigenvalues. These can be associated with the eigenvalues of the dispersion relation Eq. (32) at each grid surface.

In Fig. 3a, we plot the square of the frequency versus the wall position a_w/a for the circular, pressureless case with $\ell = 2$, $q(\psi_0) = 1.5$ and constant density. Note that as the wall is moved closer to the plasma surface the kink, depicted by the solid line, is stabilized, passes through the Alfvén continuum and then moves up towards the fast magnetosonic spectrum. It is interesting that when the wall is far from the plasma the lowest mode in this part of the spectrum has a node in ξ_ψ inside the plasma. When the wall is brought in, the stable kink has no such node and it thus completes the fast spectrum. In Fig. 3b, we introduce some density variation, $\rho = (1.0 - 0.75\psi/\psi_0)$, to show

some details of the interaction of the kink with the continuum. Each point in the figure represents actual data obtained from the numerical code and we identify the passage of the kink through the continuum by the locus of points in the spectrum whose eigenfunctions are characterized by $A = 0$; in other words, the sign of A is different on opposite sides of the solid line in Fig. 3b. Note the nonanalytic behavior of the kink near the top of the continuum predicted by Eq. (37). In Fig. 3c, we show the behavior of the eigenfunction ξ_ψ for the five spectral points neighboring the discrete kink for $a_w/a = 1.1475$. Note that as the frequency increases the strength of the logarithmic singularity decreases, vanishes at the discrete mode, changes sign, and then increases negatively. At the same time, the singularity moves outward in ψ in accordance with Eq. (29) evaluated for a density profile which decreases outwards.

Similar situations occur under other circumstances. For example, in the long-wavelength, pressureless limit with constant density and no conducting wall, the dispersion relation for the kink mode is given by

$$\Omega^2 = 2(\ell - nq)(\ell - \ell/|\ell| - nq); \quad (38)$$

on the other hand, the continuous Alfvén spectrum is given by

$$\Omega^2 = (\ell - nq)^2. \quad (39)$$

These two curves intersect at $nq = \ell$ and $\ell - 2$ with $\Omega^2 = 0$ and 4 respectively; this is depicted in Fig. 4 for $\ell = 3$ where again a diffuse density profile, $\rho = (1.0 - \psi/3.0\psi_0)$, is used to spread the continuum. Note that the density variation steepens the dispersion curve for the kink mode. In Fig. 4b, we show the region where the kink passes through the continuum on an expanded scale. Again, the break in $d\Omega^2/dq$ noted in connection with Eq. (37) is clear.

One obvious advantage gained by using the reduced matrix formulation to describe the continua is the smaller amount of computer storage and time required for the calculation; this is so because the calculation of the vacuum and integrations over ψ need not be performed, and, since we have only two components of the polarization, ζ, τ , the size of the resultant matrices is, at most, double the number of Fourier modes. This is particularly advantageous because it provides a convenient check on the convergence with the number of Fourier modes included in the representation. Also, because of the local property of the method, the limits on the continua can be more accurately defined, whereas the full matrix eigenvalue system can only define the limits to within an accuracy which is of the order of the grid spacing in ψ . This accuracy is, in fact, achieved in our calculations.

4. INTERCHANGE MODES

In the previous section we saw that the continua reach the origin ($\Omega^2 = 0$) from the positive side if $\ell - nq$ vanishes somewhere in the plasma. The continua never extend to the negative side. However, when the interchange criterion [11-13] is violated, discrete unstable modes accumulate to the lower bound of the continuum. This is the subject of this section.

We assume a parabolic pressure profile, $p(\psi) = p_0(1 - \psi/\psi_0)$, so that the plasma is unstable if [21]

$$\frac{p_0}{B_0^2} > \frac{\Delta^2(\psi_s/\psi_0)}{8q^2(\psi_0)q^2(\psi_s)} \frac{ab}{(a^2 + b^2)} \quad (38)$$

Here, ψ_s is the position of the rational surface within the plasma, $\Delta \equiv q^2(\psi) - q^2(0)$ characterizes the shear, and a and b are the minor and major radii of the plasma cross section.

First we examine a situation where $\Delta = 0$ so that the plasma is always unstable for any $p_0 > 0$. Then we introduce a modest amount of shear and describe in some detail the localization of the modes and our numerical technique for handling such localization efficiently.

It is characteristic of interchange modes that, with shear, the modes can become extremely localized at the rational surface in the vicinity of the marginal point. Such localization naturally can generate numerical problems. However, if there is no shear present the modes are not localized and can be studied

quite easily with our numerical code. Thus, we show in Fig. 5 for $\ell = 2$, the full spectrum of a shearless plasma column with a circular cross section of length $L/a = 10\pi$, versus different values of nq lying between 1.9 and 2.1. We see distinctly the three branches of the spectrum. The fast magnetosonic branch is virtually constant and is discrete whereas that of the slow sound branch is continuous. The latter always reaches the origin because the pressure vanishes at the plasma boundary. On the other hand, the Alfvén branch is discrete with an accumulation point given by the local dispersion relation, Eq. (29), which is constant across the plasma. Note that at $nq = 2$ both branches touch the origin. Here we have as many unstable modes as there are radial expansion functions (twenty in this calculation). The spectrum of these unstable modes is discrete and is Sturmian in behavior [22]. Away from the point where $nq = 2$, we can easily trace the behavior of these modes as they pass through the virtually undisturbed sound continuum and approach the solid curve in the figure, which represents the locus of the accumulation point, $\Omega^2 = (\ell - nq)^2$. The connection between the four most unstable of these modes is shown by the dashed lines; the others can similarly be connected. The mode seen just above the right-hand portion of this branch is the stable part of the $\ell = 2$ current-driven kink which is destabilized at $nq = 2$. It then couples with the interchanges, and exits at the lower left of the figure to become stabilized eventually in the vicinity of $nq = 1$.

It is difficult to obtain as many unstable modes as appear in Fig. 5 without taking special care. Since these growth rates

are very much smaller than the frequencies of the fast magnetosonic branch, the decoupling of these branches must be precise. This is possible only if the functions describing δ can be chosen to make $\vec{v} \cdot \vec{\xi}$ arbitrarily small for a given ζ . We accomplish this by using the finite elements of Eq. (24).

This result is in complete agreement with that of Appert et al. [15]; our curves differ in detail because we have taken a compressible case. In Appendix B, we examine the behavior of the eigenvalues near the accumulation point using WKB-type approximations and show that modes accumulate from below if $\gamma_p < B^2$, and from above when $\gamma_p > B^2$.

