**Calculation of a Self-Consistent,
Low Frequency Electrostatic Field
in Toroidal Geometry**

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Printed in the United States of America. Available from
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road, Springfield, Virginia 22161
Price: Printed Copy \$4.00; Microfiche \$3.00

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Contract No. W-7405-eng-26

FUSION ENERGY DIVISION

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ELECTROSTATIC FIELD IN TOROIDAL GEOMETRY

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Date Published - March 1977

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CALCULATION OF A SELF-CONSISTENT, LOW FREQUENCY
ELECTROSTATIC FIELD IN TOROIDAL GEOMETRY*

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ABSTRACT

We derive an asymptotic series in ω_p^{-2} , the inverse-square plasma frequency, for the self-consistent, curl-free, low frequency electric field in tori. The derivation is consistent with the drift-kinetic ordering and may be used in either instability or equilibrium calculations. We also provide an interpretation of past formulations of the statement of quasi-neutrality as applied to neoclassical equilibria.

I. INTRODUCTION

The creation of toroidally confined plasmas naturally results in the creation of spatial inhomogeneities in densities, currents, and temperatures, as well as in the confining magnetic field. These inhomogeneities drive species-dependent particle currents which cause charge separations, thus setting up electric fields. These electric fields in turn influence the currents, completing the cycle.

While the physically interesting phenomena relating to equilibrium or instability behavior of such plasmas occur on relatively long drift, collision, or bounce time scales, the mathematical coupling of the kinetic equations for the individual species and the electric field by use of Poisson's equation results in an extremely rapid oscillation at or near the plasma frequency ω_p . Aside from the mathematical and numerical difficulties this oscillation introduces, it is of little physical interest in toroidal problems, and is also, of course, completely inconsistent with the drift-kinetic ordering used to describe the species kinematics in such problems.

To eliminate (or in effect average out) this rapid oscillation, the familiar concept of quasi-neutrality has been used. A number of applications appropriate to toroidal problems are discussed.¹⁻³ The procedure involves a coupling which correctly models the low frequency response and at the same time precludes plasma oscillations. While relatively simple formulations used in the past have been suitable for analytic approximations, computation of equilibrium and instability (for instance, with the Collisional Plasma Model^{4,5}) necessitates a mathematically rigorous formulation consistent with the ordering

used in the kinetic equation and an improved understanding of the concept of quasi-neutrality.

In treatments of trapped-particle instabilities, the formulation of quasi-neutrality (see, for example, Ref. 1) was used in dealing with a linearized instability in which the unperturbed distributions of electrons and ions were taken to be equal-density Maxwellians. To obtain a dispersion relation, it is almost always sufficient to state that the perturbed ion and electron densities are very nearly equal and to set $\tilde{N}_i = \tilde{N}_e$. In the process of obtaining the resulting dispersion relation, the electric field was eliminated from the problem, except for the integral operator

$$\hat{K}\tilde{\phi} \equiv \frac{\oint \frac{ds}{v_{||}} \tilde{\phi}}{\oint \frac{ds}{v_{||}}}$$

which was argued to be very nearly unity. Thus, good use was made of the notion of quasi-neutrality, but no discussion of its physical interpretation was offered.

For use in calculations of neoclassical transport, quasi-neutrality was formulated in more precise terms by Hazeltine and Hinton² by setting

$$\vec{\nabla} \cdot \vec{j} = 0 ,$$

where $\vec{j}$ is the total electric current density,

$$\vec{j} = \sum_{s=\text{species}} e_s \vec{j}_s = \sum_s e_s N_s \vec{V}_s ,$$

and $e_s = Z_s e$, $\vec{J}_s$, N_s and $\vec{V}_s$ are the particle charges, fluxes, densities, and mass-flow velocities. By recognizing that most of the current is in the direction parallel to the magnetic field and imposing number and momentum conservation, one obtains

