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Magnetic Monopole Solution in Non-Abelian Gauge Theory*

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Abstract

We propose an approximate analytic solution of the equations of motion of the 't Hooft magnetic monopole model. Virial type global tests are carried out for the solution. Then, the monopole mass, energies of the vector field A_μ^a , Higgs field ϕ^a and interaction are computed in closed form. The form factors of A_1^a and ϕ^a in a quantized version are also calculated.

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1. INTRODUCTION

There is a great interest in the localized classical solutions or solitary waves of the Higgs type Lagrangians, because they may represent extended objects that have some features of elementary particles. One interesting model of that type was proposed by 't Hooft.¹ In it a magnetic monopole arises as a static solution of a non-Abelian $SO(3)$ gauge theory with spontaneous symmetry breaking. Solutions have been provided for various versions of the 't Hooft model.²⁻⁴ The purpose of this paper is to present an approximate analytic solution which allows us to compute physical quantities such as monopole mass, and form factors of a quantized version.

The plan of the paper is as follows: An approximate analytic solution of 't Hooft's model is presented and its accuracy tested in Sec. 2, monopole mass and its parameter dependence are obtained in Sec. 3, the form factors are computed in Sec. 4, and finally remarks are given in Sec. 5.

2. APPROXIMATE ANALYTIC SOLUTION AND ITS TESTS

We start from the Lagrangian density of the model

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a + c_2 (\phi^a)^2 - c_4 (\phi^a)^4 - \frac{c_2^2}{4c_4} \quad (1)$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + e \epsilon^{abc} A_\mu^b A_\nu^c \quad (2)$$

$$D_\mu \phi^a = \partial_\mu \phi^a + e \epsilon^{abc} A_\mu^b \phi^c \quad (3)$$

The fields A_μ^a and ϕ^a are $SO(3)$ triplets of vector fields and scalar fields, respectively, $\mu = 0, 1, 2, 3$, $a = 1, 2, 3$, and ϵ^{abc} is the antisymmetric tensor of the third rank. The parameter $c_2 > 0$, so we have the Higgs mechanism and $c_4 > 0$ for stability. A constant $c_2^2/4c_4$ has been added so that the vacuum expectation value has a vanishing action integral.

Nielsen-Olesen Lagrangian⁵ is obtained from Eq. (1) using the ansatz $\phi^3 = 0$, $\phi = e^{i\theta} \phi(\rho) = \phi^1(\rho) + i\phi^2(\rho)$, $A_\mu^1 = A_\mu^3 = 0$ and $A_\mu^2 = A_\mu$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} |(\partial_\mu + ieA_\mu)\phi|^2 + \frac{c_2}{2} |\phi|^2 - c_4 |\phi|^4 - \frac{c_2^2}{2c_4}, \quad (4)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (5)$$

The Higgs-type Lagrangian (4) has a string-like solution that is related to a localized classical solution of the Landau Ginzburg Lagrangian. In a previous article⁶ we obtained an approximate analytic solution of the Nielsen-Olesen model. As we see later, this solution also provides an approximate analytic solution of the 't Hooft's monopole model in spite of the additional terms due to the non-Abelian gauge theory.

We now return to Eq. (1) and consider time-independent solutions of the form¹

$$A_i^a(r) = \epsilon^{iab} r^b A(r)/r, \quad A_0^a = 0, \quad (6)$$

$$\phi^a(r) = r^a \phi(r)/r,$$

and obtain

$$\mathcal{L}(r) = -\left(\frac{1}{r} \frac{d}{dr} rA\right)^2 - \frac{1}{2} A^2 \left(eA + \frac{2}{r}\right)^2 - \left(eA + \frac{1}{r}\right)^2 \phi^2 - \frac{1}{2} \left(\frac{d\phi}{dr}\right)^2 - c_4 \left(\phi^2 - \frac{c_2}{2c_4}\right)^2, \quad (7)$$

$$E = 4\pi \int_0^\infty r^2 dr [-\mathcal{L}(r)]. \quad (8)$$

We note that ϕ gets a nonvanishing vacuum expectation value $(c_2/2c_4)^{1/2}$ and the vector field and the Higgs particle acquire masses m_A and m_H given, respectively, by

$$m_A = e(c_2/2c_4)^{1/2}, \quad m_H = 2c_2^{1/2} \quad (9)$$

The equations of motion for $\phi(r)$ and $A(r)$ are

$$\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d\phi}{dr} = 2 \left(eA + \frac{1}{r} \right)^2 \phi + 4c_4 \left(\phi^2 - \frac{c_2}{2c_4} \right) \phi, \quad (10)$$

$$\frac{1}{r^2} \frac{d}{dr} r^2 \frac{dA}{dr} = \left(eA + \frac{1}{r} \right) \left[A \left(eA + \frac{2}{r} \right) + e\phi^2 \right]. \quad (11)$$