When shear is introduced, we find it necessary to modify our numerical code to represent efficiently the localization of the extremizing eigenfunctions. Thus, we localize our computational mesh in ψ such that the grid points are concentrated about the rational surface ψ_s . Quantitatively, the localized grid is generated by

$$\psi_i = \hat{\psi}_i \left[1 - \sigma \frac{(\hat{\psi}_i - \psi_s)(\hat{\psi}_i - \psi_o)}{\psi_s (\psi_s - \psi_o)} \right], \quad (41)$$

where we take equal intervals in the stretched coordinate

$\hat{\psi}_i = \psi_o(i/i_{\max})$; this has the property that

$$\frac{\Delta \hat{\psi}_i}{\hat{\Delta \psi}_i} = 1 - \sigma \quad \text{for } \hat{\psi}_i = \psi_s. \quad (42)$$

The effect of this transformation is shown graphically in Fig. 6 where we see that ψ_1 represents a cubic curve passing through the points where $\psi = 0$, ψ_s and ψ_o , such that the slope of the curve at ψ_s is given by $1 - \sigma$. That is, σ measures the modification of the grid such that there is no localization for $\sigma = 0$ and infinite localization for $\sigma = 1$. In the figure $\sigma = 0.5$. To see how effective this is, we illustrate with a specific example where we choose $q^2(\psi_o) = 1.2$, $\Delta = 0.4$, $a = b$, and $a_w = \infty$. Since $q^2(\psi)$ is linear in ψ , a rational surface, where $nq = 1$, exists in the plasma at $\psi = \psi_s = 0.5 \psi_o$. With these choices the plasma is unstable to interchanges if $p_o > p_c = 0.00625$. In Fig. 7 we plot, for $p_o = 0.032$, the growth rates (solid lines) of the interchanges as functions of σ , using thirty radial expansion functions. There we see dramatically the benefit obtained: the third mode emerges unstable only after σ is increased to about 0.7 where the ratio of the largest to the smallest mesh size, $(\delta\psi)_{\max}/(\delta\psi)_{\min}$, depicted by the dashed line, is about 7. The dots indicate the growth rates obtained from a highly accurate numerical scheme which integrates the full second-order differential equation. The maximum growth rate of this mode occurs when $\sigma \sim 0.9$ where $(\delta\psi)_{\max}/(\delta\psi)_{\min}$ is approximately 35. The reason for this can be seen in Fig. 8a where we plot the eigenfunctions ξ_ψ for these values of p_o and σ . There we observe the Sturmian behavior of the modes; the mode with the most oscillations has the lowest growth rate and is also the most localized. The grid spacing is depicted by the tick marks on the abscissa. From the figure one can appreciate the necessity for using a highly

localized grid to obtain the modes with smaller growth rates; on the other hand, we cannot localize too much since the eigenfunctions also have some structure outside the singular region. We must therefore compromise and this accounts for the relative maxima in the curves of Fig. 7. That the modes are more localized as the marginal pressure is approached is illustrated in Fig. 8b where we show the wavefunctions of the most unstable mode for $p_0 = 0.032$ and 0.008 . We place the wall at infinity.

To study the behavior of these modes near the marginal point and their relationship with the continua, we plot in Fig. 9 the normalized growth rates versus p_0 for values extending below and above the critical pressure required for instability. There we see the lower part of the stable Alfvén and sound continua (both of them touch zero since $\ell - nq = 0$ in the plasma), three interchange modes and, at the bottom of the figure, the free-boundary current-driven kink. We can make several observations about the interchanges. First, one sees that the code is able to represent at least the most unstable interchange very close to the marginal point; this is due to our ability to localize our grid and to make $\vec{v} \cdot \vec{\xi}_1 = 0$ by means of the elements of Eq. (24). Second, for fixed values of p_0 the modes tend to be widely separated and each mode tends to creep in very slowly towards the stable continuum near the marginal pressure. Third, although we use thirty radial expansion functions and a highly localized grid we find only three unstable modes; this, we suspect, is largely due to the fact that the extreme local oscillations and vanishingly small growth rates are prohibitive factors in obtaining more accurate results with our numerical procedure.

On the other hand, localization properties make possible analytic evaluation of the growth rates. This is done in Appendix C.

5. DISCRETE MODES

In this section, we use the code to study various aspects of the discrete parts of the MHD spectrum. In the previous sections, we encountered some discrete modes inasmuch as they served to help in illustrating the properties of the continua and the interchanges with their associated accumulation points. Here, we concentrate on the free boundary kinks, axisymmetric ($n = 0$) modes, and the fast spectrum.

Dewar et al. [5] have determined the growth rates for free boundary kinks in the long wavelength, zero-pressure limit. In the circular limit, the instability boundaries are $m - 1 < nq < m$. For a constant density profile the growth rate is given by Eq. (38). The stabilizing effect due to the presence of a conducting wall was discussed in section 3. Finite pressure with a diffuse radial profile (e.g., parabolic) can narrow the stability window around $nq = \ell$; this can be understood by consideration of the interchange criterion near the boundary of the plasma, Eq. (40). Finite ellipticity widens the instability boundary at $nq = \ell - 1$ [5,7]. We illustrate these effects in Fig. 10a where we plot the lower part of the spectrum versus $nq(\psi_0)$ for $b/a = 2$ with $p_0 = 0.1$ and $L/a = 20\pi/3$. In order to represent the modes accurately, we include in the calculation

even Fourier modes ranging between -2 and 6. In this case the shear, which is determined by equilibrium pressure balance, Eq. (13), is quite low so that the interchange criterion is violated about $nq = 2$ and $nq = 4$. For purposes of comparison, we also show in the figure the growth rates for the case of zero pressure and infinite wavelength and the corresponding cases for a circular cross section. The nature of the enhanced unstable regions at $nq(\psi_0) = 2$ and 4 is shown in more detail in Fig. 10b; there we show the three most unstable modes. The current-driven unstable $\ell = 2$ and $\ell = 4$ kinks couple with the interchange modes to effectively widen the instability boundary of the kinks.

We turn now to the axisymmetric ($n = 0$) mode. The growth rate in the infinite wavelength, pressureless limit is [5]

$$\Omega^2 = - \frac{b - a}{b}, \quad (b > a) \quad (43)$$

and the perturbation associated with the instability consists of equal amounts of $\ell = \pm 1$ Fourier components which correspond to a vertical rigid displacement of the plasma. The instability is due to the external currents required for the elliptical cross section of the plasma; these external currents flow along z and in the same direction as the axial plasma current. Because of the mutual attraction between the currents, any vertical displacement of the plasma results in an unstable situation. One might thus expect the growth rate to be a strong function of the plasma current. The results of our numerical investigation

indicate that this is the case. In Fig. 11, we plot the normalized growth rate versus the axial current J_z for $b/a = 2$ and three values of the pressure using odd Fourier modes between $\ell = -3$ and $\ell = 3$. Note that the increase of the growth rate with plasma current is even more rapid than is shown since the growth rate ω^2 is normalized to a quantity which is proportional to the square of the axial current. The figure also indicates that the growth rate is insensitive to the value of the pressure. Since the perturbation is axially symmetric, L and f always occur in the combination L/f in our model so that the axial current, given by Eq. (6), and the ellipticity are the critical parameters for this case.