$$\frac{\partial J}{\partial t} = \frac{ZeEN}{M} - \frac{\partial}{\partial z} \left(\frac{NkT}{M} \right) + v^2 \frac{\partial N}{\partial z} + 2v \frac{\partial N}{\partial t} ,$$

where E is the electric field, T the temperature, M the mass, k the Boltzmann constant, and the z direction is parallel to the magnetic field. At equilibrium, $\dot{N} = \dot{J} = 0$; for a Maxwell-Boltzmann distribution,

$$ZeEN = \frac{\partial}{\partial z} (NkT) ,$$

so that the expression used by Hazeltine and Hinton² is obtained:

$$\frac{\partial \rho}{\partial z} = 0 , \quad (I.1)$$

where $\rho = \sum_s e_s N_s$ is the charge density. Earlier, Hinton and Rosenbluth³ had used the flux-surface integral form of (I.1).

While the forms described above^{2,3} eliminated the plasma oscillations and were suitable for obtaining the zeroth order electric field, the electric field is not obtained to an order in Ω^{-1} consistent with the drift-kinetic approximation. More specifically, the electric field resulting from perpendicular motion is ignored. Also, the formulation is not suitable for use in time-dependent problems. Finally, only the zeroth order in an ω_p^{-2} expansion is obtained, with no prescription for obtaining higher order corrections.

In Sec. II, we obtain in a general geometry the proper ordering in ω_p^{-2} for the electric field, charge, and current to first order in Ω^{-1} . In Sec. III, we apply this formulation to a toroidal geometry.

II. ASYMPTOTIC SERIES DERIVATION

The electric field consists of the externally applied ring field $\vec{E}_r$ and the plasma self-consistent field $\vec{E}$, which is the solution of Poisson's equation. An equation equivalent to Poisson's equation is the time-derivative equation

$$\vec{\nabla} \cdot \frac{\partial^2 \vec{E}}{\partial t^2} = 4\pi \frac{\partial^2 \rho}{\partial t^2} \quad (\text{II.1})$$

coupled with the initial conditions

$$[\vec{\nabla} \cdot \vec{E} = 4\pi\rho]_{t=0} \quad , \quad (\text{II.2a})$$

$$[\vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t} = 4\pi \frac{\partial \rho}{\partial t}]_{t=0} \quad (\text{II.2b})$$

Charge continuity,

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{j} \quad , \quad (\text{II.3})$$

can be used to eliminate ρ from (II.1):

$$\vec{\nabla} \cdot \left(\frac{\partial^2 \vec{E}}{\partial t^2} + 4\pi \frac{\partial \vec{j}}{\partial t} \right) = 0 \quad (\text{II.4})$$

Further, the momentum-balance equation, multiplied by charge and summed over species, can be used to eliminate $\vec{j}$ from (II.4). This momentum equation can be written in the form

$$4\pi \frac{\partial \vec{j}}{\partial t} = (\omega_p^2) \vec{E} - \vec{G} \quad , \quad (\text{II.5})$$

where

$$\omega_p^2 = 4\pi \sum_s \frac{e_s^2 N_s}{M_s} \quad , \quad (\text{II.6})$$

$$\vec{G} = -\omega_p^2 \vec{E}_r - 4\pi \sum_s e_s \left[(\vec{J}_s \times \vec{\Omega}_s) + \int d^3v \vec{v} C_s - \vec{v} \cdot \underline{\underline{K}}_s \right], \quad (\text{II.7})$$

$$\vec{\Omega}_s = \frac{e_s}{M_s c} \vec{B},$$

$$\underline{\underline{K}}_s = \int d^3v \vec{v} \vec{v} f_s,$$

and C_s is the collision operator. Clearly, the electron term will dominate the plasma frequency ω_p defined by (II.6).

