LA-6955

334
5-3-78

MASTER

16. 77

UC-32 and UC-34

Issued: March 1978

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John C. Goldstein
D. O. Dickman

los alamos
scientific laboratory
of the University of California
LOS ALAMOS, NEW MEXICO 87545

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UNITED STATES
DEPARTMENT OF ENERGY
CONTRACT W-7408-ENG. 36

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LAPU2: A LASER PULSE
PROPAGATION CODE WITH DIFFRACTION

by

John C. Goldstein and D. O. Dickman

ABSTRACT

Complete descriptions of the mathematical models and numerical methods used in the code LAPU2 are presented. This code can be used to study the propagation with diffraction of a temporally finite pulse through a sequence of resonant media and simple optical components. The treatment assumes cylindrical symmetry and allows nonlinear refractive indices. An unlimited number of different media can be distributed along the propagation path of the pulse. A complete users guide to input data is given as well as a FORTRAN listing of the code.

I. INTRODUCTION

In 1972 work was begun on a code which was designed to serve as a numerical tool in theoretical studies of some optical propagation problems which arose in connection with the development of lasers for the Los Alamos Laser Fusion Project. The code was originally written to handle only diffraction problems. Later, a series of modified versions was written with each succeeding version capable of treating a somewhat broader class of laser problems than the previous one. The current version of the code is known as LAPU2. It is the purpose of this report to document LAPU2 as it now exists.

The numerical methods used in LAPU2 are due to B. R. Suydam. These methods were presented in a previous report¹ to which reference will frequently be made. However, since that report was primarily devoted to an exposition of the numerical treatment of diffraction used in the code and did not reflect the actual content of the code or present any examples of calculations, it was felt that

another document which emphasized these latter two points would be useful at the present time.

LAPU2 represents the current version of a code that has been developed to handle specific propagation problems. It was not initially conceived to be a general optical design code. Rather, it was primarily intended to have the capability of handling problems involving the propagation of temporally finite pulses which are of interest for laser fusion. Various additional features were added in order to treat circumstances of interest as they developed. Hence, the content and structure of the code is biased towards adequate treatment of short pulse problems at the expense of features which might be more useful in a comprehensive optical design code.

The organization of this report is as follows. Section II presents the specific mathematical models of different physical processes which the code can treat; the forms of these mathematical models in finite difference representations, as well as the specific solutions of the finite difference equations and other operations used in the code, are explicitly stated. Section II.A presents the numerical treatment of diffraction in an axisymmetric geometry. Section II.B contains descriptions of the code's models of responses of materials to the intense fields of a short laser pulse. Section II.C elucidates the special sub-routines which are used as diagnostic features or models of simple optical components. Section III contains a description of the input-output mechanics of the code. Sections III.A, III.B, III.C, and III.D describe, respectively, the general information data input, the medium information data input, the end card, and the restart and modify data input. Examples of various features of the code are given in Section IV, while a flow chart of the structure of the code is given in Section V. Five appendices contain specific difference equation operations and supplementary information.

Although this report represents the first complete exposition of the contents and methods of LAPU2, the results of various applications to specific problems have appeared elsewhere.²⁻⁶

II. PHYSICAL CONTENT AND NUMERICAL METHODS

LAPU2 can numerically evaluate the propagation of a temporally finite pulse of radiation through a sequence of resonant media and simple optical components. We shall discuss how the code treats (A) propagation of the electromagnetic wave, (B) response of a resonant medium to the light pulse, and (C) the action of

various "lens" subroutines. Some of the latter routines are purely diagnostic devices which do not modify the pulse, while others simulate optical components (apertures or thin lenses) which do modify the amplitude or phase of the electric field of the pulse.

A. Propagation of the Electromagnetic Wave

1. Fundamental propagation equation. We take, as in Ref. 7, Maxwell's displacement vector $\vec{D}$ to be given by

$$\vec{D} = \epsilon_0 (\vec{E} + \vec{X}) \quad , \quad (1)$$

where ϵ_0 is (the scalar part of) the linear dielectric constant and the medium response $\vec{X}$ may be an arbitrary function of the electric field E . The phase velocity of light v in the medium is given by

$$v = \frac{c}{\sqrt{\epsilon_0}} \quad . \quad (2)$$

We impose the condition that

$$|\vec{X}| \ll |\vec{E}| \quad . \quad (3)$$

We shall be interested in problems in which the electric field propagates primarily in one direction - which we shall label the $\hat{z}$ direction. We may then resolve $\vec{X}$ and $\vec{E}$ into parallel and perpendicular components

$$\vec{X} = X_z \hat{z} + \vec{X}_\perp \quad (4a)$$

and

$$\vec{E} = E_z \hat{z} + \vec{E}_\perp \quad . \quad (4b)$$

We assume next that $\vec{E}_\perp$ is a quasi-monochromatic wave of frequency $\omega/2\pi$ and use the following slowly varying complex envelope representation of the perpendicular components:

$$\vec{E}_\perp = \hat{e} \left[\frac{1}{2} E(\vec{r}_\perp, z, \tau) e^{i\omega\tau} + \text{c.c.} \right] \quad (5a)$$

and

$$\vec{X}_1 = \hat{e} \left[\frac{1}{2} X(\vec{r}_1, z, \tau) e^{i\omega\tau} + \text{c.c.} \right] . \quad (5b)$$

Here, τ is the "proper time" variable

$$\tau = t - z/v , \quad (6)$$

c.c. means complex conjugate, and we have made the unnecessary but convenient simplifying assumption that $\vec{E}_1$ and $\vec{X}_1$ are linearly polarized. One can then show, as in Ref. 7 or Ref. 8, that in the lowest approximation the z components of the fields may be neglected and the electric field envelope function $E(\vec{r}_1, z, \tau)$ satisfies the following quasi-optical or paraxial wave equation

$$\frac{2i\omega}{v} \frac{\partial}{\partial z} E(\vec{r}_1, z, \tau) = \nabla_1^2 E(\vec{r}_1, z, \tau) + \left(\frac{\omega}{v} \right)^2 X(\vec{r}_1, z, \tau) . \quad (7)$$

Here, ∇_1^2 is the part of the Laplacian that operates only on the transverse coordinates $\vec{r}_1$. Equation (7) is thus expected to be valid for the study of a linearly polarized quasi-monochromatic field $\vec{E}_1$ which propagates primarily in the $\hat{z}$ direction and whose complex amplitude E varies in z and t very little over one optical wavelength or period. That is,

$$\frac{\partial}{\partial z} E \ll \frac{\omega}{c} E \quad (8a)$$

and

$$\frac{\partial}{\partial t} E \ll \omega E . \quad (8b)$$

We choose to neglect problems in which different polarizations are coupled and thus restrict ourselves to the scalar propagation Eq. (7).

We shall further restrict our attention to cylindrically symmetric problems. Thus, in terms of the radial coordinate r which measures displacement from the $\hat{z}$ axis, Eq. (7) becomes

$$\frac{2i\omega}{v} \frac{\partial}{\partial z} E(r, z, \tau) = \frac{1}{r} \frac{\partial}{\partial r} r \left[\frac{\partial}{\partial r} E(r, z, \tau) \right] + \left(\frac{\omega}{v} \right)^2 X(r, z, \tau) , \quad (9)$$

Since axisymmetry is assumed, the general type of problem that can be handled by LAPU2 is the propagation of a pulse through a sequence of different media, all of which occupy consecutive locations along the initial direction of propagation. Thus, although the various media can have different characteristics (laser or attenuating media or lenses) and possibly different radial dimensions, it is assumed that the longitudinal axis of each medium is coincident with the initial direction of propagation. Hence, problems involving off-axis mirrors or other nonaxisymmetric components cannot be treated by this code.

The following two points can be made:

(a) In the case of pure diffraction [$X(r,z,\tau) \equiv 0$], according to Eq. (9) different time slices $E(r,z,\tau = \text{constant})$ of a temporally finite pulse propagate independently of each other. The variable τ serves only to label different time slices within the pulse. In this case the Kirchhoff-Fresnel integral provides a solution which allows propagation over an arbitrary distance in a single step. Although we shall find it useful in special circumstances to use this solution of scalar diffraction theory, it obviously is unsuited whenever $X \neq 0$ and therefore whenever the reaction of a medium on the pulse changes over small distances and thus requires small propagation steps.

(b) If $E(r,z,\tau) = 0$ at some point (r,z,τ) but $X(r,z,\tau) \neq 0$, an electric field may grow from zero amplitude since, according to Eq. (9), $\frac{\partial}{\partial z} E(r,z,\tau) \neq 0$. However, in all cases currently treated in LAPU2, the medium response is proportional to the acting electric field: $X(r,z,\tau) \propto E(r,z,\tau)$. Therefore, this situation cannot occur, and one cannot generate an electric field in a region in which the field is initially absent. Thus, if one starts with a finite pulse,

$$\begin{aligned}
 E(r,z,\tau) &\neq 0, & \tau_1 \leq \tau \leq \tau_2 \\
 &= 0, & \text{all other values of } \tau.
 \end{aligned}
 \tag{10}$$

of total duration $\Delta\tau = \tau_2 - \tau_1$, the field will always exist only over this same interval in τ . One can therefore introduce a "window" (of fixed longitudinal length $v\Delta\tau$) which defines the region over which the light interacts with a medium. The length of this window does not change throughout the problem, although of course coordinates of the leading and trailing edge do change with time as the finite pulse propagates down the system. One therefore in effect

views the propagating pulse from a frame moving at the pulse's velocity so that the electric field envelope appears to change shape only within the fixed length window as the active media affect it. LAPU2 ignores the time evolution of the rest of the system outside the window, although relaxation or other processes may of course be going on both before the pulse reaches some section of active medium and after the pulse leaves that section. The temporal evolution of the electric field and the medium are calculated for only the section of the system within the window.

We emphasize that the existence of a fixed-duration window holds for the media responses presently considered in LAPU2. It does not hold for responses in general. For example, spontaneous emission from an inverted medium or the coherent (density matrix) response of a resonant medium to a light pulse both lead to the generation of a field which can grow from initially zero amplitude. The current models of laser media included in LAPU2 (discussed in Section II) are based either on rate equations or an instantaneous nonlinear response to the applied field - both of which preserve the duration of the initial temporal window.

2. Numerical Treatment. The precise numerical method developed by B. R. Suydam to compute the solution of Eq. (9) is specified in detail in Ref. 1. Here we shall discuss only the salient features of his method, although for completeness we have included in Appendix A the exact operations evaluated by the code in computing the solution. These are taken directly from Ref. 1.