The effect of ellipticity on the fast magnetosonic waves is interesting. Since the waves propagate isotropically, one expects that there is a high degree of degeneracy in their dispersion relation, in particular, between modes with $\ell = \pm m$. The numerical results indicate that this is in fact only true in the circular limit; this is shown in Fig. 12 where we plot the fast spectrum as a function of ellipticity with $L/a = 40\pi$ and $p_0 = 0.05$. There we see that this degeneracy is quickly resolved and, in fact, the different modes in the spectrum interact quite strongly in the presence of ellipticity.

6. DISCUSSION

In this paper we have discussed the complete spectrum of ideal magnetohydrodynamics for a useful special model. The ideal equations are often amenable to analysis if one makes relevant physical assumptions about time scales, incompressibility, localization, etc. In this way one can get information about isolated parts of the spectrum within limited ranges of the parameter space of the equilibrium. More often than not, however, the realistic situation occurs in some region which is outside the space that can be treated analytically. The numerical code used in deriving the results of this paper, though limited to the basic assumptions of a straight system with constant axial current and elliptical cross section, is nevertheless sufficiently flexible to bridge the gaps between analytically tractable regions. We determine not just stability boundaries, or the fastest growing modes, but the whole spectrum of modes. This is perhaps the most useful consequence of our approach to the problem, since it gives us a rather panoramic view of the properties of magnetohydrodynamics under different equilibrium conditions. For example, the spectrum associated with the diffuse linear pinch is usually treated in several separate analyses. One can calculate the fast spectrum analytically to within a few percent. One can also predict the existence of the sound continuum and the Alfvén accumulation points as well as the stability boundaries of the current-driven kink. Finally, one can also predict the existence of unstable modes using the interchange criterion and

determine the growth rates in various approximations. Figure 5, generated by our numerical code, gives the overall integrated picture and puts these observations into their proper perspective. Note also how the current-driven kink passes through the Alfvén continuum and completes the fast magnetosonic spectrum as seen in Fig. 3.

From the computational point of view the usefulness of our method can be appreciated if we imagine the spectrum being generated by other means such as solving the second-order differential equation given by Eqs. (25) and (26) for the circular case, or by time dependent methods. Such a task would be prohibitive not only because of the multiplicity of modes but also because some of the eigenfunctions are rather pathological and make direct integration very difficult. The method used here also does not require any predetermination of approximate eigenvalues or eigenfunctions upon which an iterative procedure is based.

From the practical standpoint one might wonder whether the stable parts of the spectrum are important since one is usually concerned only with instability limits and growth rates. However, supplementary heating of plasmas is of paramount importance in tokamak research and viable prospects via radio-frequency methods include heating by the Alfvén continuum and the fast magnetosonic waves. Furthermore, the ability to follow an unstable mode into the stable region as a parameter is changed is useful for determining stability boundaries accurately. Finally, approximation techniques which truncate the spectrum can introduce errors.

In the future we shall present results using a fully numerical toroidal code [10] with which we can study more realistic situations such as tokamak discharges with diffuse current profiles and more complicated cross sections. That code is modeled after the one upon which this paper is based and may be viewed as having several realistic extensions beyond the present equilibrium parameter space. That code will be more complicated due to the difficulties involved in determining the equilibrium. Our special model gives us great flexibility in our numerical procedure without an essential loss of physics. We expect that all or most of what we have learnt from the content of this paper will be carried over into the more general toroidal case. Thus we will find the same classes of modes with spectra varying over many orders of magnitude, and with eigenfunctions possessing great variation in localization and oscillatory properties. We will also see phenomena such as the continuous spectra with their related accumulation points, discrete eigenvalues passing through the continua, and the interconnectedness of these classes in the parameter space of the equilibrium.

A good discussion of the nature of the MHD spectrum in toroidal configurations was given by Goedbloed [23]. The work reported here provides a good illustration of the concepts enunciated there. The present model contains all the basic features of a toroidal system including geodesic curvature of the magnetic field lines. At the same time, the basic simplicity of this model makes numerical evaluation and interpretation possible.

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APPENDIX A. Treatment of the Vacuum Region.

The vacuum contribution to the Lagrangian is obtained by extremizing

$$2\delta W_V = \int |\vec{\nabla}\chi|^2 d\tau \quad (A1)$$

with respect to χ , keeping the normal component of the perturbed magnetic field $\vec{n} \cdot \vec{\nabla}\chi$ equal to $\vec{n} \cdot \vec{Q}$ on the plasma-vacuum interface and $\vec{n} \cdot \vec{\nabla}\chi = 0$ at the wall. Lüst and Martensen [24] have shown that this can be expressed in terms of surface integrals over the plasma-vacuum interface

$$2\delta W_V = - \frac{1}{2\pi} \int_0^L dz \int_0^{2\pi} \chi^* \vec{Q} \cdot \vec{\nabla}\psi d\theta \\ + \sum_{i=1,2} L_i^{-1} \left| \frac{1}{2\pi} \int_0^L dz \int_0^{2\pi} \vec{\xi} \cdot \vec{\nabla}\psi \vec{B} \cdot \vec{\mathcal{D}}_i d\theta \right|^2. \quad (A2)$$

Here χ^* is a singlevalued scalar potential that satisfies $\nabla^2 \chi^* = 0$, and the $\vec{\mathcal{D}}_i$'s are two orthogonal vectors such that $\vec{\nabla} \times \vec{\mathcal{D}}_i = \vec{\nabla} \cdot \vec{\mathcal{D}}_i = 0$ in the vacuum region, $\vec{\mathcal{D}}_i \cdot \vec{n} = 0$ at the interface and on the wall, and $\oint \vec{\mathcal{D}}_i \cdot d\vec{\ell}_k = \delta_{ik}$ where the contours ($k = 1, 2$) are closed curves going the long or short way around the plasma. The induction matrix elements are given by $L_i = 4\pi \oint \mathcal{D}_i^2 d\tau$. This last term in Eq. (A2) arises from the condition that the perturbed magnetic field cannot be given completely in terms of a single-valued scalar potential and is

associated with flux conservation.

If the conducting shell is placed on an elliptical surface that is confocal with the plasma-vacuum interface, then

$$\vec{r}_1 = \vec{e}_z/L, \quad \vec{r}_2 = \vec{r}_0/2\pi, \quad (A3)$$

so that

$$L_1 = \frac{2\pi^2 a^2}{L \sinh^2 \mu_0} \left(\sinh 2\mu_w - \sinh 2\mu_0 \right), \quad L_2 = 2L(\mu_w - \mu_0). \quad (A4)$$

Here μ, θ, z form an orthogonal confocal elliptic coordinate system such that $\mu = \mu_0$ on the plasma-vacuum interface and $\mu = \mu_w$ on the wall. Then

$$\begin{aligned} 2\delta W_V = & - \frac{1}{2\pi} \int_0^L dz \int_0^{2\pi} \chi^* \vec{Q} \cdot \vec{\nabla} \psi d\theta \\ & + \frac{B_0^2 L^3}{8\pi^4 a^2} \left[\sinh^2 \mu_0 (\sinh 2\mu_w - \sinh 2\mu_0)^{-1} \right. \\ & \left. + \frac{\pi^2 a^2}{L^2 q^2} (\mu_w - \mu_0)^{-1} \right] \left| \xi_\psi(\ell=0, n=0) \right|^2. \end{aligned} \quad (A5)$$

It follows from Eq. (A5) that only axisymmetric modes with $\ell = 0, n = 0$ are affected by these flux constraints. Thus they can modify only the even modes. It has been shown that computations of stability criteria [5,7] are not affected by this constraint since the even axisymmetric modes are all stable. On the other

hand, if the walls are close to the plasma so that $\mu_w - \mu_0$ is small, the stable spectrum in noncircular configurations could be modified significantly.