The motivation for solving Poisson's equation in the form (II.4) is apparent from (II.5); it introduces the plasma frequency explicitly. However, in the drift-kinetic approximation, (II.5) is not satisfied because of the order mixing produced by the $\vec{J} \times \vec{\Omega}$ terms in (II.7). Therefore, we must calculate $(\partial \vec{J} / \partial t)$ from the drift-kinetic $\vec{J}$ using the drift-kinetic equation, if the ordering of the drift-kinetic approximation is to be preserved. Details of the implications of the drift-kinetic approximation for this purpose are given in the appendix for the case of a static magnetic field. We obtain

$$4\pi \frac{\partial \vec{J}}{\partial t} = \hat{n} (\omega_p^2 E_{||} - G_0) - \vec{G}_1, \quad (\text{II.8})$$

where

$$\hat{n} = \frac{1}{B} \vec{B},$$

$$E_{||} = \hat{n} \cdot \vec{E},$$

$$G_0 = -\omega_p^2 \hat{n} \cdot \vec{E}_r - 4\pi \sum_s c_s g_{0s}, \quad (\text{II.9a})$$

$$\vec{G}_1 = -4\pi \sum_s e_s \vec{g}_{1s}. \quad (\text{II.9b})$$

The terms g_0 and $\vec{g}_1$ are fully defined in the appendix; here, it is sufficient to say that the 0 and 1 subscripts indicate the Ω^{-1} ordering and that g_0 has no explicit dependence on the self-consistent electric field $\vec{E}$.

We can now eliminate $(\partial \vec{j} / \partial t)$ from (II.4) by using (II.8), so that the equation to be solved is

$$\vec{\nabla} \cdot \left(\frac{\partial^2 \vec{E}}{\partial t^2} + \hat{n} \omega_p^2 E_{\parallel} \right) = H_0 + H_1, \quad (\text{II.10})$$

where

$$H_0 = \vec{\nabla} \cdot (\hat{n} G_0), \quad (\text{II.11a})$$

$$H_1 = \vec{\nabla} \cdot \vec{G}_1. \quad (\text{II.11b})$$

It is immediately obvious from (II.10) that any plasma oscillation effects in the perpendicular direction occur as a result of the $\vec{E}$ dependence in H_1 and are therefore reduced by a factor of order Ω^{-1} compared to the parallel plasma oscillation. This result simply reflects the familiar fact that, in devices with a strong magnetic field, particles can move much more freely along the field lines than perpendicular to them, so that charge separation and neutralization motions tend naturally to be in the parallel direction.

Appropriate to the assumed static magnetic field, the electrostatic case is taken,

$$\vec{E} = -\vec{\nabla} \phi, \quad (\text{II.12})$$

and ϕ is separated into its ordered parts:

$$\phi = \phi_0 + \phi_1 \quad . \quad (II.13)$$

We now Fourier-analyze the ϕ_m and H_m with respect to both position $\vec{x}$ and time t :

$$\phi_m(\vec{x}, t) = \int d^3k \int d\omega e^{i(\vec{k} \cdot \vec{x} + \omega t)} \phi_m(\vec{k}, \omega) \quad , \quad (II.14a)$$

$$H_m(\vec{x}, t) = \int d^3k \int d\omega e^{i(\vec{k} \cdot \vec{x} + \omega t)} H_m(\vec{k}, \omega) \quad . \quad (II.14b)$$

Equations (II.10), (II.12), (II.13), and (II.14) readily yield the relationship

$$(k_{||}^2 \omega_p^2 - k^2 \omega^2) \phi_m(\vec{k}, \omega) = H_m(\vec{k}, \omega, \phi_{m-1}) \quad , \quad (II.15)$$

where $\phi_{-1} = 0$. In deriving (II.15), fluctuations in time and space of ω_p^2 have been neglected, as has the variation of $B^{-1} \nabla B$ parallel to the magnetic field lines (since these approximations do not affect the content of the ω_p^{-2} ordering). Equation (II.15) can be solved to give the time dependence of ϕ , using the initial conditions (II.2):