The numerical method begins with the introduction of a discrete mesh of points to replace the continuous coordinates r , z , and t . The radial mesh spacing is Δr and the coordinate is specified by

$$r_K = \left[K - \frac{1}{2} \right] \Delta r, \quad 1 \leq K \leq K_{\max}(N) \quad (11)$$

Here $K_{\max}(N)$ refers to the maximum value of the radial index K for medium number N . We defer a discussion of which quantities can be specified differently for different media in the input data for the code and only note here that Δr is not such a variable parameter.

Although r , z , and t are the independent variables of Eq. (9), the code reverts to r , z , and t . A temporally finite pulse is represented by a sequence of $J_{\max}$ "z-slices": at a fixed point in time, the pulse extends in space over

a finite interval of the z-axis (the window discussed above whose length is fixed for the entire problem).

$$\begin{aligned} E(r, z, t) &= E(r_K, z_J, t_n) \quad , \quad 1 \leq K \leq K_{\max}(N) \\ & \quad 1 \leq J \leq J_{\max} \\ &= E_{K,J}^n \quad . \quad 1 \leq n \end{aligned} \quad (12)$$

In particular, at $t = t_1$ the initial pulse is set up as $J_{\max}$ radial arrays, each of width Δz . These arrays represent "z-slices" of finite total energy rather than wave fronts of finite total power.

As already noted above, in pure diffraction problems the propagation step size can in principle be arbitrary, although this does not obtain if active media are involved. Nonetheless, one sees that a second longitudinal step size enters the problem:

$$\Delta z(N) = v \Delta t \quad . \quad (13)$$

Here v is the light phase velocity in medium N and $\Delta z(N)$ is the propagation step size for that medium.

The distinction between the thickness Δz and the propagation step size $\Delta z(N)$ should be clear. In principle, they can be very different in value. The pulse Δz is determined by resolution of the anticipated temporal structure on the envelope of the pulse, whereas $\Delta z(N)$ is determined from either diffraction or active medium considerations. Restrictions on their relative sizes are discussed below.

The specific numerical operations - as given in Ref. 1 - are repeated in Appendix A. The equation is solved with the boundary condition that $E[K = K_{\max}(N), J, n] \equiv 0$; that is, it is assumed that the cylindrical boundary surface is a perfect conductor.* A secondary boundary condition is a user option: it is supposed to be absorbing (nonreflecting) but does not work well. The boundary condition at $r = 0$ is that appropriate for an even function of r .

* Note that this means that diffraction ripples on the pulse will be reflected by the wall back toward the axis. If a problem simulating free space is to be done, the zoning must be chosen so as to put the wall far enough away such that its reflections do not affect the results.

Suydam has given an error analysis of his numerical method in Ref. 1. This provides a connection between the mesh size $[\Delta r, \Delta z(N)]$ and the accuracy of the numerical calculation. It may be seen that if $X(r, z, t) \equiv 0$ in Eq. (9), then an approximate solution is given by

$$E(r, z, t) = E_0 e^{i(\Omega z - k_t r)} , \quad (14)$$

where $k_t = 2\pi/\lambda_t$ is the wave vector for radial ripples and

$$\Omega = \frac{k_t^2}{2k} \quad (15)$$

where $k = \omega/v$. This solution is approximately valid if $k_t r \gg 1$; in particular it fails for $r \rightarrow 0$. Thus one can obtain a "group velocity" V for transverse ripples of wave vector k_t given by

$$V = \frac{\delta r}{\delta z} = \frac{d\Omega}{dk_t} = \frac{k_t}{k} = \frac{\lambda}{\lambda_t} . \quad (16)$$

This "group velocity" V is really a dimensionless quantity that relates the distance of propagation of energy in transverse ripples in the radial direction δr per unit of longitudinal propagation δz . In particular, if the transverse ripples are caused by an aperture, Eq. (16) predicts the existence of a diffraction pattern involving sufficiently small λ_t a very short longitudinal distance beyond the plane of the aperture. It is shown in Ref. 1 that the numerical scheme leads to a "group velocity" $\bar{V}$ different from Eq. (16):

$$\bar{V} = \frac{k_t}{k} f(\phi) , \quad (17a)$$

where

$$f(\phi) = \frac{\left(\frac{\sin \phi}{\phi} \right)}{\left[1 - 2\eta(1 - \cos \phi) \right]^2 + \frac{N^2}{4} (1 - \cos \phi)^2} , \quad (17b)$$

$$\phi = k_t \Delta r \quad , \quad (17c)$$

and

$$N = \frac{\lambda}{2\pi} \frac{\Delta z(N)}{(\Delta r)^2} \quad . \quad (17d)$$

In general, $f(\phi) \rightarrow 1$ as $\phi \rightarrow 0$, while $f(\phi) \rightarrow 0$ as $\phi \rightarrow \pi$. Hence the numerical method usually works well for long wavelength transverse ripples but eventually deviates from the ideal curve [Eq. (16)] as λ_t decreases. Ultimately, the numerical scheme does not propagate at all waves of transverse wavelength $\lambda_t = 2\Delta r$. Choosing $\eta = 1/6$ and $N \approx 1$ generally results in waves of $\lambda_t \gtrsim 4\Delta r$ ($\phi \leq \pi/2$) being propagated with reasonable ($\geq 90\%$) accuracy.

General restrictions on mesh sizes are as follows. The radial step Δr is the same for every medium, but the total radial dimension $K_{\max}(N)$ can be different for each medium. The longitudinal step sizes are fixed as follows: a $\Delta z_{\min}$ is entered via a data card. $\Delta z_{\min}$ must be of the form 2^p , with p a positive or negative integer (if the number on the data card is not of this form, the code replaces it by the next smallest number representable as a power of 2). No longitudinal step size in the problem, either read in as initial data or computed or rezoning, can be smaller than $\Delta z_{\min}$. The propagation step size in medium N , $\Delta z(N)$, must be chosen to be an integral multiple of $\Delta z_{\min}$; that is, $\Delta z(N) = L_N \Delta z_{\min}$ with L_N a positive integer. The code will adjust $\Delta z(N)$ if it does not satisfy this criterion. The pulse z -slice thickness Δz is set to $\Delta z(1)$, the propagation step size of the first medium in which the pulse is initialized. If N is an active medium with two level kinetics, $\Delta z(N) \leq \Delta z$. If N is an active medium with four level kinetics, $\Delta z(N) = \Delta z$. If N is either type of active medium, $\Delta z(N)$ cannot be larger than Δz .

The numerical scheme leads to a tridiagonal matrix in the radial index K . A parameter θ is related to the stability: $\theta < 1/2$ gives unconditional instability; $\theta \geq 1/2$ gives unconditional stability; $\theta > 1/2$ leads to damping of transverse waves; whereas $\theta = 1/2$, the usual value used, leads to neither growth nor damping; and a parameter η is useful to improve the "group velocity" curve of Eq. (17). The code separates all complex quantities into real and imaginary components on which algebra is done.

B. Nonlinear Medium Models

The medium response term $X(r,z,t)$ of Eq. (9) has the following form:

$$-\frac{1}{2} \left(\frac{\omega}{v} \right) X(r,z,t) = -\frac{1}{2} \left(\frac{v}{\omega} \right) \gamma |E|^2 E + g_A(r,z,t) E - \beta |E|^2 E \quad (18)$$

The three terms correspond, respectively, to a nonlinear index of refraction related to the constant γ , a resonant saturable gain or loss with electric field amplitude gain coefficient g_A , and a nonsaturable two-photon loss specified by β . There are two different resonant medium models in LAPU2: we discuss them first.

1. Resonant Medium Models. The objective of this code is the study of the amplification of short (nanosecond or less) pulses in a sequence of laser amplifiers. Over the duration of such a short pulse (and the transit time of the pulse through a meter long amplifier) most of the complex kinetic processes involved in pumping the laser medium can hardly proceed at all due to the slow rate constants which characterize these processes. Hence, we adopt the approximation that the short pulse reaches an amplifier only after the pumping process has been completed. Thus, the medium is in an inverted state when the short pulse reaches the amplifier and only those kinetic (relaxation) processes which are characterized by time scales comparable to or shorter than the pulse duration are of importance to the amplification process.

We generally use semiclassical radiation theory: the electric field is treated classically but the atomic medium is treated quantum mechanically. Hence, spontaneous emission is neglected. Although in principle it is not necessary to do so, we assume that all amplifiers have initial gains that are constant throughout their volumes. The response of active media to the incident laser pulse is limited to the rate equation approximation (no atomic coherence or density matrix treatment). Two types of laser medium models are included in LAPU2: (a) a two-level atomic rate equation model and (b) a four-level atomic rate equation model. These models have been applied to Nd:glass and CO_2 laser amplifier systems, respectively. In principle - due to the subroutine coding used - other types of gain models can be substituted into the code in order to study other types of lasers.

a. Two-Level Rate Equation Model. We consider two isolated levels, say a and b , of an atomic system and denote the population densities of these levels

by n_a and n_b respectively. The model is summarized by the following equation for the inversion density $\Delta = n_a - n_b$ (cm^{-3}):

$$\frac{\partial}{\partial \tau} \Delta(r, z, \tau) = - \frac{\Delta - \Delta^0}{T} - 2\alpha \sigma I(r, z, \tau) \Delta(r, z, \tau) , \quad (19a)$$

where

$$I(r, z, \tau) = \frac{1}{\hbar \omega} \frac{c \sqrt{\epsilon_0}}{8\pi} |E(r, z, \tau)|^2 \left(\frac{\text{photons}}{\text{cm}^2 \text{ s}} \right) , \quad (19b)$$

$$\frac{1}{T} = \text{energy relaxation rate (s}^{-1}\text{)} , \quad (19c)$$

$$\sigma = \text{stimulated emission cross section (cm}^2\text{)} , \quad (19d)$$

$$\alpha = \text{dimensionless constant (usually } \alpha = 1 \text{)} , \quad (19e)$$

$$\Delta^0 = \text{initial inversion density (cm}^{-3}\text{)} . \quad (19f)$$

This model corresponds to a homogeneously broadened transition between two atomic states each of which has a relaxation rate, or T_1 time, $1/T$. The amplitude gain g_A of Eq. (18) is given by

$$g_A = \frac{1}{2} \sigma \Delta \quad (20)$$

so that Eq. (19a) can be rewritten as

$$\frac{\partial}{\partial \tau} g_A(r, z, \tau) + \frac{1}{T} (1 + q |E(r, z, \tau)|^2) g_A(r, z, \tau) = \frac{g_A^0}{T} , \quad (21a)$$

where

$$q = 2\alpha \sigma \frac{1}{\hbar \omega} \frac{c \sqrt{\epsilon_0}}{8\pi} T . \quad (21b)$$