It remains to evaluate $\chi^*(\vec{r})$ on the plasma-vacuum interface. We do this using a general formulation involving Green's theorem: [25]

$$\chi^*(\vec{r}_s) = \frac{1}{2\pi} \oint d\ell_t \left[\chi^*(\vec{r}_t) \vec{n} \cdot \vec{\nabla}_t G(\vec{r}_s, \vec{r}_t) - G(\vec{r}_s, \vec{r}_t) \vec{n} \cdot \vec{\nabla}_t \chi^*(\vec{r}_t) \right], \quad (\text{A6})$$

where we take the principal part of the integral and evaluate the contour on a line of constant z over both the interface and the wall. Here $G(\vec{r}_s, \vec{r}_t)$ satisfies

$$\nabla_t^2 G(\vec{r}_s, \vec{r}_t) \exp(-2\pi i n z/L) = -4\pi \delta(\vec{r}_s - \vec{r}_t) \exp(-2\pi i n z/L), \quad (\text{A7})$$

$\vec{n} \cdot \vec{\nabla}_t$ denotes the normal derivative pointing outward from the vacuum, and $d\ell_t$ is a line element. A convenient solution of Eq. (A7), which vanishes at infinity and thus necessitates integration of Eq. (A6) over both the plasma-vacuum interface and the wall, is

$$G(\vec{r}_s, \vec{r}_t) = 2K_0(2\pi n |\vec{r}_s - \vec{r}_t|/L) \quad (\text{A8})$$

with K_0 the modified Bessel function of the second kind.

Using Eq. (A6) and Fourier decomposing in θ at the interface,
 $\chi^*(\mu_o, \theta) = \sum_{\ell} \eta_{\ell}^* \exp i \ell \theta$, and at the wall, $\chi^*(\mu_w, \theta) = \sum_{\ell} v_{\ell}^* \exp i \ell \theta$,

we obtain a set of coupled matrix equations

$$(I - A) \eta^* = -B \xi^* + C v^* , \quad (A9)$$

$$(I - D) v^* = E \eta^* - F \xi^* , \quad (A10)$$

with

$$A_{\ell \ell'} \equiv \frac{1}{4\pi^2} \oint d\theta_p \oint d\theta'_p J_p \vec{\nabla} \psi_p \cdot \vec{\nabla} G(\vec{r}_p, \vec{r}'_p) \exp i(\ell \theta_p - \ell' \theta'_p) ,$$

$$B_{\ell \ell'} \equiv - \frac{i f D_o(\ell' - n q)}{4\pi^2} \oint d\theta_p \oint d\theta'_p G(\vec{r}_p, \vec{r}'_p) \exp i(\ell \theta_p - \ell' \theta'_p) ,$$

$$C_{\ell \ell'} \equiv \frac{1}{4\pi^2} \oint d\theta'_w \oint d\theta_p J_w \vec{\nabla} \psi_w \cdot \vec{\nabla} G(\vec{r}_p, \vec{r}'_w) \exp i(\ell \theta_p - \ell' \theta'_w) ,$$

$$D_{\ell \ell'} \equiv \frac{1}{4\pi^2} \oint d\theta_w \oint d\theta'_w J_w \vec{\nabla} \psi_w \cdot \vec{\nabla} G(\vec{r}_w, \vec{r}'_w) \exp i(\ell \theta_w - \ell' \theta'_w) ,$$

$$E_{\ell \ell'} \equiv \frac{1}{4\pi^2} \oint d\theta'_p \oint d\theta_w J_p \vec{\nabla} \psi_p \cdot \vec{\nabla} G(\vec{r}_w, \vec{r}'_p) \exp i(\ell \theta_w - \ell' \theta'_p) ,$$

$$F_{\ell\ell'} \equiv - \frac{ifB_0(\ell' - nq)}{4\pi^2} \int d\theta'_p \int d\theta_w G(\vec{r}_w, \vec{r}'_p) \exp(i(\ell\theta_w - \ell'\theta'_p)) ,$$

and I is the unit dyadic. Here $\vec{r}_p$ and $\vec{r}_w$ denote, respectively, position vectors on the plasma surface and the wall in the plane perpendicular to the ignorable coordinate z .

Eliminating v^* from Eqs. (A9) and (A10), we get

$$\eta^* = - [(I - A) - C(I - D)^{-1}E]^{-1} [B + C(I - D)^{-1}F]\xi \quad (A11)$$

so that Eq. (A2) can be given in terms of ξ_ψ on the interface. For the confocal wall case the matrix elements can be evaluated directly.

APPENDIX B. WKB Analysis for Shearless Systems.

It was noted in the discussion associated with Fig. 5 that our results differ from those of Appert et al. [15] because we treat a compressible plasma where they treat an incompressible one. It is useful to note that considerable understanding of the difference can be obtained from a WKB analysis. Such an analysis is appropriate for studying behavior near accumulation points where the eigenfunctions are oscillating over a significant portion of the plasma. That is true of the shearless configurations considered here. Within this class there are many different WKB problems, depending on compressibility, closeness of q to an integer, etc. We consider two illustrative examples.