$$\begin{aligned} k^2 \phi_m(\vec{k}, t) = & 4\pi \rho_m(\vec{k}, t) - P \int d\omega \frac{H_m(\vec{k}, \omega, \phi_{m-1}) e^{i\omega t}}{\tilde{\omega}_p^2 - \omega^2} \\ & + \left[4\pi \rho_m(\vec{k}, 0) - P \int d\omega \frac{H_m(\vec{k}, \omega, \phi_{m-1})}{\tilde{\omega}_p^2 - \omega^2} \right] \cos \tilde{\omega}_p t \\ & + \frac{1}{\tilde{\omega}_p} \left[4\pi \dot{\rho}_m(\vec{k}, 0) - P \int d\omega \frac{i\omega H_m(\vec{k}, \omega, \phi_{m-1})}{\tilde{\omega}_p^2 - \omega^2} \right] \sin \tilde{\omega}_p t, \quad (II.16) \end{aligned}$$

where

$$\tilde{\omega}_p = \frac{k_{||}}{k} \omega_p \quad (II.17)$$

The function $H_0(\vec{x}, t)$ is determined by the ring field, collision, and pressure terms [see (II.9a), (II.11a), and the appendix], and therefore will vary on a time scale that is much longer than ω_p^{-1} . Also, even the modified plasma frequency (II.17) is still very large. As an example, in toroidal devices, for all except flux-surface-averaged quantities (corresponding to $k_{||} = 0$ here and discussed in Sec. III), $|\frac{k_{||}}{k}| \geq \epsilon$, the inverse aspect ratio. Consequently, we can obtain from (II.16) the low frequency (zeroth order) portion of the potential in the form

$$k^2 \bar{\phi}_0(\vec{k}, t) = 4\pi \bar{\rho}_0(\vec{k}, t) = P \int d\omega \frac{H_0(\vec{k}, \omega) e^{i\omega t}}{\tilde{\omega}_p^2 - \omega^2}, \quad (II.18a)$$

where $\bar{\phi}_0$ and $\bar{\rho}_0$ are in effect time averages over several modified plasma periods. Except for its dependence on the high frequency part of ϕ_0 , H_1 is also a low frequency function, and therefore the low frequency, first-order-correction potential is given by

$$k^2 \bar{\phi}_1(\vec{k}, t) = 4\pi \bar{\rho}_1(\vec{k}, t) = P \int d\omega \frac{H_1(\vec{k}, \omega, \bar{\phi}_0) e^{i\omega t}}{\tilde{\omega}_p^2 - \omega^2}. \quad (II.18b)$$

An asymptotic form exists for (II.18). Because the $H_m(\vec{x}, t)$ terms vary slowly in time, the numerators of the integrands will be vanishingly small unless $|\omega/\tilde{\omega}_p| \ll 1$, and consequently

$$P \int d\omega \frac{H_m(\vec{k}, \omega) e^{i\omega t}}{\tilde{\omega}_p^2 - \omega^2} = P \int d\omega \left[\sum_{j=0}^N \frac{\omega^{2j}}{\tilde{\omega}_p^{2j+2}} + \frac{\omega^{2N}}{\tilde{\omega}_p^{2N+2}(\tilde{\omega}_p^2 - \omega^2)} \right] H_m(\vec{k}, \omega) e^{i\omega t} = \sum_{j=0}^N \frac{(-1)^j}{\tilde{\omega}_p^{2j+2}} \frac{\partial^{2j}}{\partial t^{2j}} H_m(\vec{k}, t) + R_N(\vec{k}, t),$$

where the remainder R_N will be negligibly small, provided N is not too large. We therefore have

$$k^2 \bar{\phi}_m(\vec{k}, t) = 4\pi \bar{\rho}_m(\vec{k}, t) = \sum_{j=0}^N \frac{(-1)^j}{\tilde{\omega}_p^{2j+2}} \frac{\partial^{2j}}{\partial t^{2j}} H_m(\vec{k}, t). \quad (\text{II.19})$$