The difference equation solution, originally supplied by Suydam, is given explicitly in Appendix B. A discussion of the physical interpretation of this model is given in Section II.B.1.e.

b. Four-Level System Model. This model consists of equations for the population densities of four atomic levels, two of which are directly coupled to the electromagnetic field: we label these by n_a and n_b . The other two levels, n_0 and n_1 , act as reservoirs and are collisionally coupled to n_a and n_b , respectively. We thus obtain ($\dot{x} = \frac{\partial}{\partial \tau} x$):

$$\dot{n}_0 = -\frac{1}{N_1 \tau_1} n_0 + \frac{1}{\tau_1} n_a, \quad (22a)$$

$$\dot{n}_a = -\frac{1}{\tau_1} n_a + \frac{1}{N_1 \tau_1} n_0 - \alpha \sigma I (n_a - n_b), \quad (22b)$$

$$\dot{n}_b = -\frac{1}{\tau_2} n_b + \frac{1}{N_2 \tau_2} n_1 + \alpha \sigma I (n_a - n_b), \quad (22c)$$

and

$$\dot{n}_1 = -\frac{1}{N_2 \tau_2} n_1 + \frac{1}{\tau_2} n_b. \quad (22d)$$

Here I is the number of photons per s per cm^2 as in Eq. (19b),

$$I = \frac{1}{\hbar \omega} \frac{c \sqrt{\epsilon_0}}{8\pi} |E(r, z, \tau)|^2,$$

σ is the stimulated emission cross section, α is a dimensionless constant, and τ_1 (τ_2) is the relaxation time of the upper (lower) laser level. In order for Eq. (22) to maintain steady state populations (reflecting thermal equilibrium) in the absence of an external driving field, the constants N_1 and N_2 are related to the initial time population ratios:

$$N_1 = \frac{n_0(t=0)}{n_a(t=0)}, \quad (23a)$$

$$N_2 = \frac{n_1(t=0)}{n_b(t=0)}. \quad (23b)$$

We define f by

$$f = \frac{n_b(t=0)}{n_a(t=0)} . \quad (23c)$$

These equations are solved in LAPU2 using a method given by B. R. Suydam. The explicit solution is given in Appendix C. The amplitude gain g_A which appears in Eq. (18) is computed in this model from

$$g_A = \frac{1}{2} \sigma(n_a - n_b) .$$

c. Special Case. Where diffraction is primarily the object of scrutiny - and thus only one z-slice need be carried in the calculation - but where some effect of saturable gain is desired, the following simple form for gain is available:

$$g_A = \frac{g_A^0}{1 + q|E|^2} . \quad (24)$$

This form can be used only with one-z-slice calculations.

d. Special Propagation Step Size Restrictions. There are special restrictions on the propagation step size $\Delta z(N)$ for any active medium with nonzero initial gain. For example, suppose z-slice J in the window is at $z = z_0$ at time step n ; it propagates to $z = z_0 + \Delta z(N)$ on the next time step. The amplification it receives depends on the medium saturated gain at $z = z_0$ at time step n [Eqs. (A.21), (A.22), and (A.23) of Appendix A]. The saturated gain is lower than the initial gain due to the prior passage of the leading edge of the pulse (z-slices $J+1, J+2, \dots, J_{\max}$) through $z = z_0$ and the consequent transfer of the energy from the medium to those z-slices. Thus, the amplification of $E_{K,J}^n$ is determined by $(g_A)_{K,J}^n$.

The saturated gain at $z = z_0$ at time step n , $(g_A)_{K,J}^n$, obviously evolved from another value $(g_A)_{K,J+1}^{n-1}$ at an earlier time step $n-1$. If the active medium is a four-level system, this evolution is specified in Appendix C. There, the assumption is made that the propagation step size $\Delta z(N)$ equals the pulse

z-slice thickness Δz : this condition $[\Delta z(N) = \Delta z]$ must always hold in using the four-level gain model. The code does not automatically check to see if this condition is satisfied by the input data. The user is warned that the code will run even if this condition is not satisfied, but it will produce incorrect results.

For the two-level gain model, a little more freedom is available. If $\Delta z(N) < \Delta z$, the gain at step n , $(g_A)_{K,J}^n$, is determined by linear interpolation between $(g_A)_{K,J}^{n-1}$ and $(g_A)_{K,J+1}^{n-1}$:

$$(g_A)_{K,J+1}^{n-1} \xrightarrow[\text{by}]{\text{replaced}} \left(1 - \frac{\Delta z(N)}{\Delta z}\right) (g_A)_{K,J}^{n-1} + \frac{\Delta z(N)}{\Delta z} (g_A)_{K,J+1}^{n-1} .$$

Hence, the propagation step size $\Delta z(N)$ in a two-level medium can be less than the pulse z-slice thickness Δz . The opposite case is not allowed: if, for example, $\Delta z(N) = 2\Delta z$, the gain $(g_A)_{K,J}^n$ would have to be calculated from $(g_A)_{K,J+2}^{n-1}$. The code does not consider this, so always $\Delta z(N) \leq \Delta z$ and for a four-level medium necessarily $\Delta z(N) = \Delta z$.

Of course, all of the statements made with respect to an active medium include (with obvious modifications where needed) the case of an absorber in which the initial gain is negative. The code runs perfectly well with either two- or four-level saturable absorbers if the initial gain is specified to be a negative number.

One may also note that the maximum radial resolution is limited to either 1000 or 750 points depending on the choice of a two- or four-level gain model, respectively. This choice is made in the initial data of a problem and implies that only one gain model type will be used in the problem; both gain models cannot be used in a single problem, although of course the parameters of the four-level system may be chosen to effectively convert it into a two-level system. The limit for the single z-slice gain (Section II. B.1.c) is also 1000 radial points.

e. Physical Interpretations. The two-level model of Eq. (21) has different physical interpretations depending on the values chosen for the various parameters of the model.

(a) The full Eq. (21) represents a two-level system collisionally coupled to infinite reservoirs. The coupling is through the relaxation time T , and use of the full equation implies that the rate $1/T$ is comparable to the stimulated

emission rate $(q/T)|E|^2$ or to the reciprocal of the pulse temporal width. The fact that infinite reservoirs are implied is seen from the right hand side of Eq. (21a) which is a constant rate of driving the gain back to its initial value g_A^0 . This corresponds to a constant supply of population into (out from) the upper (lower) laser levels by reservoirs of infinite capacity. Equation (21a) can be derived from the four-level model Eq. (22) by setting $\tau_1 = \tau_2 = T$ and $N_1 = N_2 = N$ and subtracting Eq. (22c) from Eq. (22b). This gives

$$\frac{\partial}{\partial \tau} \left(\frac{g_A}{\frac{1}{2}\sigma} \right) = \dot{n}_a - \dot{n}_b = \frac{n_0 - n_1}{NT} - 2\alpha\sigma I(n_a - n_b) - \frac{n_a - n_b}{T} ,$$

which is equivalent to Eq. (21) if n_0 and n_1 are constants.

(b) If the relaxation time $T \rightarrow \infty$ (also $q \rightarrow \infty$ but q/T remains finite), the model reduces to that solved by Frantz and Nodvik⁹ provided that the transverse diffraction term in Eq. (9) is dropped. The equation for the light may then be written as an equation for the intensity I . This model, which includes no diffraction effects and assumes a two-level system relaxation time T which is much longer than either the pulse width or the reciprocal of the stimulated emission rate and therefore can effectively be set to infinity, is contained in the following two equations:

$$\dot{g} = -2\sigma I g$$

and

$$\frac{\partial I}{\partial z} = \sigma g I .$$

Here I is the number of photons/cm²/s, as given in Eq. (19b), and $g = 2g_A$. Note that we have set α of Eq. (21b) equal to one. As is well known, in this model the total energy per cm² of an amplified pulse $\left[\int d\tau \omega I(z, \tau) \right]$, E_{out} , is related to the energy per cm² of the input pulse, E_{in} , and to a saturation energy $E_s = \frac{\hbar\omega}{2\sigma}$ by

$$E_{out} = E_s \ln \left[1 + e^{g_0 L} \left(e^{E_{in}/E_s} - 1 \right) \right] ,$$

where the amplifier's initial intensity gain is g_0 and its length is L . This result is independent of pulse shape because the dynamical equations of the model contain no characteristic time. Of course, explicit solutions for $g(z,t)$ and $I(z,t)$ were derived by the original authors. This solution provides the possibility of an approximate check of LAPU2 when the two-level model is used in this limit.

(c) The parameter α in Eq. (21b) was included for the following reason. If we again set $N_1 = N_2 = N$ and $\tau_1 = \tau_2 = T$ in Eq. (22) and then add Eq. (22a) to Eq. (22b) and Eq. (22c) to Eq. (22d) we get [set $\alpha = 1$ in Eq. (22) also]

$$\dot{n}_0 + \dot{n}_a = -\sigma I(n_a - n_b)$$

and

$$\dot{n}_1 + \dot{n}_b = \sigma I(n_a - n_b) \quad .$$

If the relaxation rate $1/T$ is much faster than the stimulated emission rate, Eqs. (22a) and (22d) may be approximately solved by

$$n_a(t) = \frac{n_0(t)}{N} \longrightarrow \dot{n}_0 = N \dot{n}_a$$

and

$$n_b(t) = \frac{n_1(t)}{N} \longrightarrow \dot{n}_1 = N \dot{n}_b \quad .$$

Hence, substituting for $\dot{n}_0$ and $\dot{n}_1$ we obtain

$$(\dot{n}_a - \dot{n}_b) = \frac{\partial}{\partial t} \left(\frac{g_A}{\frac{1}{2}\sigma} \right) = - \frac{1}{1+N} 2\sigma I(n_a - n_b) \quad .$$

That is, $\alpha = 1/(1+N)$ in Eq. (19a) or Eq. (21b).