We restrict consideration to systems with a circular cross section. Then, from Eqs. (25) and (26),

$$\frac{d}{dr} \frac{D}{rC_2} \frac{dy}{dr} - \left[\frac{d}{dr} \left(\frac{C_1}{rC_2} \right) + \frac{(C_1^2 - C_2C_3)}{rC_2D} \right] y = 0 \quad (B1)$$

with

$$D \equiv (\rho\omega^2 - F^2) \left[\rho\omega^2 (B^2 + \gamma P) - \gamma P F^2 \right] ,$$

$$C_1 \equiv -2\ell F B_\theta \left[\rho\omega^2 (B^2 + \gamma P) - \gamma P F^2 \right] / r^2 + 2\rho^2 \omega^4 B_\theta^2 / r ,$$

$$C_2 \equiv -(\ell^2 + n^2 k^2 r^2) \left[\rho\omega^2 (B^2 + \gamma P) - \gamma P F^2 \right] / r^2 + \rho^2 \omega^4 ,$$

$$(C_1^2 - C_2 C_3)/D \equiv 4n^2 k^2 B_\theta^2 (\rho \omega^2 B^2 - \gamma p F^2)/r^2$$

$$- C_2 [\rho \omega^2 - F^2 + 2B_\theta d(B_\theta/r)/dr],$$

$$F \equiv \ell B_\theta/r - nk B_z,$$

$$y \equiv r \xi_r.$$

Setting $y = \hat{y} \sin[\int \eta(r) dr - (2\ell - 1)\pi/4]$, we see that

$$\eta^2 = - \left[\frac{r C_2}{D} \frac{d}{dr} \left(\frac{C_1}{r C_2} \right) + \frac{(C_1^2 - C_2 C_3)}{D^2} \right]. \quad (B2)$$

The phase factor $(2\ell - 1)\pi/4$ comes from matching to Bessel function behavior for small r . This leads to the dispersion relation

$$\int_0^a \eta dr = j\pi + (2\ell - s)\pi/4, \quad j = 1, 2, 3, \dots \quad (B3)$$

where $s = 1$ for the boundary condition $y(a) = 0$ and $s = 3$ for $y'(a) = 0$. The former condition is required in most cases since $y'/y \sim O(\eta) \ll D^{-1}$ inside the plasma. We consider the configuration $B_\theta = \hat{B}_\theta r/a$, $B_z = \hat{B}_\theta q/ka$, $p = \hat{B}_\theta^2 (1 - r^2/a^2)$, $\Omega^2 \equiv \rho \omega^2 a^2 / \hat{B}_\theta^2 =$ constant, $ka = 0.2$, and $\ell = 2$; the spectrum for this system is shown in Fig. 5 for various values of q .

We first set $n = 1$ and $q = 2$, so that $F = 0$, and assume that $\Omega^2 \ll 1$. Then

$$\eta^2 = -4k^2 B^2 / \Omega^2 (B^2 + \gamma p). \quad (B4)$$

Here we need the boundary condition $y'(a) = 0$ since the vacuum energy vanishes identically. Integrating Eq. (B3), we find

$$\Omega^2 = - \left(\frac{2ka}{(j+1/4)\pi(\gamma-1)} \right)^2 \left\{ \left[\gamma(1+(q/ka)^2) \right]^{1/2} \times E \left[\sin^{-1} \left(\frac{\gamma}{\gamma+(q/ka)^2} \right)^{1/2}, \left(\frac{\gamma+(q/ka)^2}{\gamma[1+(q/ka)^2]} \right)^{1/2} \right]^{-1} \right\}^2 \quad (B5)$$

For long thin systems, $(q/ka)^2 \gg 1 \sim \gamma$, this reduces to

$$\Omega^2 = - \left[2ka/(j+1/4)\pi \right]^2 \quad (B6)$$

In the nearly incompressible case, $\gamma \gg (q/ka)^2 \gg 1$, it becomes

$$\Omega^2 = -q^2 / \left[\gamma(j+1/4)^2 \right] + 0 \quad (B7)$$

Equation (B6) is in good agreement with the numerical results for $F = 0$ as shown in Table 1. Note that the computer program is accurate for small j and the WKB calculation is good for large j . Equation (B7) shows that the spectrum coalesces into the point $\Omega^2 = 0$ for an incompressible fluid, in agreement with the results of Appert et al. [15].

A second useful case is where $|\Omega^2 - (\ell-nq)^2| \ll (\ell-nq)^2$.

Here

$$\eta^2 = - \frac{4nk^2 [\ell B_z^2 + nq(B_\theta^2 - \gamma p)]}{[\Omega^2 - (\ell-nq)^2] qB^2} \quad (B8)$$

Again, taking a long thin system, $(q/ka)^2 \gg 1 \sim \gamma$, we find

$$\Omega^2 = (\ell - nq)^2 - \frac{\ell}{q} \left(\frac{2ka}{(j + 3/4)} \right)^2 ; \quad (B9)$$

the accumulation point is approached from below. Comparison with the numerical results is shown in Table 2. If γ is sufficiently large the numerator in Eq. (B8) changes sign somewhere inside the plasma and the accumulation point is approached from either side.

APPENDIX C. The Unstable Spectrum Near the Suydam Accumulation Point.

In our discussion of interchange modes, we noted that shear localizes the modes, making numerical treatment difficult. On the other hand, this localization makes analytic effort possible through the techniques of boundary-layer theory following Kulsrud [26]. Here we describe an analytic calculation of the unstable spectrum associated with the Suydam accumulation point at $\omega^2 = 0$ for a diffuse linear pinch. The results are useful for comparison with the computational work as a check on its accuracy, and as a complement to it.

We start with Eq. (B1). Away from the singular surface where $\ell = nq(r_s)$ the inertial terms are small for the small growth rates and localized modes of interest. By looking at the indicial equation for solutions near the surface, $r = r_s$, we see that

$$y_{\pm} \sim |G_{\pm}| \exp i[(u/2) \ln|r - r_s| + \gamma_{\pm}] + c. c. \quad (C1)$$

as one approaches this layer from either side. Here

$$u \equiv [-(1 + 4A)]^{1/2}, \quad (C2)$$

$$A \equiv 2q^2(r_s)p'(r_s)/n^2aB_z^2(r_s)q'^2(r_s) < -\frac{1}{4} \quad (C3)$$

for the system to be Suydam unstable. The constants $G_{\pm}$ and $\gamma_{\pm}$ must be evaluated by integrating Eq. (B1) inward from the wall

and G_- and γ_- by integrating outward from the origin.

We introduce a stretched coordinate for the region of rapid variation, such that

$$t \equiv [nq'/\Omega] (r - r_s) , \quad (C4)$$

to make the inertial terms the same order as the others, and approximate the equilibrium quantities by the leading terms in their Taylor expansions. Then, inside the layer, Eq. (B1) reduces to

$$\frac{d}{dt} (1 + t^2) \frac{dy}{dt} - [A + B/(C + t^2)] y = 0 \quad (C5)$$

with A given by Eq. (C4),

$$B \equiv 8\pi^2/n^2 L^2 q'^2 , \quad (C6)$$

and

$$C \equiv 1 + B^2/\gamma_p . \quad (C7)$$

In the limit $t \rightarrow \infty$ the solution of Eq. (C5) varies as

$$y_{\pm} \sim |E_{\pm}| \exp i[(u/2) \ln t + \epsilon_{\pm}] + c.c. . \quad (C8)$$

Since this behavior must match that described by Eq. (C1), we see that

$$\Omega = nq' \exp \frac{2}{u} (\varepsilon_{\pm} - \gamma_{\pm} - j\pi) . \quad (C9)$$