Differentiating (II.19) with respect to time, we get

$$\begin{aligned} \vec{\nabla} \cdot \vec{j}_m(\vec{k}, t) &= -\dot{\bar{\rho}}_m(\vec{k}, t) \\ &= -\frac{1}{4\pi} \sum_{j=0}^N \frac{(-1)^j}{\tilde{\omega}_p^{2j+2}} \frac{\partial^{2j+1}}{\partial t^{2j+1}} H_m(\vec{k}, t). \end{aligned} \quad (\text{II.20})$$

Solving Poisson's equation (II.19), we obtain $\bar{E}$ to order $(N+1)$ in ω_p^{-2} .

It is then immediately apparent from (II.8) and (II.3) that the current and density calculated from the drift-kinetic equation will be determined only to order ω_p^{-2N} . This illustrates explicitly how, with $N=0$, an electric field exists in an apparently neutral plasma (the plasma is "polarized"). Of course, (II.19) and (II.20) give the density and current to order $\omega_p^{-2(N+1)}$.

For the application considered in Sec. III, we take $N = 0$, and so it follows from (II.19), (II.17), (II.12), and (II.11) that the low frequency electrostatic field must satisfy the equations

$$\vec{\nabla} \cdot [\hat{n}(\omega_p^2 \bar{E}_{\parallel,0} - G_0)] = 0, \quad (\text{II.21a})$$

$$\vec{\nabla} \cdot [\hat{n} \omega_p^2 \bar{E}_{\parallel,1} - \vec{G}_1] = 0. \quad (\text{II.21b})$$

We can calculate $\vec{G}_1$ in (II.21b) with $\bar{E}_0 = 0$ because $\bar{E}_0$ would contribute a term of second order in ω_p^{-2} to $\bar{E}_{\parallel,1}$ and with $N = 0$ such higher order terms are to be neglected. Equations (II.21a) and (II.21b) can be combined to give

$$\vec{\nabla} \cdot [\hat{n}(\omega_p^2 \bar{E}_{\parallel} - G_0) - \vec{G}_1] = 0. \quad (\text{II.22})$$

Comparing this with (II.10) and (II.11), we see that we can obtain the ($N = 0$) low frequency field by neglecting $\partial^2 \vec{E} / \partial t^2$ in (II.10). However, a breakdown of (II.22) occurs for its flux-surface average. This problem is considered in Sec. III.

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III. TOROIDAL GEOMETRY

We model the static magnetic field by the vacuum toroidal field and the plasma current poloidal field, taking care to keep $\vec{\nabla} \cdot \vec{B} = 0$. In the usual toroidal geometry (R_0 = major radius, r = minor radius, θ = poloidal angle, ζ = toroidal angle), the model field is given by

$$\vec{B} = \frac{\hat{n} B_0}{\eta(1 + \epsilon \cos \theta)} = \hat{n} B \quad , \quad (III.1a)$$

where

$$\hat{n} = \eta \delta \hat{\theta} + \eta \hat{\zeta} \quad , \quad (III.1b)$$

$$\eta = (1 + \delta^2)^{-1/2} \quad , \quad (III.1c)$$

$$\delta = \frac{\epsilon}{q(r)} \quad , \quad \epsilon = \frac{r}{R_0} \quad . \quad (III.1d)$$

If we flux-surface average (II.22), we obtain

$$\frac{\partial}{\partial r} [r \langle \hat{r} \cdot \vec{G}_1 \rangle] = 0 \quad . \quad (III.2)$$