Let us introduce K as

$$K = 1/r + eA \quad (12)$$

and substitute it into (10) and (11) and obtain

$$\frac{d^2\phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} - 2 \left[K^2 + 2c_4 \left(\phi^2 - \frac{c_2}{2c_4} \right) \right] \phi = 0 \quad (13)$$

$$\frac{d^2K}{dr^2} + \frac{2}{r} \frac{dK}{dr} + \frac{1}{r^2} K - (K^2 + e^2\phi^2) K = 0 \quad (14)$$

The expressions

$$\phi = \left(\frac{c_2}{2c_4} \right)^{1/2} \tanh \frac{y}{2} \quad (15)$$

$$K = \left(\frac{4c_2}{y \sinh y} \right)^{1/2} \quad (16)$$

where $y = 2c_2^{1/2}r$, satisfy Eq. (13) exactly and Eq. (14) approximately.

The expressions ϕ and A are plotted in Fig. 1 as functions of y .

The nature of the approximation can be understood by the following arguments: If $e/(2c_4)^{1/2}$ is not a constant but has the y dependence

$$\left(\frac{e}{(2c_4)^{1/2}}\right)(y) = \left[\left(\frac{1}{y} - \frac{1}{\tanh y}\right)^2 + 2\left(\frac{1}{y} - \frac{1}{\sinh y}\right)^2\right]^{1/2} / \tanh \frac{y}{2} \quad (17)$$

then ϕ and K defined in Eqs. (15) and (16) is an exact solution of Eqs. (13) and (14). The above function is smooth and has asymptotic values

$$\frac{e}{(2c_4)^{1/2}} \rightarrow \begin{cases} \left(\frac{2}{3}\right)^{1/2} & (y \rightarrow 0), \\ 1 & (y \rightarrow \infty). \end{cases} \quad (18)$$

It is plotted in Fig. 2 as a function of y . From Fig. 2, one would expect the approximate solution (15) and (16) to be reasonably good when the constant is chosen in the interval

$$\left(\frac{2}{3}\right)^{1/2} \lesssim e/(2c_4)^{1/2} \lesssim 1. \quad (19)$$

This condition restricts the usefulness of our approximate solution.

In order to obtain physical quantities reliably with use of the approximate solution (15) and (16), it is necessary to carry out some global tests. Let us consider the virial type theorems⁷ which arise from the variations

$$(i) \quad A \rightarrow A_\lambda = \lambda A(\lambda r), \quad \phi \rightarrow \phi_\lambda = \phi(\lambda r) \quad (20)$$

$$(ii) \quad A \rightarrow A_\xi = A(r), \quad \phi \rightarrow \phi_\xi = \xi \phi(r) + (1-\xi)\left(\frac{c_2}{2c_4}\right)^{1/2} \quad (21)$$

These variations satisfy the asymptotic conditions $erA_\lambda, erA_\xi \rightarrow -1$ and $\phi_\lambda, \phi_\xi \rightarrow (c_2/2c_4)^{1/2}$ as $r \rightarrow \infty$. The former condition arises from the magnetic charge conservation. When A_λ and ϕ_λ of Eq. (20) are substituted into E given in Eq. (8), we get