This model corresponds to simulating a four-level system with a two-level system and the parameter α . The collisional coupling between the laser levels n_a and n_b and their reservoirs n_0 and n_1 is so strong that the ratios of these populations remain locked at their initial (thermal equilibrium) values as the pulse interacts with the system. Using $g = 2g_A$ and neglecting the diffraction term in the quasi-optical equation Eq. (9) yields a Frantz-Nodvik-like model:

$$\dot{g} = 2\alpha\sigma I g$$

and

$$\frac{\partial}{\partial z} I = \sigma g I \quad .$$

Notice that these equations imply that, when one photon is added to the pulse (from the second equation), the population inversion does not decrease by two but rather by two α . This obviously takes into account the interaction with the reservoirs, which, in contrast to (a) above, are now finite. That is, the gain in (a) does not saturate fully no matter how high the light intensity may be because infinite reservoirs are coupled to the two laser levels; here, the gain g does saturate, and the energy extraction formula reflects the finite total energy stored in an amplifier:

$$E_{out} = E'_s \ln \left[1 + e^{g_0 L} \left(e^{E_{in}/E'_s} - 1 \right) \right] \quad .$$

Here the symbols have the same meaning as in (b) above but $E'_s = \frac{\hbar\omega}{2\alpha\sigma}$. This form of two-level system was used in amplifier calculations in an early version of LAPU2 before the four-level model Eq. (22) was included in the code.

It may be anticipated that the four-level model, Eq. (22), might be used, in the spirit of (c) above, to simulate a six-level system. Unfortunately, this cannot be done because it requires modification of the first terms on the right side of Eqs. (22a) and (22d). If these equations are rewritten as

$$\dot{n}_0 = - \frac{\gamma}{N_1 \tau_1} n_0 + \frac{1}{\tau_1} n_a$$

and

$$\dot{n}_1 = - \frac{\gamma}{N_2 \tau_2} n_1 + \frac{1}{\tau_2} n_b$$

where $\gamma < 1$, one could approximately account for the pumping of n_0 and n_1 by two other finite reservoir levels as these equations along with Eqs. (22b) and (22c) would imply that when n_a (n_b) increased by one, n_0 (n_1) would decrease by $\gamma < 1$. Unfortunately, the code contains only the model of Eq. (22), so this

cannot be done. Appendix E contains a somewhat more general model than that of Eq. (22) which however has not yet been included in LAPU2.

f. CO₂ Amplifiers. The 10.6 micron infrared transition in CO₂ is actually a series of lines in two branches (P and R) which correspond to transitions between different rotational levels of the 001 and 100 vibrational states. At high pressures collisions closely couple the populations in each of the two rotational manifolds. If the amplifying pulse contains just a single frequency which is resonant with one of the P or R branch lines, the four-level model works well. The model lumps all of the rotational levels other than the two directly coupled to the light into just two states whose populations are labeled by n_0 and n_1 . An approximate expression for the energy extracted by a pulse from a gain medium which obeys the equations of motion of the four-level model is derived in Appendix D. That saturation calculation shows that more energy can be extracted from the active medium if γ/τ or T_p or both are increased. In CO₂ these quantities can be increased by amplifying a pulse which contains several frequencies, each of which is resonant with one P or R branch transition. This is known as a multiline pulse.¹⁰

One can crudely model energy extraction by a multiline pulse with the four-level scheme of Eq. (22) by using the parameter α and interpreting the other quantities appropriately. If N frequencies ($N \sim 2-5$) are assumed to be in the pulse, $\bar{n}_a^*$ would correspond to the population of N rotational levels (rather than just one) and $\bar{n}_0$ would refer to the total population of the remaining rotational levels of the 001 vibrational state (and similarly for the lower-state quantities $\bar{n}_b$ and $\bar{n}_1$). Thus, initially, $\bar{n}_0(0)/\bar{n}_a(0) = \bar{N}_1 = \bar{\gamma}^{-1}$. Clearly, $\bar{N}_1 < N_1$ and $\bar{\gamma} > \gamma$; in fact, $\bar{\gamma} \sim N\gamma$ in typical circumstances, provided $N_1 \gg (N-1)$, and usually $\gamma = (1/N_1) \ll 1$. If σ is the stimulated emission cross section of one line, $\bar{\sigma} = \alpha\sigma = \frac{1}{N}\sigma$ would be the approximate** multiline cross section. Since $\bar{\delta} \sim \bar{n}_a - \bar{n}_b = N(n_a - n_b)$, the initial small signal gain $G = \exp(\bar{\sigma}\bar{\delta}L)$ for an amplifier of length L would be unchanged. This obviously assumes approximately equal small signal gains for each different frequency in the pulse.

The code could thus simulate energy extraction by a multiline pulse, using the barred quantities in the four-level model. Obviously, dispersion effects,

* All quantities of the four-level model in this application are denoted by a bar.

** The variation of σ over different lines is neglected.

which might affect different frequencies differently, would not be included in such a calculation. The saturation calculation of Appendix D would yield the following extracted energy per unit cross-sectional area of the gain medium:

$$\bar{E}\bar{N} = \frac{\hbar\omega}{2\bar{\sigma}\bar{\gamma}} (\ln G) \left[1 - e^{-(\bar{\gamma}/\tau)\bar{T}_p} \right] .$$

Since $\bar{\sigma}\bar{\gamma} = \sigma\gamma$, the maximum extractable energy is unchanged from the single line case for the same gain G . This obviously ignores small differences between the resonant frequencies of the different lines in the pulse. However, both $\bar{\gamma}$ and $\bar{T}_p$ are increased, the latter via $\bar{\sigma}$ [see Eq. (D.25)], so that a significant increase in extracted energy is possible with the multiline pulse. A more complete analysis of multiline energy extraction, ignoring diffraction effects, is given in Ref. 10.

2. Nonlinear Index of Refraction. The first term on the right-hand side of Eq. (18) corresponds to a nonlinear refractive index. The total dielectric constant ϵ is written as¹

$$\epsilon = \epsilon_0 + \epsilon_2 \vec{E}_\perp \cdot \vec{E}_\perp . \quad (25)$$

Note that we always use cgs units so that ϵ_0 is a dimensionless number and not the vacuum dielectric constant of mks units. Also, we have expressed ϵ in terms of the complete transverse electric field $\vec{E}_\perp$. If the contribution of this term at frequencies different from ω is dropped we obtain the following constant γ , which enters the medium response equation Eq. (18):

$$\gamma = \frac{3}{4} \left(\frac{\omega}{c} \right)^2 \epsilon_2 . \quad (26)$$

The conventional nonlinear index n_2 is related to ϵ_2 and the linear refractive index $n_0 = \sqrt{\epsilon_0}$ by

$$n_2 = \frac{4}{3} \frac{\epsilon_2}{n_0} . \quad (27)$$

Notice that this model assumes that the medium responds instantaneously to the electric field. Hence the origin of the nonlinearity cannot be electrostriction, which has a finite response time.

3. Nonsaturable Two-Photon Absorption. The third term on the right side of Eq. (18) is meant to model a two-photon (or intensity-dependent) absorption. It was included primarily to study Nd:glass systems in which such an effect is possible. The code includes only a characteristic constant β in this mechanism and does not provide any means of saturating this absorption.

This concludes a discussion of the possible types of medium response included in LAPU2. It should be noted that, due to machine storage limitations, the radial resolution can be lower for the case of four-level gain equations than for the case of two-level gain equations. This occurs obviously because of the added storage requirements needed to use the four-level model.

Another significant point to note is the way diffractive effects interact with active media. Both active medium models respond only to the local intensity $|E(r,z,t)|^2$ and thus are not sensitive to the phase of the light wave. The active media sense only the local energy density of the light wave. Diffraction ripples are induced only by apertures in the system or by reflections from the enclosing cylindrical wall. Thus, the coupling of diffraction with active media is done in a very simple way in this code.

C. Special Subroutines

In this section we shall discuss various special subroutines that are used in LAPU2 either as intrinsic to the general numerical solution or as special components.

1. Medium Rescaling. The "moving window" method of calculation used in LAPU2 assumes that the light pulse has a unique phase velocity. If the pulse enters a medium with a different linear refractive index, the phase velocity of the front of the pulse in the new medium is different from the phase velocity of the trailing part of the pulse. This situation persists until the entire pulse has entered the new medium, although it appears again of course as the pulse leaves that medium and enters a third one. Inasmuch as two pulse velocities are contrary to the basic formulation of this problem, it may be anticipated that this situation will cause some numerical difficulties. To avoid these, Suydam has devised a rescaling procedure¹ which we summarize here.

The full quasi-optical equation including the most general medium response is given by Eqs. (9) and (18):

$$\frac{\partial E}{\partial z} = \frac{c}{2i\omega\sqrt{\epsilon_0}} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E}{\partial r} \right) + \gamma |E|^2 E \right\} + g_A E - \beta |E|^2 E . \quad (28)$$

This equation assumes a medium of linear dielectric constant ϵ_0 . We wish to scale all quantities so that the form of the equation is unchanged even if the linear dielectric constant changes from medium to medium. We write the scaled equation as:

$$\frac{\partial \bar{E}}{\partial \bar{z}} = \frac{c}{2i\omega\sqrt{\epsilon_{00}}} \left\{ \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left(\bar{r} \frac{\partial \bar{E}}{\partial \bar{r}} \right) + \bar{\gamma} |\bar{E}|^2 \bar{E} \right\} + \bar{g}_A \bar{E} - \bar{\beta} |\bar{E}|^2 \bar{E} . \quad (29)$$

Here, $\bar{x}$ is a machine quantity and x is the corresponding physical quantity. The precise scaling relations used are

$$\bar{r} = r , \quad (30)$$

$$\bar{z} = \sqrt{\frac{\epsilon_{00}}{\epsilon_0}} z , \quad (31)$$

$$\bar{E} = \left(\frac{\epsilon_0}{\epsilon_{00}} \right)^{3/4} E , \quad (32)$$

$$\bar{\gamma} = \left(\frac{\epsilon_{00}}{\epsilon_0} \right)^{3/2} \gamma , \quad (33)$$

$$\bar{g}_A = \sqrt{\frac{\epsilon_0}{\epsilon_{00}}} g_A , \quad (34)$$

and

$$\bar{\beta} = \left(\frac{\epsilon_{00}}{\epsilon_0} \right) \beta . \quad (35)$$