However, from the definition of $\vec{G}_1$ [see (II.9b) and the appendix] (III.2) is not satisfied in general. We must therefore use the exact equation (II.10) for the flux-surface average. We then obtain

$$\frac{\partial}{\partial r} \left[r \langle \hat{r} \cdot \left(\frac{\partial^2 \vec{E}}{\partial t^2} - \vec{G}_1 \right) \rangle \right] = 0 \quad , \quad (III.3)$$

a low frequency equation which relates $\langle \hat{r} \cdot \vec{E} \rangle$ to $\langle \hat{r} \cdot \vec{j} \rangle$ and corrects (III.2) by including the low frequency content of $\vec{E}$ in the flux average. We may thus obtain the complete low frequency electric field, including

the flux-averaged part, by using (III.3). This implies using an improved version of (II.22):

$$\vec{\nabla} \cdot \left[\hat{n}(\omega_p^2 \bar{E}_{\parallel} - G_{\parallel}) - \vec{G}_{\perp} \right] = 0, \quad (\text{III.4})$$

where

$$G_{\parallel} = G_0 + \hat{n} \cdot \vec{G}_1,$$

$$\vec{G}_{\perp} = \vec{G}_1 - \hat{n}(\hat{n} \cdot \vec{G}_1) - \hat{r} \langle \hat{r} \cdot \vec{G}_1 \rangle.$$

We may write (III.3) and (III.4) in terms of $\bar{\phi}$ by using (II.12) and (III.1):

$$\frac{\partial}{\partial r} \left[r < \frac{\partial^2}{\partial t^2} \cdot \frac{\partial \bar{\phi}}{\partial r} + \hat{r} \cdot \vec{G}_1 > \right] = 0, \quad (\text{III.5a})$$

$$\begin{aligned} & \frac{\partial^2 \bar{\phi}}{\partial \theta^2} + \left(\frac{2q}{1 + \epsilon \cos \theta} \right) \frac{\partial^2 \bar{\phi}}{\partial \theta \partial \zeta} + \left[\frac{q \epsilon \sin \theta}{(1 + \epsilon \cos \theta)^2} \right] \frac{\partial \bar{\phi}}{\partial \zeta} + \left[\frac{q^2}{(1 + \epsilon \cos \theta)^2} \right] \frac{\partial^2 \bar{\phi}}{\partial \zeta^2} \\ & = - \frac{1}{\langle \omega_p^2 \rangle} \left(\frac{q R_0}{\eta} \right)^2 \vec{\nabla} \cdot (\hat{n} G_{\parallel} + \vec{G}_{\perp}). \end{aligned} \quad (\text{III.5b})$$

Because $\bar{\phi}$ must satisfy toroidal periodicity, we can expand it in a Fourier series,

$$\bar{\phi} = \sum_{\mu\lambda} \bar{\phi}_{\mu\lambda} \exp [i(\mu\theta - \lambda\zeta)],$$

and solve (III.5) for all $\bar{\phi}_{\mu\lambda}$.

For the zeroth order potential we obtain the simpler equations

$$\frac{\partial}{\partial r} \left[r < \frac{\partial^2 \bar{\phi}}{\partial t^2} > \right] = 0, \quad (\text{III.6a})$$

$$\frac{\partial \bar{\phi}}{\partial \theta} + \left(\frac{q}{1 + \epsilon \cos \theta} \right) \frac{\partial \bar{\phi}}{\partial \zeta} = \frac{q R_0}{\eta \langle \omega_p^2 \rangle} \left(\frac{K(r)}{1 + \epsilon \cos \theta} - G_0 \right), \quad (\text{III.6b})$$

where the integration constant K is given by

$$K = \frac{\sqrt{1 - \epsilon^2}}{4\pi^2} \oint d\theta \oint d\zeta G_0 .$$

To this order, $[(\partial/\partial r) \langle r(\partial\bar{\phi}/\partial r) \rangle]$ remains zero for an initially neutral plasma.