Here j is an arbitrary integer. From Eq. (C4) we see that the exponent measures the degree of localization. Thus, j must be sufficiently large that the exponent is negative for the procedure to be valid. Furthermore, the value of Ω must be identical when evaluated for $r \lesssim r_s$, so

$$\varepsilon_+ - \varepsilon_- = \gamma_+ - \gamma_- . \quad (C10)$$

The left side can be adjusted by choosing the proper solution of Eq. (C5). We are led to

$$\begin{aligned} \varepsilon_+ = \tan^{-1} & \left\{ \left[\sin \varepsilon_e \tan(\gamma_+ - \gamma_-) - \sin \varepsilon_o \sin(\varepsilon_o - \varepsilon_e) \right. \right. \\ & \left. \pm \sin \varepsilon_o (\sin^2(\varepsilon_o - \varepsilon_e) + \tan^2(\gamma_+ - \gamma_-))^{1/2} \right] \\ & / \left[\cos \varepsilon_e \tan(\gamma_+ - \gamma_-) - \cos \varepsilon_o \sin(\varepsilon_o - \varepsilon_e) \right. \\ & \left. \left. \pm \cos \varepsilon_o (\sin^2(\varepsilon_o - \varepsilon_e) + \tan^2(\gamma_+ - \gamma_-))^{1/2} \right] \right\} . \quad (C11) \end{aligned}$$

Here ε_o and ε_e are the phase constants evaluated from the odd and even solutions of Eq. (C5). Equation (C11) is obtained by writing $y_{\pm} = y_{e\pm} + Zy_{o\pm}$ and solving the matching equations for the constant Z . This reduces to a quadratic equation for Z so that the $\pm$ sign in Eq. (C11) denotes two branches of solutions. It is straightforward to determine ε_o and ε_e by solving Eq. (C5) numerically.

Although Eq. (C6) does not have an exact analytic solution in closed form it is useful to note that in typical tokamak configurations the parameter C is large so that a further asymptotic expansion can be employed. We set

$$v \equiv [-(1 + 4A + 4B/C)]^{1/2} \quad (C12)$$

and assume that B/C is sufficiently small that v is real. For $t^2 \sim 1 \ll C$, Eq. (C5) can be transformed into a hypergeometric equation with the solution

$$y = |H| F\left(\frac{1}{2}(1 - iv), \frac{1}{2}(1 + iv); 1; \frac{1}{2}(1 + it)\right) \exp i\eta + \text{c.c.} \quad (C13)$$

For large t (inside this inner ordering)

$$y \sim |H| \frac{|\Gamma(iv)| 2^{1/2}}{|\Gamma^2(\frac{1}{2}(1 + iv))|} \exp i \left[A(iv) - 2A\left(\frac{1}{2}(1 + iv)\right) + \frac{v}{2} \ln \frac{|t|}{2} \pm \tan^{-1} \left(\tan(\eta + \pi/4) \tanh \frac{\pi v}{2} \right) \right] + \text{c.c.} \quad (C14)$$

where $\Gamma(z) \equiv |\Gamma(z)| \exp iA(z)$. It is clear from Eq. (C14) that $\eta = 0$ for the even solution and $\eta = \pi/2$ for the odd one. Further, for $t^2 \sim C \gg 1$, Eq. (C5) can be written as another hypergeometric equation so that

$$y = \hat{E}(t^2/C)^{-1/4(1-iv)} F\left(\frac{i}{4}(v+u), \frac{i}{4}(v-u); 1 + \frac{iv}{2}; -t^2/C\right) + \text{c.c.} \quad (\text{C15})$$

The hypergeometric function goes to one as $t \rightarrow 0$ so that the argument of $\hat{E}$ can be determined by matching to Eq. (C14).

For large t

$$y \sim \hat{E} \left[\frac{\Gamma(1+\frac{iv}{2}) \Gamma(-\frac{iu}{2}) (t^2/C)^{-1/4(1+iu)}}{\Gamma(\frac{i}{4}(v-u)) \Gamma(1+\frac{i}{4}(v-u))} + \frac{\Gamma(1+\frac{iv}{2}) \Gamma(\frac{iu}{2}) (t^2/C)^{-1/4(1-iu)}}{\Gamma(\frac{i}{4}(v+u)) \Gamma(1+\frac{i}{4}(v+u))} \right] + \text{c.c.} \quad (\text{C16})$$

so that

$$\epsilon_{e,o} = -\frac{u}{4} \ln C + A\left(\frac{iu}{2}\right) - A\left(\frac{i}{4}(u+v)\right) - A\left(\frac{i}{4}(u-v)\right) + \tan^{-1} \left[\frac{\tan \frac{\pi u}{4}}{\tanh \frac{\pi v}{4}} \tan\left(\Xi \pm \tan^{-1} \tanh \frac{\pi v}{4}\right) \right] \quad (\text{C17})$$

with

$$\Xi \equiv \frac{v}{4} \ln \frac{C}{4} + A(iv) + A\left(\frac{iv}{2}\right) - 2A\left(\frac{1}{2}(1+iv)\right) + A\left(\frac{i}{4}(u-v)\right) - A\left(\frac{i}{4}(u+v)\right). \quad (\text{C18})$$

In Eq. (C17), ϵ_e is obtained with the plus sign in the last term, ϵ_o with the minus sign.

A different solution is obtained if B/C is large enough that

$$\hat{v} \equiv [(1 + 4A + 4B/C)]^{1/2} \quad (C19)$$

is real ($\hat{v} = iv$). Then for $t^2 \sim 1 \ll C$

$$y = H F\left(\frac{1}{2}(1-\hat{v}), \frac{1}{2}(1+\hat{v}); 1; \frac{1}{2}(1+it)\right) + c.c., \quad (C20)$$

and for $t^2 \sim C \gg 1$

$$y = \hat{E}(t^2/C)^{-1/4(1-\hat{v})} F\left(\frac{1}{4}(\hat{v}+iu), \frac{1}{4}(\hat{v}-iu); 1 + \frac{\hat{v}}{2}; -t^2/C\right) \\ + \tilde{E}(t^2/C)^{-1/4(1+\hat{v})} F\left(-\frac{1}{4}(\hat{v}+iu), -\frac{1}{4}(\hat{v}-iu); 1 - \frac{\hat{v}}{2}; -t^2/C\right). \quad (C21)$$