We want $\delta \bar{z} = (c/\sqrt{\epsilon_{00}}) \delta t$ which gives

$$\bar{t} = \frac{\epsilon_{00}}{\epsilon_0} t \quad . \quad (36)$$

The rate equation for the two-level system [Eq. (21)] can be rewritten in terms of $\bar{t}$, $\bar{g}_A$, and $\bar{E}$ using Eqs. (36), (34), and (32):

$$\frac{\epsilon_{00}}{\epsilon_0} \frac{\partial}{\partial \bar{t}} \bar{g}_A + \frac{1}{\bar{T}} \left[1 + \frac{2\alpha\sigma}{\hbar\omega} \frac{c\sqrt{\epsilon_0}}{8\pi} T \left(\frac{\epsilon_{00}}{\epsilon_0} \right)^2 \sqrt{\frac{\epsilon_0}{\epsilon_{00}}} |\bar{E}|^2 \right] \bar{g}_A = \frac{\bar{g}_A^0}{\bar{T}} \quad . \quad (37)$$

Clearly we should scale the relaxation time T via Eq. (36):

$$\bar{T} = \frac{\epsilon_{00}}{\epsilon_0} T \quad . \quad (38)$$

If we then decide to scale the cross section σ by

$$\bar{\sigma} = \sigma \quad , \quad (39)$$

Eq. (37) becomes

$$\frac{\partial}{\partial \bar{t}} \bar{g}_A + \frac{1}{\bar{T}} \left[1 + \bar{q} |\bar{E}|^2 \right] \bar{g}_A = \frac{\bar{g}_A}{\bar{T}} \quad (40)$$

where

$$\bar{q} = \frac{2\alpha\sigma}{\hbar\omega} \frac{c\sqrt{\epsilon_{00}}}{8\pi} \bar{T} \quad . \quad (41)$$

Thus, if we use the variables $\bar{r}$, $\bar{z}$, $\bar{t}$, and $\bar{E}$, Eqs. (29) and (40) are form invariant as long as the other quantities γ , g_A , β , and T change discontinuously, as prescribed by Eqs. (33), (34), (35), and (38), when going into a medium with a different ϵ_0 .

One can show that this scaling leaves the total energy stored in a laser amplifier unchanged. Even though this procedure has been used primarily in Nd:glass calculations, there is provision made for using the four-level system model in a medium with linear dielectric constant ϵ_0 . The scaling relations

are given above with the obvious additions,

$$\bar{f} = f \quad , \quad (42)$$

$$\bar{N}_1 = N_1 \quad , \quad (43)$$

and

$$\bar{N}_2 = N_2 \quad . \quad (44)$$

These follow from the usual physical interpretation of these quantities as population ratios [see Eqs. (23) and (C.5)].

As discussed above at the end of Section II.A.2. each medium length must be an integral multiple of the propagation step of that medium. Since according to Eq. (31), the rescaling procedure for media with $\epsilon_0 \neq 1$ also involves changing the length of those media, the code actually does the following:

$$(a) \quad L'_N = \sqrt{\epsilon_{00}/\epsilon_0} L_N, \text{ where } L_N \text{ is the read-in value of the medium length;}$$

$$(b) \quad L''_N = I_N \Delta z(N), \text{ where } I_N \text{ is a positive integer;} \quad (45)$$

$$(c) \quad \alpha = L''_N / L'_N \quad ;$$

and

$$(d) \quad \text{Rescale all other appropriate quantities for that medium with } \alpha, \text{ for example,}$$

$$\bar{\gamma} = \alpha^3 \gamma \quad , \quad (46)$$

$$\bar{g}_A = \frac{1}{\alpha} g_A \quad , \quad (47)$$

$$\bar{\beta} = \alpha^2 \beta \quad , \quad (48)$$

and

$$\bar{T} = \alpha^2 T \quad . \quad (49)$$

Notice that α may be interpreted in terms of an effective dielectric constant ϵ_{eff} , which differs from ϵ_0 due to the length change in (b):

$$\alpha = \sqrt{\frac{\epsilon_{00}}{\epsilon_{eff}}}.$$

This rescaling procedure keeps the physical problem unchanged [$\exp(2g_{AN}) = \exp(2\tilde{g}_A L''_N)$, for example] whether the rescaling is done because $\epsilon_0 \neq 1$ or because the integral multiple condition (b) is not initially satisfied or both.

The code automatically carries out the above procedure for all media. This is done for the initially read-in data as well as for new media data read in at a restart (Section III.D below).

2. Rezoning Subroutines. Rezoning subroutines are used to change Δr and $\Delta z(N)$ when the pulse's radial size, as determined by the mean square radius, $\langle r^2 \rangle$ changes:

$$\langle r^2 \rangle = \frac{\int 2\pi r dr dz \, r^2 \frac{\sqrt{\epsilon_0}}{8\pi} |E(r, z, t)|^2}{\int 2\pi r dr dz \, \frac{\sqrt{\epsilon_0}}{8\pi} |E(r, z, t)|^2}.$$

If $\langle r^2 \rangle$ changes by a factor of 4 from its previous value (after the last rezoning, or at the initial time if there have been no rezonings), Δr is changed by a factor of 2; it is multiplied by 2 if $\langle r^2 \rangle$ quadrupled, divided by 2 if $\langle r^2 \rangle$ diminished. According to the numerical "group velocity" expression Eq. (17), $\Delta z(N)$ must also be changed to keep the correction factor $f(\phi)$ unchanged. Hence, $\Delta z(N)$ and the propagation step sizes for all succeeding media are either multiplied or divided by 4 at a rezoning to keep the quantity N of Eq. (17d) unchanged. All medium lengths are readjusted to be integral multiples of the new propagation step sizes, and the rescaling procedure of Section II.C.1. is carried out to leave the physical problem unchanged after the length of each medium has been changed.

We note that, even though the diffraction accuracy factor $f(\phi)$ in Eq. (17b) has been kept unchanged by this rezoning procedure, the actual transverse wavelengths (in cm) in the diffraction pattern which are propagated well have changed by a factor of 2 due to the change in Δr via Eq. (17c). Also, even though the new lengths of each medium are integral multiples of the new $\Delta z(N)$, there is no guarantee that after the rezoning the distance from the right

boundary of the window to the right boundary of the current medium (that the window is in) is exactly divisible by the new $\Delta z(N)$ of that medium. Hence, the pulse may miss the boundary of the current medium by some fraction of the new $\Delta z(N)$ (which may be 4, 16, 64, 256, $\dots$ times the original $\Delta z(N)$ of that medium; obviously this difficulty does not occur if $\Delta z(N)$ is reduced in the rezoning). Additional violence occurs in succeeding media. The chances are one in four that, for a one z-slice pulse, the medium boundary will be hit exactly (if only one rezoning occurs while the pulse is in that medium). One can repeat a problem and adjust lengths so that this happens, but computer storage requirements have made it impossible to have it done automatically.

Hence, for the above reason, one should never rezone in both r and z when using a finite (many z-slice) pulse. Such a rezoning - even if by chance the right window boundary should exactly hit the medium right boundary - would yield different errors for each z-slice (if the rezoning were done in an active medium). With appropriate input data the user may choose to rezone in r only [$\Delta z(N)$ left unchanged], or not at all. Each time a rezoning of any kind is done, an appropriate print is produced to mark the event.

The radial rezoning is accomplished as follows: on rezoning upward, the known fields $E_{K,J}^n$ for $K = 2, 4, 6, \dots, K_m$ [which we associated with radii $(K - \frac{1}{2})\Delta r$] are placed on the new mesh (at radii $(K - 1)\Delta r$, $K = 2, 4, 6, \dots, K_m$). This process shrinks the whole pulse width by $\frac{1}{2}\Delta r$, a totally negligible amount if several hundred radial points are used. This has been found to work better than linear interpolation for new field values in this case. It does do some violence to the first point on the new mesh. From $r = K_m\Delta r$ to $r = 2\Delta r K_m$, the last field value E_{K_m} , for $K = K_m$ [at $r = (K_m - \frac{1}{2})\Delta r$], is repeated.

On rezoning downward, new points are obtained by interpolation. We note here that the rezoning described in this section is initiated only by changes in the pulse's rms radial dimension $\langle r^2 \rangle^{\frac{1}{2}}$. It is distinct from the procedure used to reduce $\Delta z(N)$ by a factor of 2 if certain denominators of the basic diffraction difference scheme become too small.

3. Lens Subroutines. The following special subroutines are all called lens routines. Some of them modify the pulse - such as the thin lens model - while others merely sample but do not alter it. The sampling routines are useful diagnostic aids. Each lens should be assigned a length of one $\Delta z(N)$; otherwise they will act more than once on the pulse.

a. Lens 1. Lens 1 simulates the effect of an ideal thin lens of focal length F with an annular aperture of inner (outer) radius R_{in} (R_{out}). The plane of the lens is at some location $z = z_0$ and it has the following effect on each z -slice of a pulse as the pulse propagates past that point:

$$E(r, z_0, t) \rightarrow E(r, z_0, t) e^{ikr^2/2F} F(r) , \quad (50a)$$

$$F(r) = \exp \left\{ - \left(1 - \frac{1}{f(r)} \right)^2 \right\} , \quad (50b)$$

and

$$f(r) = \left[\frac{1}{2} - \frac{1}{\pi} \tan^{-1} \left(\frac{r - R_{out}}{\eta \Delta r} \right) \right] \left[\frac{1}{2} - \frac{1}{\pi} \tan^{-1} \left(\frac{R_{in} - r}{\eta \Delta r} \right) \right] . \quad (50c)$$

If $\eta > 1$, the edges of the aperture are somewhat softened to simulate an apodizing aperture. Note that $F > 0$ ($F < 0$) corresponds to a converging (diverging) lens so that this phase convention yields a concave (convex) curve when the phase is plotted as a function of radius.

There are some limits on the phase front curvature that can be imposed by this lens routine. From Eq. (50a) the value of the phase function increases quadratically in radius. The code cannot handle curvatures which increase the phase by more than $\pi/2$ over the last radial zone of the lens. That is, $\Delta\psi < \pi/4$ where

$$\Delta\psi = \frac{k}{2F} \left[R_{max}^2 - (R_{max} - \Delta r)^2 \right] \cong \frac{2\pi}{\lambda F} R_{max} \cdot \Delta r .$$

This condition implies that the focal length F^* should satisfy the condition

$$|F| < \frac{8 \cdot R_{max} \Delta r}{\lambda} .$$

Severe violation of this condition will cause the diffraction calculation to break down in accord with Eq. (17) and the accompanying comments.

* We use $|F|$ as the condition applies to both converging and diverging lenses.