$$E_\lambda = \lambda I_4 + \lambda^{-1} I_2 + \lambda^{-3} I_0, \quad (22)$$

where

$$\begin{aligned} I_4 &= 4\pi \int_0^\infty r^2 dr \left[\left(\frac{1}{r} \frac{d}{dr} rA \right)^2 + \frac{1}{2} A^2 \left(eA + \frac{2}{r} \right)^2 \right] = \frac{2\pi c_2^{1/2}}{e^2} (3 \ln 2 - 2 + \frac{\pi^2}{8}), \\ I_2 &= 4\pi \int_0^\infty r^2 dr \left[\left(eA + \frac{1}{r} \right)^2 \phi^2 + \frac{1}{2} \left(\frac{d\phi}{dr} \right)^2 \right] = \frac{\pi c_2^{1/2}}{3 c_4} (2 + \frac{\pi^2}{6}), \\ I_0 &= 4\pi \int_0^\infty r^2 dr c_4 \left(\phi^2 - \frac{c_2}{2c_4} \right)^2 = \frac{\pi c_2^{1/2}}{3 c_4} \left(\frac{\pi^2}{6} - 1 \right), \end{aligned}$$

where the numerical values are calculated with the approximate solution (15) and (16).

For an exact solution the energy should be stationary under small variations, which now gives

$$\left. \frac{\partial E_\lambda}{\partial \lambda} \right|_{\lambda=1} = 0$$

i.e.

$$I_4 = I_2 + 3I_0 \quad (23)$$

For our approximate solution, this gives

$$\frac{e}{(2c_4)^{1/2}} = \left[9(3 \ln 2 - 2 + \frac{\pi^2}{8}) / (2\pi^2 - 3) \right]^{1/2} \approx 0.83$$

which is in the interval (19).

From (8) and (21), we obtain similarly the condition

$$\begin{aligned} 4\pi \int_0^\infty r^2 dr \left[\left(\frac{d\phi}{dr} \right)^2 + 4c_4 \phi \left(\phi^2 - \frac{c_2}{2c_4} \right) \left(\phi - \left(\frac{c_2}{2c_4} \right)^{1/2} \right) \right] \\ = 4\pi \int_0^\infty r^2 dr \, 2\phi \left(\left(\frac{c_2}{2c_4} \right)^{1/2} - \phi \right) \left(eA + \frac{1}{r} \right)^2. \end{aligned} \quad (24)$$

After partial integration of the term $(d\phi/dr)^2$ in (24), the integrand reduces to Eq. (13) multiplied by the factor $(c_2/2c_4)^{1/2} - \phi$. Since our solution (15) and (16) satisfies Eq. (13) exactly, Eq. (24) is automatically satisfied.

The electromagnetic tensor proposed by 't Hooft¹ is

$$F_{\mu\nu} = F_{\mu\nu}^a \phi^a / |\phi| - \epsilon^{abc} \phi^a (D_\mu \phi^b) (D_\nu \phi^c) / e |\phi|^3,$$

where $|\phi| = (\phi^a \phi^a)^{1/2}$ so that

$$F_{ij} = -\epsilon^{ija} r^a / e r^3.$$

This is independent of the particular form of ϕ and A . By Gauss's theorem the magnetic charge g satisfies Schwinger's condition $eg = 1$.

3. THE ENERGY OF THE SYSTEM

Let us calculate the energy of the system which is identified with the mass of the magnetic monopole. Energy can be divided into three parts, energy of the ϕ field E_ϕ , interaction energy E_I , and energy of the A field E_A . They are

$$\begin{aligned} E_\phi &= 4\pi \int_0^\infty r^2 dr \left[\frac{1}{2} \left(\frac{d\phi}{dr} \right)^2 + c_4 \left(\phi^2 - \frac{c_2}{2c_4} \right)^2 \right] \\ &= \frac{\pi c_2^{1/2}}{4c_4} \int_0^\infty y^2 dy \left(\cosh \frac{y}{2} \right)^{-4} = \frac{\pi c_2^{1/2}}{c_4} \frac{2}{3} \left(\frac{\pi^2}{6} - 1 \right) \approx \frac{\pi c_2^{1/2}}{c_4} 0.43, \end{aligned} \quad (25)$$

$$\begin{aligned}
E_I &= 4\pi \int_0^\infty r^2 dr \phi^2 \left(eA + \frac{1}{r} \right)^2 \\
&= \frac{\pi c_2^{1/2}}{2c_4} \int_0^\infty y dy \sinh \frac{y}{2} \left(\cosh \frac{y}{2} \right)^{-3} = \frac{\pi c_2^{1/2}}{c_4}, \quad (26)
\end{aligned}$$