Matching solutions in the overlap region, we obtain

$$\epsilon_{e,o} = -\frac{u}{4} \ln C + A\left(\frac{iu}{2}\right) + \epsilon_{e,o} \quad (C22)$$

where

$\Xi_{e,0} \equiv$

$$\tan^{-1} \left[\left(\frac{2^{\hat{v}} C^{\hat{v}/2} \Gamma(\hat{v}) \Gamma(1 + \frac{\hat{v}}{2}) \sin\{A[\frac{1}{4}(\hat{v}+iu)] + A(1 + \frac{\hat{v}}{4} + \frac{iu}{4})\}}{\Gamma^2[\frac{1}{2}(1+\hat{v})] |\Gamma[\frac{1}{4}(\hat{v}+iu)]| |\Gamma(1 + \frac{\hat{v}}{4} + \frac{iu}{4})|} \cos(\eta + \frac{\pi}{4}(1-\hat{v})) \right. \right. \\ \left. \left. - \frac{\Gamma(-\hat{v}) \Gamma(1 - \frac{\hat{v}}{2}) \sin\{A[-\frac{1}{4}(\hat{v}+iu)] + A(1 - \frac{\hat{v}}{4} - \frac{iu}{4})\}}{\Gamma^2[\frac{1}{2}(1-\hat{v})] |\Gamma[-\frac{1}{4}(\hat{v}+iu)]| |\Gamma(1 - \frac{\hat{v}}{4} + \frac{iu}{4})|} \cos[\eta - \frac{\pi}{4}(1+\hat{v})] \right) \right. \\ \left. / \left(\frac{2^{\hat{v}} C^{\hat{v}/2} \Gamma(\hat{v}) \Gamma(1 + \frac{\hat{v}}{4}) \cos\{A[\frac{1}{4}(\hat{v}+iu)] + A(1 + \frac{\hat{v}}{4} + \frac{iu}{4})\}}{\Gamma^2[\frac{1}{2}(1+\hat{v})] |\Gamma[\frac{1}{4}(\hat{v}+iu)]| |\Gamma(1 + \frac{\hat{v}}{4} + \frac{iu}{4})|} \cos[\eta + \frac{\pi}{4}(1-\hat{v})] \right. \right. \\ \left. \left. + \frac{\Gamma(-\hat{v}) \Gamma(1 - \frac{\hat{v}}{2}) \cos\{A[-\frac{1}{4}(\hat{v}+iu)] + A(1 - \frac{\hat{v}}{4} - \frac{iu}{4})\}}{\Gamma^2[\frac{1}{2}(1-\hat{v})] |\Gamma[-\frac{1}{4}(\hat{v}+iu)]| |\Gamma(1 - \frac{\hat{v}}{4} + \frac{iu}{4})|} \cos[\eta - \frac{\pi}{4}(1+\hat{v})] \right) \right] \quad (C23)$$

with $\eta = 0$ for ϵ_e and $\eta = \pi/2$ for c_0 .

The difference between the cases where v or $\hat{v} = iv$ is real has physical significance. In the first case, where Eq. (C17) applies, the destabilizing terms driving an interchange instability are strong enough that inertial effects associated with flow along the field lines to equalize the perturbed pressure over the surface are unimportant. If these curvature forces are weak or the lines are nearly closed so

that the flow is important [19,27,28], then the growth rates obtained from Eq. (C22) may be much smaller. Although the term associated with $C \neq 0$ can strongly affect the growth rate, it does not modify the critical pressure for stability.

As an indication of the accuracy of the numerical program, the growth rates of several modes as given by it, and by Eqs. (C9) and (C11), using direct numerical integration and also Eq. (C17), are compared in Table 3.

REFERENCES

- [1] HAIN, K. and LUST, R., Z. Naturforsch 13a, (1958) 936.
- [2] NEWCOMB, W.A., Ann. Phys. (N.Y.) 10, (1960) 232.
- [3] GOEDBLOED, J.P. and HAGEBEUK, H.J.L., Phys. Fluids 15, (1972) 1090.
- [4] APPERT, K., GRUBER, R., and VACLAVIK, J., Phys. Fluids 17, (1974) 1471.
- [5] DEWAR, R.L., GRIMM, R.C., JOHNSON, J.L., FRIEMAN, E.A., GREENE, J.M., and RUTHERFORD, P.H., Phys. Fluids 17, (1974) 930.
- [6] DEWAR, R.L., GREENE, J.M., GRIMM, R.C., and JOHNSON, J.L., J. Comp. Phys. 18, (1975) 132.
- [7] LAVAL, G., PELLAT, R., and SOULE, J.L., Phys. Fluids 17, (1974) 835.
- [8] GAJEWSKI, R., Phys. Fluids 15, (1972) 70.
- [9] CHANCE, M.S., DEWAR, R.L., GLASSER, A.H., GREENE, J.M., GRIMM, R.C., JARDIN, S.C., JOHNSON, J.L., ROSEN, B., SHEFFIELD, G.V., and WEIMER, K.E. In "Plasma Physics and Controlled Nuclear Fusion Research (1974)." (International Atomic Energy Agency, Vienna, 1975), Vol. 1, p. 463.
- [10] GREENE, J.M., GRIMM, R.C., and JOHNSON, J.L. In "Methods in Computational Physics" (ADLER, B., FERNBACH, S., KILLEEN, J., and ROTENBERG, M., eds. Academic Press, New York, 1976), Vol. 16, p. 253.
- [11] SUYDAM, B.R., in "Proc. 2nd Int. Conf. on the Peaceful Uses of Atomic Energy" (United Nations, Geneva, 1958), Vol. 31, p. 157.

- [12] MERCIER, C., Nucl. Fusion Suppl., Pt. 2, (1962) 801.
- [13] GREENE, J.M., and JOHNSON, J.L., Phys. Fluids 5, (1962) 510.
- [14] BERNSTEIN, I.B., FRIEMAN, E.A., KRUSKAL, M.D. and KULSRUD, R.M., Proc. Roy. Soc. (London) A244 (1958) 17.
- [15] APPERT, K., BERGER, D., GRUBER, R., and RAPPAPORT, J., J. Comp. Phys., 18, (1975) 284.
- [16] ROSEN, B., GREENE, J.M., GRIMM, R.C., and JOHNSON, J.L., Bull. Am. Phys. Soc. II 19, (1974) 940.
- [17] COURANT, R. and HILBERT, D., "Methods of Mathematical Physics," (Interscience, New York, 1962), Vol. II, p. 486.
- [18] FRIEDMAN, B., "Principles and Techniques of Applied Mathematics," (John Wiley & Sons, New York, 1956) p. 125.
- [19] GRAD, H., Proc. Natl. Acad. Sci. 70, (1973) 3277.
- [20] PAO, Y.P., Nucl. Fusion 15, (1975) 631.
- [21] GLASSER, A.H., GREENE, J.M., and JOHNSON, J.L., Phys. Fluids 18, (1975) 875.
- [22] GOEDBLOED, J.P., and SAKANAKA, P.H., Phys. Fluids 17, (1974) 908.
- [23] GOEDBLOED, J.P., Phys. Fluids 18, (1975) 1258.
- [24] LUST, R. and MARTENSEN, E., Z. Naturforsch 15a, (1960) 706.
- [25] MARDER, B.M., Phys. Fluids 17, (1974) 634.
- [26] KULSRUD, R.M., Phys. Fluids 6, (1963) 904.
- [27] WARE, A.A., Phys. Rev. Lett. 12 (1964) 439.
- [28] GREENE, J.M., and JOHNSON, J.L., in "Advances in Theoretical Physics", (K.A. Brueckner, ed., Academic Press, New York, 1965) Vol. 1, p. 195.