One notes that problems may also arise if the change in beam size due to a lens-induced phase curvature occurs more rapidly than the numerical scheme can propagate transverse diffraction ripples. Thus, the beam may collapse near a focus before any edge-induced diffraction ripples can propagate into the axis. Under such circumstances it is possible to calculate focal patterns using the Lens 4, 5, and 6 subroutines described below.

b. Lens 2. A radial "burn pattern" is available through this routine which computes the energy per unit area $A(r, z_0)$ as a function of radius summed over the entire pulse as it passes through the plane $z = z_0$.

$$A(r, z_0) = \int_{-\infty}^{\infty} dt \frac{c}{8\pi} |E(r, z_0, t)|^2 \quad (51a)$$

and

$$A_K = \frac{\Delta z}{8\pi} \sum_{\ell=0}^{J_{\max}-1} |E_{K, J_{\max}-\ell}^{n+\ell}|^2 \quad (51b)$$

$$B(r, z_0) = 2\pi \int_0^r dr' r' A(r', z_0) \quad (52A)$$

and

$$B_K = 2\pi(\Delta r)^2 \sum_{\ell=0}^K (\ell - \frac{1}{2}) A_{\ell} \quad (52b)$$

$B(r, z_0)$ is obviously the total energy in the pulse within a radius r . This lens routine only samples field values of the pulse, it does not modify them.

c. Lens 3. The temporal shapes of the total power of the pulse $S(t)$ and of the total energy $T(t)$, as measured by a detector in the plane $z = z_0$, are provided by this subroutine.

$$S(t) = 2\pi \int_{R_{in}}^{R_{out}} dr' r' \frac{c}{8\pi} |E(r, z, t)|^2, \quad (53a)$$

$$S_{\ell} = 2\pi(\Delta r)^2 \frac{c}{8\pi} \sum_{K=K_{in}}^{K_{out}} \left| E_{K, J_{max}}^{n+\ell} - \ell \right|^2, \quad (53b)$$

$$R_{in} = (K_{in} - \frac{1}{2})\Delta r, \quad R_{out} = (K_{out} - \frac{1}{2})\Delta r, \quad (53c)$$

$$T(t) = \int_0^t dt' S(t'), \quad (54a)$$

and

$$T_n = \frac{\Delta z}{c} \sum_{m=1}^n S_{\ell}. \quad (54b)$$

Obviously, this information can be provided over either the entire pulse or just an annular section specified by R_{in} and R_{out} .

d. Lens 4. The energy per unit area as a function of radius for each z-slice of the pulse after it passes through an ideal thin lens of focal length F located at $z = z_0$ is computed by this subroutine. The Kirchhoff-Fresnel integral is used. The energy density of each z-slice is computed in a plane located a distance z_1 beyond the plane of the lens $z = z_0$; it is not necessary to choose $z_1 = F$.

$$E(R, z_0 + z_1, t) = \left(-\frac{ia}{\pi} \right) e^{iaR^2} \int_0^{\infty} dr r 2\pi J_0(2arR) e^{-i(a-b)r^2} E(r, z_0, t), \quad (55a)$$

$$a = \frac{\pi}{\lambda z_1}, \quad (55b)$$

and

$$b = \frac{\pi}{\lambda F}. \quad (55c)$$

We use a script E and capital R to indicate that these quantities are not the electric field of a z-slice E nor the usual radial coordinate r ; in particular, this lens subroutine samples but does not modify z-slice electric fields E .

Note that if $F > 0$ and $z_1 \approx F$, the focal pattern will be appreciable only over a very small radial extent; hence, a new radial grid, which starts at $R = 0$, is used:

$$R = (\ell - 1)\Delta R, \quad \ell = 1, 2, \dots, \ell_{\max} \quad (55d)$$

Also, J_0 is the zero-order Bessel function. In terms of actual machine quantities we have

$$E_{J,\ell} = (-2ia) e^{ia(\Delta R)^2(\ell-1)^2} \sum_{K=1}^{K_{\max}} (\Delta r)^2 (K - \frac{1}{2}) J_0[2a\Delta r\Delta R(K - \frac{1}{2})(\ell - 1)] \\ \times e^{-i(a-b)(\Delta r)^2(K-\frac{1}{2})^2} E_{K,J}^n \quad (55e)$$

This is done for every z -slice, labeled by J , in the pulse as it passes $z = z_0$. Then the focused energy density EN of each z -slice is computed from

$$EN(R, z_0 + z_1, t) = \frac{\Delta z}{8\pi} |E(R, z_0 + z_1, t)|^2 \quad (56a)$$

$$EN_{J,\ell} = \frac{\Delta z}{8\pi} |E_{J,\ell}|^2 \quad (56b)$$

Lens 4 also computes the enclosed energy $\delta(R, z_0 + z_1, t)$ for the focal pattern of each z -slice and sums it over the whole pulse $q(R)$:

$$\delta(R, z_0 + z_1, t) = 2\pi \int_0^R dR' R' EN(R', z_0 + z_1, t) \quad (57a)$$

$$\delta_{J,\ell} = 2\pi(\Delta R)^2 \sum_{m=1}^{\ell} EN_{J,m} \quad (57b)$$

$$q(R) = \int_{-\infty}^{\infty} dt \delta(R, z_0 + z_1, t) \quad , \quad (58a)$$

and

$$q_\ell = \sum_{J=1}^{J_{\max}} \delta_{J,\ell} \quad . \quad (58b)$$

This information is displayed on $J_{\max} + 1$ film frames with both $EN_{J,\ell}$ and $\delta_{J,\ell}$ on the first $J_{\max}$ frames and q_ℓ on the last frame. The phase of the focused electric field of each z -slice, derived from $E_{J,\ell}$, is displayed along with the energy density and enclosed energy. Again we note that this subroutine only samples $E_{K,J}^n$ values from the pulse but does not change them. By computing the focal patterns, one can use Lens 4 as a test of the optical quality of the pulse anywhere along the optical system through which it is propagating.

e. Lens 5. An obvious disadvantage of using Lens 4 (rather than the regular numerical scheme) to compute the focal pattern of a pulse is that the best focus may not coincide with the nominal focal point of the ideal thin lens because of aberrations present in the pulse. To locate the best focus of a finite pulse, we first look only at the on-axis focused energy per unit area for each z -slice and for the total pulse via the Lens 5 subroutine. That is, we look at $EN(R=0, z_0 + z_1, t)$, Eq. (56a), for each z -slice for some range of values of z_1 near the estimated best focus. Unless severe aberration is anticipated, the range of values of z_1 usually includes the nominal focus $z_1 = F$. Therefore, Lens 5 plots $EN_{J,1}$ [Eq. (56b)] for various values of z_1 and for all z -slices J . It also plots q_1 [Eq. (58b)] for the same set of z_1 values and thus presents the focused energy density on axis summed over the whole pulse. One can then decide whether the peak of this latter curve or the peak of some particular z -slice curve represents the best focus of the pulse and choose that value of z_1 to evaluate the radial focal patterns with Lens 4. This of course involves stopping the calculation while one examines the Lens 5 output and then continuing through a Lens 4 with the appropriate value of z_1 . We emphasize again that neither Lens 4 nor Lens 5 actually modifies the values $E_{K,J}^n$; they only sample them.

f. Lens 6. We have repeatedly emphasized that the Lens 2 to 5 sub-routines do not modify the values of $E_{K,J}^n$ but only sample them. Lens 1 obviously does modify these values. It is occasionally convenient to use the Kirchhoff-Fresnel integral simply to propagate the pulse itself over large air gaps. This is done via the Lens 6 subroutine which is formally similar to Lens 4 except that the field values $E_{K,J}^n$ of each z-slice are replaced by the newly computed (propagated) values - which we denote by $E_{K,J}^{n'}$. Note that the radial mesh is unchanged during the propagation.

$$E_{K,J}^{n'} = (-2ia) e^{ia(\Delta r)^2(K-\frac{1}{2})^2} \sum_{m=1}^{K_{\max}} (\Delta r)^2 (m - \frac{1}{2}) J_0 \left[2a(\Delta r)^2 (m - \frac{1}{2}) (K - \frac{1}{2}) \right] \\ \times e^{-i(a-b)(\Delta r)^2(m-\frac{1}{2})^2} E_{m,J}^n . \quad (59)$$

Here, a and b are given by Eqs. (55b) and (55c), and z_1 is the air gap length over which the propagation is done. If no lens is involved, $b = 0$ ($F = \infty$). After propagation, the $E_{K,J}^{n'}$ are propagated by the usual numerical scheme of Section II.A.

One can prove easily that the Kirchhoff-Fresnel integral [Eq. (55a)] is the solution of Eq. (9) in vacuum [$X(r,z,\tau) \equiv 0$] by writing both the quasi-optical equation and the Kirchhoff-Fresnel integral in rectangular coordinates.

$$2i \frac{\omega}{c} \frac{\partial E}{\partial z} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) E \quad (60a)$$

$$E(x,y;z) = \left(\frac{-ia}{\pi} \right) \int dx' dy' E(x',y';0) e^{ib(x'^2+y'^2)} e^{-ia[(x-x')^2+(y-y')^2]} . \quad (60b)$$

Again, a and b are given by Eqs. (55b) and (55c). Direct differentiation of Eq. (60b), using $\omega/c = 2za$, shows that it is a solution in free space of Eq. (60a).*

* Recall that we have omitted a plane wave factor in Eq. (5a) so that $E(r,z,t)$ is only the envelope function.

This terminates the discussion of the special subroutines in LAPU2. There are of course many other subroutines (28 in all) which handle various input-output problems, such as two-dimensional and three-dimensional plots. The input-output data are the subject of the next section of this report. In addition to the subroutines of LAPU2, the code of course does use many system routines which are unique to the LASL computing facility.

III. INPUT-OUTPUT DATA FOR LAPU2

In this section we reveal how a problem is specified to LAPU2. The general method is through data cards, of which there are four types: (A) general information, (B) medium information, (C) END card, (D) RESTART information. Since the input data specifies what output the user desires, the output forms will be included in the discussion. All variables have default values, which the code uses automatically if either a blank or a zero is encountered in the appropriate fields of the data cards unless these values are changed by the data.

A. General Information Data

This consists of up to 11 cards, each of which is numbered and therefore can appear in any order in the data deck. We discuss each card separately.

CARD 1: This is an 80-column header card; all alphanumeric data in 80 columns are printed. Problem name and number and other relevant identification features may be entered here.