$$\begin{aligned}
E_A &= 4\pi \int_0^\infty r^2 dr \left[\left(\frac{1}{r} \frac{d}{dr} rA \right)^2 + \frac{1}{2} A^2 \left(eA + \frac{2}{r} \right)^2 \right] \\
&= \frac{2\pi c_2^{1/2}}{e^2} \int_0^\infty y dy \left[\frac{1}{\sinh y} \left(\frac{1}{y} - \frac{1}{\tanh y} \right)^2 + \frac{2}{y} \left(\frac{1}{y} - \frac{1}{\sinh y} \right)^2 \right] \\
&= \frac{2\pi c_2^{1/2}}{e^2} \left(3 \ln 2 - 2 + \frac{\pi^2}{8} \right) \approx \frac{\pi c_2^{1/2}}{e^2} 2.62. \quad (27)
\end{aligned}$$

We plot in Fig. 3 the energy densities $\mathcal{E}_A(r)$, etc. (defined by $E_A = \int d^3r \mathcal{E}_A(r)$, etc.) as functions of $y = 2c_2^{1/2}r$. Note that the tail contribution is important only for E_A .

Let us express the total energy as

$$E = (m_A/\alpha) C(\beta) \quad (28)$$

where $\alpha = e^2/4\pi$ and $\beta = (m_H/m_A)^2 = 8 c_4/e^2$, with the aid of (9). For our solution, we find

$$\begin{aligned}
C(\beta) &= \beta^{-1/2} \left[\frac{1}{3} \left(\frac{\pi^2}{3} + 1 \right) + \frac{1}{4} \left(3 \ln 2 - 2 + \frac{\pi^2}{8} \right) \beta \right] \\
&\approx \beta^{-1/2} (1.43 + 0.33 \beta) \quad (29)
\end{aligned}$$

The function $C(\beta)$ has a minimum at $\beta = (4/9) (\pi^2 + 3) [3 \ln 2 - 2 + (\pi^2/8)]^{-1} \approx 4.34$. This is in the interval (19), which can be expressed in terms of β as $6 \gtrsim \beta \gtrsim 4$.

The $C(\beta)$ has a singularity at $\beta = 0$ and therefore our approximate solution is not applicable when β is small. In Fig. 4, $C(\beta)$ is plotted. The calculated energy agrees well with the numerical results^{1,4} in the range $3 < \beta < 10$. From the global test the preferred value is $\beta \equiv 4 (e^2/2c_4)^{-1} \approx 5.8$.

In order to compare the energy distributions of 't Hooft's model with that of Nielsen and Olesen we choose β to have the same value $\beta = 4$ (or $e^2/2c_4 = 1$) in both cases. In the Nielsen-Olesen model the partial energies E_A , E_I and E_ϕ were 20%, 30% and 45% of the total energy, respectively, whereas for 't Hooft's model they are 48%, 36% and 16%, respectively. In the former model 50% of the energy was contained in the scalar field energy E_ϕ but in the latter model 50% is in the vector field energy E_A . The reason for the different distributions can be traced to the additional self-interaction of A in E_A which occurs for non-Abelian gauge theories. If other values $\beta > 4$ are used then E_A contains larger share of the total energy in 't Hooft's model.

4. FORM FACTORS

An approximate analytic solution to the classical static field equations of the 't Hooft model has been given in Sec. 2. We now wish to interpret this solution as the form factor of the quantum field between localized states of heavy particles.^{8,9} Let us consider the localized

fields $\chi^a = (c_2/2c_4)^{1/2}(r^a/r) - \phi^a$ and A_i and consider their expectation values with respect to heavy scalar fields. The matrix elements of χ^a and A_i^a are parametrized as

$$\int d^3r' \langle \vec{r}' | \chi^a(\vec{r}) | \vec{0} \rangle = r^a \chi(r)/r, \quad (30)$$

$$\int d^3r' \langle \vec{r}' | A_i^a(\vec{r}) | \vec{0} \rangle = \epsilon^{iab} r^b A(r)/r. \quad (31)$$

In the momentum representation, the matrix element of ϕ and A_i^a are expressed in terms of form factors

$$\langle p | \chi^a(0) | p' \rangle = i q^a G(q^2)/2M, \quad (32)$$

$$\langle p | A_i^a(0) | p' \rangle = i \epsilon^{iab} q^b F(q^2)/2M. \quad (33)$$

where M is the mass of the heavy scalar field, $\vec{q} = \vec{p}' - \vec{p}$ and $q^2 = -|\vec{q}|^2$ because $q_0 = (M^2 + p'^2)^{1/2} - (M^2 + p^2)^{1/2} \rightarrow 0$ for heavy mass particles.