TABLE I. COMPARISON OF EIGENVALUES
OBTAINED FROM A WKB CALCULATION WITH
THE NUMERICAL PROGRAM AND A DIRECT
INTEGRATION OF EQ. (B1) FOR THE
SYSTEM OF FIG. 5 WITH $q = 2$.

<u>Fig.5</u>	<u>Eq. (B1)</u>	<u>Eq. (B6)</u>
-0.017006	-0.017014	-0.010375
-0.003484	-0.003519	-0.003202
-0.001533	-0.001592	-0.001535
-0.000835	-0.000912	-0.000898
-0.000504	-0.000592	-0.000588
-0.000324	-0.000416	-0.000415
-0.000217	-0.000308	-0.000308
-0.000150	-0.000237	-0.000238

TABLE II. COMPARISON OF EIGENVALUES
OBTAINED FROM A WKB CALCULATION WITH
THE NUMERICAL PROGRAM AND A DIRECT
INTEGRATION OF EQ. (B1) FOR THE
SYSTEM OF FIG. 5 WITH $q = 2.1$.

<u>FIG.5</u>	<u>EQ. (B1)</u>	<u>EQ (B9)</u>
0.004149	0.004126	0.004959
0.007897	0.007850	0.007959
0.008945	0.008877	0.008902
0.009390	0.009309	0.009316
0.009619	0.009531	0.009533
0.009750	0.009661	0.009661
0.009831	0.009744	0.009743
0.009882	0.009799	0.009798

TABLE III. COMPARISON OF EIGENVALUES ASSOCIATED WITH THE SUYDAM UNSTABLE SPECTRUM ASSOCIATED WITH FIG. 7, OBTAINED FROM THE NUMERICAL PROGRAM, A CALCULATION USING DIRECT INTEGRATION OF EQ. (B1), AND THE ASYMPTOTIC RESULTS OF EQS. (C9) AND (C11). THE RESULTS OBTAINED BY NUMERICAL INTEGRATION OF EQ. (C5) ARE THE SAME AS THOSE GIVEN BY EQ. (C22) FOR THIS CASE.

FIG. 7		EQ. (B1)		EQS. (C9) (C11)	
2.28	10^{-1}	2.28	10^{-1}	3.62	10^{-2}
1.78	10^{-3}	1.78	10^{-3}	1.78	10^{-3}
4.92	10^{-5}	6.30	10^{-5}	6.56	10^{-5}
1.55	10^{-6}	3.20	10^{-6}	3.22	10^{-6}
		1.19	10^{-7}	1.19	10^{-7}

762064

Fig. 1. The continuous spectrum for a circular column of length $L = 40\pi$ with zero pressure and constant J_z for even Fourier modes from $\ell = -2$ to 8. The density profile is given by $\rho = 1.0 - (\psi/3.0\psi_0)$ so that the upper and lower limits on the continuum correspond to the edge and center of the plasma column.

762063

Fig. 2. The continuous spectrum for a column with the same parameters as in Fig. 1, but with an elliptical cross section, $b/a = 2.0$.

762065

Fig. 3. The complete spectrum of a circular pressureless plasma with constant density as a function of wall position a_w/a (3a). Here $q = 1.5$ and $\ell = 2$. The current-driven kink is stabilized, passes through the continuum and then moves up to the fast spectrum as the wall is brought to the plasma boundary. In (3b) a density variation, $\rho = (1.0 - 0.75\psi/\psi_0)$, is introduced in order to show how the discrete kink passes through the continuum. The spectrum appears discrete because of the finiteness of the representation of the expansion functions. The shapes of the radial eigenfunction for five spectral points neighboring that of the discrete kink is shown in (3c), illustrating the change in sign of the logarithmic contribution to the eigenfunction.

752353 (a)
762062 (b)

Fig. 4. The passage of the free boundary kink through the Alfvén continuum as nq is varied (4a). Here we treat a circular cross section. The continuum has a finite width because of a diffuse density profile, $\rho = (1.0 - \psi/3.0\psi_0)$. More details of the interaction are shown in an expanded scale in (4b). The break is the slope as the kink enters the continuum can be understood using Eq. (37).

752259

Fig. 5. The complete spectrum for a shearfree, circular plasma column of length $L/a = 10\pi$ with constant B_z and finite pressure as a function of nq for $\ell = 2$. In order that both positive and negative eigenvalues be shown over so many orders of magnitude, an inverse sine hyperbolic scale is used, i.e., $y = \sinh^{-1}(\beta\Omega^2)$. This scale is linear for small Ω^2 and logarithmic for large Ω^2 . The magnification factor β is set equal to 10^5 .

762060

Fig. 6. The localization of the radial coordinate ψ_i/ψ_0 as a function of the stretched coordinate $\hat{\psi}/\psi_0$. The dashed lines generate the variable mesh.

762066

Fig. 7. The effect of radial grid localization σ , defined by Eq. (41), on the three most unstable interchanges in a sheared plasma is depicted by the solid lines. The dotted lines indicate the degree of grid localization $(\delta\psi)_{\max}/(\delta\psi)_{\min}$ as a function of σ . Here $a = b$, $q^2 = 1.2$, $\Delta = 0.4$, and $p_0 = 0.032$.

752280 (a)
752281 (b)

Fig. 8. The radial component ξ_ψ of the plasma displacement versus ψ for the three most unstable interchange modes in the sheared plasma of Fig. 7. The localization is increased as the radial mode number increases (8a). It also increases as the critical pressure for stability is approached as shown in (8b) for the most unstable mode. In both cases the localized grid spacing is marked on the axis.

752312

Fig. 9. The lower part of the spectrum for a plasma with the parameters of Fig. 7 where the interchange criterion is violated. Here, the three most unstable interchange modes and the current-driven kink are shown in relation to the Alfvén and sound continua. The scale is inverse sine hyperbolic.

762061 (a)
752264 (b)

Fig. 10. The effect of ellipticity and pressure on the current-driven kink. Here $b/a = 2.0$, $L/a = 20\pi/3$, $p_0 = 0.1$. The dots in (10a) represent the lower part of the continua and the kink (the dots for the kink are joined by the solid lines of curve A). For reference the behavior of the kink in the absence of pressure is depicted by curve B. The corresponding cases for the circular limit are respectively depicted by curves C and D. The enhanced destabilization at the rational surfaces, $nq = 2$ and 4 , is due to violation of the interchange criterion, as shown in (10b) on a logarithmic scale to provide more detail.

762059

Fig. 11. The growth rate $-\Omega^2$ versus axial plasma current, for the axisymmetric, $n = 0$, mode for a column with $b/a = 2.0$. The pressureless case is denoted by open circles, the solid circles denote a case with $p_0 = 0.2$, and the solid line represents $p_0 = 0.1$.

752263

Fig. 12. The effect of ellipticity on the fast branch of the spectrum for a model where $L/a = 40\pi$ and $p_0 = 0.05$. The dots are actual data points and the solid parts of the curves are hand drawn to show more clearly the breaking of the degeneracies and the interaction of the various modes.