This result provides a general expression for the self-consistent electric field in a toroidal plasma to first order in Ω^{-1} and to lowest order in an ω_p^{-2} expansion. Unlike earlier treatments, this formulation is valid for any plasma distribution function. In the zeroth order limit in both the ω_p^{-2} and Ω^{-1} expansions, it reduces to the form used by Hazeltine and Hinton² for Maxwellian plasmas. We have also seen formally how one obtains the higher-order terms in the ω_p^{-2} expansion [see Eq. (II.19)].

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APPENDIX. DRIFT-KINETIC APPROXIMATION

The drift-kinetic equation is given by⁵

$$\left(\frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla} + \vec{A} \cdot \vec{\nabla}_v\right) \bar{f} = \bar{C},$$

where

$$\vec{V} = \vec{V}_0 + \vec{V}_1,$$

$$\vec{V}_0 = \hat{n} v \cos \alpha,$$

$$\begin{aligned} \vec{V}_1 = \frac{1}{\Omega} \{ & \vec{a} \times \hat{n} + \frac{1}{3} (\hat{n} \times \vec{b} + \vec{\nabla} \times \hat{n}) v^2 \\ & - \frac{1}{3} [\hat{n} \times \vec{b} + 3 \hat{n} (\hat{n} \cdot \vec{\nabla} \times \hat{n}) - 2 \vec{\nabla} \times \hat{n}] v^2 P_2(\cos \alpha) \} \end{aligned}$$

$$\vec{A} = \vec{A}_0 + \vec{A}_1$$

$$\vec{A}_0 = \hat{n} (\hat{n} \cdot \vec{a}) + \frac{1}{2} \hat{\alpha} (\hat{n} \cdot \vec{b}) v^2 \sin \alpha,$$

$$\begin{aligned} \vec{A}_1 = \frac{1}{2\Omega} \{ & \hat{s} [\hat{n} \times \vec{b} + \hat{n} (\hat{n} \cdot \vec{\nabla} \times \hat{n})] \cdot \vec{a} v \sin \alpha \\ & + \frac{1}{\Omega} \hat{n} [\vec{\nabla} \times \hat{n} - \hat{n} (\hat{n} \cdot \vec{\nabla} \times \hat{n})] \cdot \vec{a} v \cos \alpha \\ & - \frac{1}{2} \hat{\alpha} \vec{\nabla} \cdot \{ \frac{1}{\Omega} [\vec{\nabla} \times \hat{n} - \hat{n} (\hat{n} \cdot \vec{\nabla} \times \hat{n})] \} v^3 \sin \alpha \cos \alpha, \end{aligned}$$

and $\bar{C}$ is the collision operator. In the above,

$$\hat{n} = \frac{1}{B} \vec{B} = \hat{v} \cos \alpha - \hat{\alpha} \sin \alpha,$$

$$\hat{s} = \hat{v} \sin \alpha + \hat{a} \cos \alpha,$$

$$\Omega = \frac{ZeB}{Mc} \quad , \quad \vec{a} = \frac{Ze}{M} (\vec{E} + \vec{E}_r) \quad ,$$

$$\vec{b} = \frac{1}{B} \vec{\nabla} B \quad ,$$

and it is assumed that $\vec{B}$ is static. We have chosen the velocity-space coordinates described in Refs. 4 and 5 for the reasons stated therein. These variables are the velocity

$$v = \sqrt{\frac{2E}{M}} \quad ,$$

where E is the kinetic energy, and the pitch angle α .

The particle flux is given by⁵

$$\vec{J} = \int d^3v \vec{v} f = \hat{n} J_{||} + \vec{J}_{\perp} \quad ,$$

where

$$J_{||} = \int d^3v v \cos \alpha \bar{f} \quad ,$$

$$\vec{J}_{\perp} = \frac{1}{\Omega} \hat{n} \times [-N\vec{a} + \frac{1}{2} \vec{\nabla} K_{\perp} - (\hat{n} \times \vec{\nabla} \times \hat{n})(K_{||} - \frac{1}{2} K_{\perp})] \quad ,$$