The form factors $G(q^2)$ and $F(q^2)$ are obtained by the Fourier transformations of (30) and (31) with the use of (32) and (33):

$$\frac{G(q^2)}{2M} = \frac{-i}{q^2} \int d^3r e^{-i\vec{q}\cdot\vec{r}} (\vec{q}\cdot\vec{r}) \chi(r)/r, \quad (34)$$

$$\frac{F(q^2)}{2M} = \frac{-i}{q^2} \int d^3r e^{-i\vec{q}\cdot\vec{r}} (\vec{q}\cdot\vec{r}) A(r)/r. \quad (35)$$

By substituting the solution (15) and (16) in (34) and (35), we obtain

$$\begin{aligned}\frac{G(q^2)}{2M} &= \frac{\pi^5}{4\sqrt{2}} c_2^{-3/2} c_4^{-1/2} \frac{1}{z^4} \left[2 - \frac{z}{\sinh z} - \left(\frac{z}{\sinh z} \right)^2 \cosh z \right], \\ &\rightarrow \frac{7\pi^5}{720\sqrt{2}} c_2^{-3/2} c_4^{-1/2} \quad (q^2 \rightarrow 0), \\ &\rightarrow 8\pi \left(\frac{c_2}{2c_4} \right)^{1/2} \frac{1}{q^4} \quad (q^2 \rightarrow -\infty),\end{aligned}\tag{36}$$

where $z = (\pi/m_H)(-q^2)^{1/2}$ and

$$\frac{F(q^2)}{2M} \sim \begin{cases} -\frac{2\pi^2}{e} \left(\frac{-1}{q^2} \right)^{3/2} & (q^2 \rightarrow 0^-), \\ -\frac{\pi^5}{6ec_2^{1/2} - q^2} \exp[-\pi(-q^2)^{1/2}/2c_2^{1/2}] & (q^2 \rightarrow -\infty). \end{cases}\tag{37}$$

We note that there are poles and double poles in G at $q^2 = (nm_H)^2$ for positive integer n . The spectrum of singularities corresponds to that of Goldstone and Jackiw in the one-dimensional ϕ^4 theory.⁸ When one considers the expectation value of χ^a and A_i^a with respect to heavy fermions with spin $1/2$ and isospin $1/2$, a suitable definition⁹ of the corresponding form factors leads also to (34) and (35).

5. REMARKS

We wish to emphasize the value of obtaining an analytic solution to non-linear field equations even when the solution is approximate. In this spirit we proposed an approximate analytic solution to the 't Hooft model and computed interesting quantities such as monopole mass, partial energies and form factors. We are not certain of the accuracy of our results, but we feel that they fairly well represent the situation for an exact solution.

The classical solution (15) and (16) can be regarded as a soliton solution, because the energy of the system is finite and the solution is confined. Quantum effects then provide fluctuations about the soliton solution.^{8,10} The spread in the distributions of scalar χ^a and vector fields A_i^a i.e. form factors, can also be interpreted as arising from the collective excitations of Higgs pairs and vector particle pairs.

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FIGURE CAPTION

Fig. 1; Plots of $\phi(r)(c_2/2c_4)^{-1/2}$ and $A(r)(c_2/e^2)^{-1/2}$ versus $y = 2c_2^{1/2}r$.

Fig. 2; Plot of $e(2c_4)^{-1/2}$ versus $y = 2c_2^{1/2}r$.

Fig. 3; Plots of the energy densities $\mathcal{E}_A(r)(c_2^2/e^2)^{-1}$,
 $\mathcal{E}_\phi(r)(c_2^2/2c_4)^{-1}$, and $\mathcal{E}_I(r)(c_2^2/2c_4)^{-1}$ versus $y = 2c_2^{1/2}r$.

Fig. 4; Plot of $C(\beta)$ versus β , where $E = (m_A/\alpha)C(\beta)$ and
 $\beta = (m_H/m_A)^2 = 8c_4/e^2$.

Fig. 1

Fig. 2

Fig. 3

Fig. 4