All remaining data cards are divided into 6 fields. Each field recognizes numbers in an E12.7 format; that is, +11234567+12 is recognized as +1.1234567 $\times 10^{12}$.

CARD 2:

<u>Field</u>	<u>Default</u>	<u>Function</u>
1	none	+2.0 must appear to identify card
2	0.0	active medium type: +1.0 for two-level, +0.0 for four-level
3	ZMEDA(NM)	ZLAST, total length of physical system (cm)
4	1	NM, number of media (must be ≤ 100)
5	0.0	ZBI, initial left boundary for window plot
6	0.002	Δr (cm)

The length of each medium is entered in the medium data below; the code automatically computes the right-hand boundary of each medium. If NM media are

read in, the total length, ZLAST, of the system is taken to be the right-hand boundary of medium NM, ZMEDA(NM), unless it is changed in field 3. If ZLAST is accidentally set smaller than the total length of the problem read in, the code will not propagate the pulse beyond ZLAST. The code assumes that the first medium begins at $z = 0$ unless ZBI is changed in field 5. Normally Δr is constant throughout a run.

CARD 3:

<u>Field</u>	<u>Default</u>	<u>Function</u>
1	none	+3.0 must appear to identify card
2	1	$J_{\max}$, number of z-slices in pulse
3	two-level - 1000 four-level - 750	KMAXM, maximum number of radial grid points
4	1	JLEFT, number of the z-slice on trailing edge
5	1	JRIGHT, number of the z-slice on leading edge
6	0.0	SWBD, boundary switch: 0 = reflecting, +1 = weakly absorbing

The length of the window is fixed at $(JRIGHT - JLEFT + 1)\Delta z$. The total number of mesh points in the window is limited by computer storage requirements. This number cannot exceed 50 000 if the two-level gain model is used; it cannot exceed 28 500 if the four-level gain model is used. In view of the separate limitation on the maximum number of radial grid points for these two cases (KMAXM, field 3), this means that if maximum radial resolution is used there can at most be 50 (38) z-slices in the pulse for cases using only the two-level (four-level) gain model. The J index of the z-slice increases in going from the trailing (late-time) edge toward the leading (early-time) edge of the pulse.

CARD 4:

<u>Field</u>	<u>Default</u>	<u>Function</u>
1	none	+4.0 must appear to identify card
2	1.778527×10^{14}	ω , (circular) carrier frequency of pulse
3	1.0	ϵ_{00} , scaling dielectric constant, Sec. II.C.1
4	0.5	θ , numerical stability parameter, Appendix A
5	1.0×10^5	AMPMAX, maximum allowable electric field amplitude
6	10.0	PHMAX, maximum allowable electric field phase

The carrier frequency ω is defaulted to the P(20) line of the (001)-(100) transition in CO_2 . The code stops if either AMPMAX or PHMAX is reached. Since the electric field phase is always between $-\pi$ and $+\pi$, the latter default is

never reached. The default for AMPMAX corresponds to an intensity in vacuum of 1.9285×10^{12} watts/cm² (cgs units are used throughout; $I = (c\sqrt{\epsilon_0}/8\pi) \times |E|^2$ ergs cm⁻² s⁻¹; $\epsilon_0 = 1$ in vacuum).

CARD 5:

Field	Default	Function
1	none	+5.0 must appear to identify card
2	1.0	WTOTRD, total energy of initial pulse (joules)
3	0.1	SMA, radial width of initial Gaussian pulse (cm)
4	0.0	FPERT, initial perturbation pulse amplitude ratio
5	0.1	FPERT2, initial perturbation pulse radial width ratio
6	none	none

The initial ($t = 0$) pulse is Gaussian in z and r and may have another Gaussian perturbation superimposed. The explicit form is

$$E(r, z, t = 0) = E_0 e^{-[(z-z_0)/b]^2} \left[e^{-(r/a)^2} + \text{FPERT} e^{-(r/\text{FPERT2} \cdot a)^2} \right] \times e^{i(\omega r^2 \text{CAPR}/2v)} \left[\frac{1}{2} - \frac{1}{\pi} \tan^{-1} \left(\frac{r - AC}{\Delta r} \right) \right]. \quad (61)$$

The amplitude E_0 is chosen by the code which numerically normalizes the total energy to WTOTRD in joules:

$$\text{WTOTRD(joules)} = \frac{\sqrt{\epsilon_0}}{8\pi} \int 2\pi r \, dr \, dz |E(r, z, 0)|^2 \cdot 10^{-7}. \quad (62)$$

The parameters z_0 , CAPR, and AC are entered on the next data card. Notice that the radial full width at half maximum intensity (in the absence of a perturbation) is given by $a\sqrt{2 \ln 2}$, while the temporal full width at half maximum intensity is given by $(b/v)\sqrt{2 \ln 2}$ seconds, and $v = c/\sqrt{\epsilon_0}$ is the light phase velocity in the medium. If the cutoff parameter AC is small, obviously it determines the initial radial profile.

CARD 6:

<u>Field</u>	<u>Default</u>	<u>Function</u>
1	none	+6.0 must appear to identify card
2	10.0	AC, initial pulse cutoff (cm)
3	0.0	CAPR, reciprocal of initial pulse focal length
4	1.0×10^3	SMB, determines initial pulse temporal width
5	$\frac{1}{2} J_{\max} + 1$	IZC, z-slice number at peak axial intensity of initial pulse
6	$\Delta z(1)$	DZMIN, smallest allowable Δz for problem

Referring again to Eq. (61), AC truncates the pulse in radius. If AC is larger than the radius of the first medium (read in on the first medium data card below), it has no effect. The initial pulse may have a focal length whose reciprocal is specified by CAPR, field 3, while SMB is the b in Eq. (61). IZC is related to z_0 in Eq. (61); if $J_{\max}$ is an odd number, IZC will be the middle z-slice. DZMIN is chosen automatically to be the step size $\Delta z(1)$ of the first medium.

CARD 7:

<u>Field</u>	<u>Default</u>	<u>Function</u>
1	none	+7.0 must appear to identify card
2	20	ISPRT, short print interval in cycles
3	1000	ICPRT, long print interval in cycles
4	10	IPLTI, plot interval in cycles in nonmovie mode
5	100 000	IRSDMP, dump interval in cycles
6	1000	IHOL, hologram plot in cycles

One computational cycle consists of advancing the pulse by one $\Delta z(N)$ for whatever medium N the pulse may be in. The short print, every ISPRT cycles, provides the following information: the type of problem (two-level medium is called "glass", four-level medium is called "CO₂"), the current time in seconds; $\Delta t = \Delta z(N)/v$; the cycle number; the location in z of the left (trailing edge) and right (leading edge) boundaries of the window; the initial and current total pulse energies in joules; and the initial and current values of three parameters (CAPB, CAPC, ALPHA MAX) related to the denominators of the numerical scheme. The sum of the intensity gain (CKI) over all cells of the window is also shown.

The long print, every ICPRT cycles, includes a short print and a print of the real $[E(R)]$ and imaginary $[E(I)]$ parts of the electric field of up to three specified z-slices of the pulse. The current gain (CKI), the electric field amplitude, and the energy within a given radius are also printed as a function of radius on the long print. Which z-slices are printed is controlled by parameters entered on CARD 8.

A plot on film occurs every ICPLTI cycles. This presents a display of the axial location of the window relative to the whole system as well as a plot of the pulse energy density and enclosed energy vs radius. The phase of z-slice number IZC is also plotted. Since this information is collected at a fixed time, not a fixed spatial point $z = z_0$, the energy density is somewhat artificial. Specifically, it is $EN(r)$ given by

$$EN(r) = \int dz \frac{\sqrt{\epsilon_0}}{8\pi} |E(r, z, t)|^2 , \quad (63)$$

and the enclosed energy $ENC(r)$ is simply

$$ENC(r) = \int_0^r 2\pi r' dr' EN(r') . \quad (64)$$

The time of the plot is also shown. $EN(r)$ is plotted relative to the left side of the graph; the maximum plotted value multiplied by 1.1 is recorded as E/cm^2 . $ENC(r)$ is plotted relative to the right side and its maximum value is stated as $EMAX$ (in ergs).

LAPU2 has the capability of stopping in the middle of a calculation and dumping onto tape all information necessary to restart the problem later on. The dumping occurs every IRSDMP cycles. How to restart from such a dump is explained in Section III.D. below.

The holographic plot, controlled by IHOL, is a special plot of the energy density vs radius for every z-slice. By using the technique¹¹ invented by Dickman, Winkler, and Crook these plots can be converted into a three-dimensional hologram which shows the intensity vs r and z at a fixed time for the whole pulse.

CARD 8:

Field	Default	Function
1	none	+8.0 must appear to identify card
2	$\frac{1}{2} J_{\max} + 1$	ICENT, z-slice number for long print
3	0	ISKIP, two other z-slices for long print
4	1	IPSK1, radial skip for 3D plot
5	1	IPSK2, z-slice skip for 3D plot
6	1	IPSK3, z-slice skip for hologram plot

The three z-slices whose electric field values are printed on long prints are labeled by ICENT, ICENT + ISKIP, and ICENT - ISKIP. The three-dimensional plot, described under the next card, cannot plot more than 5000 points. Hence, it may be necessary to skip some radial points or some z-slices or both in order to satisfy this limit. The hologram plot may also not involve every z-slice of the pulse.

CARD 9:

Field	Default	Function
1	none	+9.0 must appear to identify card
2	0.0	SW1, movie switch: +1 = movie plot every cycle; 0 = no movie
3	0.0	SW2, 3D: 0 = no 3D plots; +1 = front viewpoint, -1 = rear viewpoint
4	0.0	SW3, scale of energy density $EN(r)$ [Eq. (63)] in 2D plots: 0 = use computed maximum, x = fix scale at x times initial maximum
5	0.0	SW4, scale of electric field amplitude in 3D plots: 0 = use computed maximum, x = fix scale at x times initial maximum
6	0.0	SW5, long print interval: 0 = use ICPRT, 1 = use CPAR table of card 11

One possible form of film output is a movie which shows a three-dimensional plot of the pulse in the window. The electric field amplitude is plotted vs radius r and axial position z . As the pulse propagates through a sequence of different media - which can be visually differentiated by plotting in different colors - it continuously changes its shape due to diffraction and active media effects. If a movie form of output is selected by setting SW1 = 1, no other type of film output will be generated. Also, one cannot turn off the movie switch until the problem is completed.