$$N = \int d^3v \bar{f} \quad ,$$

$$K_{\perp} = K_{11} + K_{22} = \int d^3v v^2 \sin^2 \alpha \bar{f} \quad ,$$

$$K_{||} = K_{33} = \int d^3v v^2 \cos^2 \alpha \bar{f} \quad .$$

The rate of change of flux ($\partial \vec{J} / \partial t$) is obtained from the above definition of $\vec{J}$ and the appropriate moments of the drift-kinetic equation:

$$g_0 = \left(\frac{\partial J_{||}}{\partial t} \right)_0 - (\hat{n} \cdot \vec{a}) N = \int d^3v v \cos \alpha \bar{C} - \hat{n} \cdot \vec{\nabla} K_{||} \\ + (\hat{n} \cdot \vec{b}) (K_{||} - \frac{1}{2} K_{\perp}) ,$$

$$\vec{g}_1 = \hat{n} \left(\frac{\partial J_{||}}{\partial t} \right)_1 + \frac{\partial J_{\perp}}{\partial t} ,$$

$$\left(\frac{\partial J_{||}}{\partial t} \right)_1 = \frac{1}{\Omega} \vec{a} \cdot [\hat{n} x (\vec{b} - \vec{\nabla}) + 2 \vec{\nabla} x \hat{n} - \hat{n} (\hat{n} \cdot \vec{\nabla} x \hat{n})] J_{||} \\ - \frac{1}{5} \vec{\nabla} \cdot \left\{ \frac{1}{\Omega} [\vec{\nabla} x \hat{n} - \hat{n} (\hat{n} \cdot \vec{\nabla} x \hat{n})] [I(1,1) + 4I(3,0)] \right\} \\ - \frac{1}{5\Omega} [\hat{n} x \vec{b} + 2 \vec{\nabla} x \hat{n} - \hat{n} (\hat{n} \cdot \vec{\nabla} x \hat{n})] \cdot \vec{\nabla} [I(1,1) - I(3,0)] ,$$

$$\frac{\partial J_{\perp}}{\partial t} = \frac{1}{\Omega} \hat{n} x \left\{ -N \frac{\partial \vec{a}}{\partial t} + \vec{\nabla} \cdot (\hat{n} J_{||}) \vec{a} + \frac{1}{2} \vec{\nabla} \left(\frac{\partial K_{\perp}}{\partial t} \right)_0 \right. \\ \left. - (\hat{n} x \vec{\nabla} x \hat{n}) \left[\left(\frac{\partial K_{||}}{\partial t} \right)_0 - \frac{1}{2} \left(\frac{\partial K_{\perp}}{\partial t} \right)_0 \right] \right\} ,$$

where

$$I(\ell, q) = \int d^3v v^{\ell+2q} p_{\ell}(\cos \alpha) \bar{f} ,$$

$$\left(\frac{\partial K_{\perp}}{\partial t} \right)_0 = \int d^3v v^2 \sin^2 \alpha \bar{C} + \frac{4}{5} (\hat{n} \cdot \vec{b}) [I(1,1) - I(3,0)] \\ - \frac{2}{5} \hat{n} \cdot \vec{\nabla} [I(1,1) - I(3,0)] ,$$

$$\left(\frac{\partial K_{||}}{\partial t} \right)_0 = \int d^3v v^2 \cos^2 \alpha \bar{C} + 2(\hat{n} \cdot \vec{a}) J_{||}$$

$$+ \frac{1}{5} (\hat{n} \cdot \vec{b}) [I(1,1) + 4I(3,0)] - \frac{1}{5} \hat{n} \cdot \vec{\nabla} [3I(1,1) + 2I(3,0)] .$$

It is to be noted that g_0 consists of readily identifiable terms: the parallel collisional friction term, the parallel gradient of the parallel energy density or pressure, and a tensor-pressure term proportional to the parallel gradient of the magnetic field.

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