SW2 controls the three-dimensional plotting in which the field amplitude is plotted vs radius r and axial position z . Since the pulse deforms due to the action of active media, one can choose the perspective of the three-dimensional plot as viewed either from ahead of the leading edge of the pulse (SW2 = +1) or from behind the trailing edge (SW2 = -1). In either case, the three-dimensional plot appears along with the two-dimensional plot every IPLTI cycles (CARD 7, field 4).

The vertical scales of both two- and three-dimensional plots can either be based on the current maximum values of the pulse or some multiple of the initial pulse maximum values. In the latter case, the plotted amplitudes of the graphs grow as the pulse is amplified, but the plots are truncated if the values exceed the specified maximum scale values; in the former case the plotted curves are scaled to always occupy the maximum available dimensions on the film frame. The choice of which way the code will handle the vertical axis scaling is made by fixing SW3 and SW4.

Occasionally one would like to have long prints at particular cycle numbers in a problem. If SW5 = 1, the code does long prints only at those cycle numbers listed on CARD 11 below.

CARD 10:

<u>Field</u>	<u>Default</u>	<u>Function</u>
1	none	+10.0 must appear to identify card
2	0.0	none
3	0.0	none
4	0.0	none
5	0.0	none
6	0.0	automatic rezoning: 0.0 = no rezoning, +1.0 = rezone, -1.0 = rezone in r only

Most of the fields on this data card are as yet unused; they were reserved for special output functions which have not been implemented in the code. The last field controls automatic rezoning in r and z (Section II.C.2) if the radial size of the pulse changes appreciably during a problem. No rezoning is done if the default value is used; +1.0 tells the code to change both Δr and Δz to keep the numerical "group velocity" curve [Eq. (17)] unchanged (note that the diffraction resolution of transverse ripples of wavelength λ_t is changed, Eq. (17c), due to a change in Δr); -1.0 tells the code to change Δr but not Δz . The criterion for rezoning upward or downward is that the mean radius of the pulse

$\langle r^2 \rangle^{1/2}$ change by a factor of 2 from its previous value (the value at the last rezoning or, if there have been no rezonings, its initial value). Here $\langle r^2 \rangle$ is defined by

$$\langle r^2 \rangle = \frac{\int 2\pi r' dr' dz r^2 \frac{\sqrt{\epsilon_0}}{8\pi} |E(r', z, t)|^2}{\int 2\pi r' dr' dz \frac{\sqrt{\epsilon_0}}{8\pi} |E(r', z, t)|^2} \quad (65)$$

Note that the code will automatically divide $\Delta z(N)$ by 2 if the denominators in the numerical scheme get too small (Appendix A).

CARD 11:

<u>Field</u>	<u>Default</u>	<u>Function</u>
1	none	+11.0 must appear to identify card
2	none	CPAR(1)
3	none	CPAR(2)
4	none	CPAR(3)
5	none	CPAR(4)
6	none	CPAR(5)

This card is used along with SW5 on CARD 9 to specify long prints at specific cycle numbers CPAR(I). Here, $I \leq 20$, so that up to 20 specific cycle numbers on 4 cards can be given. In contrast to the general rule that each card of this group has an identification number and thus can be placed anywhere in the deck, all 11 cards must follow the 9 card if this option is selected.

B. Medium cards

Up to 100 different media can be specified at the beginning of a problem (the total number of media that the pulse propagates through is unlimited in principle although those media beyond 100 must be handled through the dump and restart procedure specified in Section III.D. below). Each medium requires four data cards. The media are not numbered on the data cards, so the user must be certain that the order of appearance of various media in the data card deck agrees with the correct order of different media along the propagation path of the pulse in the physical problem under study. We discuss each of the required four cards:

MEDIUM CARD 1:

Field	Default	Function
1	none	-1.0 must appear to identify card
2	1.0	$\Delta z(N)$, propagation step for this medium (cm)
3	KMAXM	KMAXA, number of radial points for this medium
4	1000.0	ZLA, length of this medium (cm)
5	0.0	ILENSA, medium type identifier
6	0.0	CCOLA, color of this medium for 3D plots

The number of radial points is defaulted to KMAXM depending on the active medium type specified on CARD 2, field 2; KMAXM is 1000 if the two-level gain model is used, 750 if the four-level gain model is used. ILENSA = 0 specifies that this medium is either the two- or four-level system model (of course, if the gain, specified on MEDIUM CARD 3 below, is entered as zero, the medium acts passively; if ϵ_0 , specified on MEDIUM CARD 2 below, is entered as one, the medium acts like vacuum).^{*} If ILENSA = I, where I is 1, 2, 3, 4, 5, or 6, the medium is the corresponding LENS I of Section II.C.

ZLA for the first medium can be longer, but not shorter, than the length of the window, $(JRIGHT - JLEFT + 1)\Delta z$. $\Delta z(1)$ sets the thickness Δz of each z-slice of the pulse. The code requires that $\Delta z = \Delta z(1) = 2^p$ cm, where p is a positive or negative integer. ZLA must be an integral multiple of $\Delta z(1)$ or the code readjusts it. The first medium should not be specified to be an amplifier (the initial gain on MEDIUM CARD 3 below should be set to 0 for the first medium or the calculation will be done incorrectly).

$\Delta z(N)$ for any other medium must be an integral multiple of $\Delta z = \Delta z(1)$. For all active media with nonzero initial gain, $\Delta z(N) \leq \Delta z$ necessarily (Section II.B.1). ZLA for any medium must be an integral multiple of $\Delta z(N)$ for that medium; otherwise the code readjusts it and rescales all relevant quantities. If $\epsilon_0 \neq 1$ for any medium in the problem, the rescaling procedure of Section II.C.1 is implemented for all media for which $\epsilon_0 = 1$, and the initial-conditions printout notes that this rescaling has been done. Thus the length ZLA of every medium is guaranteed to be an integral number of propagation steps $\Delta z(N)$ of that medium, and $\Delta z(N)$ in turn is an integral number (one in the case of active media) of the thickness of each z-slice Δz .

^{*} Note that either one or the other gain model, but not both, can be used in a problem; by appropriate choice of the parameters N_1 and N_2 , the four-level system can be effectively reduced to the two-level model.

MEDIUM CARD 2:

Field:	Default	Function
1	none	-2.0 must appear to identify card
2	0.0	CKIM2A, two-photon absorption constant β
3	1.0	FLKA, electric field amplitude transmission factor
4	1.0	ALPHAA, dimensionless kinetics parameter
5	0.0	ESP2A, nonlinear dielectric constant ϵ_2
6	1.0	ESPZA, linear dielectric constant ϵ_0 or Lens 1 smear factor η

Reflection at the boundary of a medium can be simulated by FLKA, which multiplies each electric field value as each z-slice enters the medium. ALPHAA is the dimensionless parameter α in the gain models of Eq. (21b) for the two-level system and Eq. (22) for the four-level system. ESP2A is the nonlinear dielectric constant ϵ_2 (cgs units) of Section II.B.2. ESPZA is the linear dielectric constant of the medium ϵ_0 (a dimensionless number in cgs units, not the dielectric constant of vacuum in mks units) except if the medium is identified on the previous card as a Lens 1; in that case, this parameter is the η parameter for that subroutine, Eq. (50). CKIM2A is the two-photon absorption constant β in Eq. (18).

MEDIUM CARD 3:

Field:	Function		
1	-3.0 must appear to identify card		
	<u>No Lens</u>	<u>Lens 1 - 3</u>	<u>Lens 4 - 6</u>
2	$g(0.0)$		
3	$\sigma(0.0)$		
4		$\frac{1}{F} (0.0)$	
5		$R_{in} (-1.0)$	$J_{in} (1)$
6		$R_{out} (R_{max})$	$J_{out} (J_{max})$

MEDIUM CARD 3 supplies data for either active media or lens routines. If the medium is identified by ILENSA on the first medium data card as an active medium, the initial small signal intensity gain coefficient g (in cm^{-1}) is read in field 2, and the stimulated emission cross section σ is read in field 3. All other fields are ignored. If the medium has been identified as Lens 1, 2, or 3, fields 2 and 3 are ignored. Field 4 contains the reciprocal $1/F$ of the

lens focal length [F in cm, > 0 (< 0) for a converging (diverging) lens]. The clear annular aperture of the lens is specified by R_{in} and R_{out} . If the medium is identified as a Lens 4, 5, or 6, fields 5 and 6 give J_{in} and J_{out} , respectively. If the medium is identified as Lens 6, the default values should always be used for J_{in} and J_{out} .

MEDIUM CARD 4:

Field	Function				
1	-4.0 must appear to identify card				
	<u>No Lens</u>	<u>Lens 1-3</u>	<u>Lens 4</u>	<u>Lens 5</u>	<u>Lens 6</u>
2	$f(0.0)$				SMFA(0.0)
3	$\tau_1(1.0)$		R4MAX	ZMIN(0.9FL)	R4MAX(0.9FL)
4	$\tau_2(1.1)$		NI(200)	NZ(200)	NI(200)
5	$N_1(1.0)$		ZBD(FL)	ZMAX(1.1FL)	ZBD(1.1FL)
6	$N_2(1.0)$		FL(100RMAXA)	FL(100RMAXA)	FL(100RMAXA)

This medium card also contains data for active media and lens routines. If the medium is identified by ILENSA on the first medium data card as an active medium, either of two things can happen: if the medium is the two-level model, all numbers are ignored except for τ_1 , field 3, which specifies the relaxation time T in Eq. (19); if the medium is the four-level model, the numbers in fields 2 to 6 are identified with the corresponding quantities f , τ_1 , τ_2 , N_1 , and N_2 as in Eqs. (22) and (23). In this latter case, we must caution the user not to make τ_1 equal τ_2 exactly if the defaults are not used; failure to do this will cause a divide by zero, as pointed out after Eq. (C.12). All relaxation times are in seconds.

The data on this card are not used at all by Lens 1-3. For Lens 4, the data in fields 3 to 6 fix values of R , ℓ_{max} , z_1 , and F of Section II.C.6, Eq. (55). The default value for R is obtained as follows: let $RMAXA = KMAXA \cdot \Delta r$, where $KMAXA$ is the number of radial points chosen on MEDIUM CARD 1. We define the effective Fresnel number N_f by

$$N_F = \frac{(RMAXA)^2}{\lambda \cdot FL} \left(\frac{FL}{ZBD} - 1 \right) ,$$

where $z_1 = \text{ZBD}$, $F = \text{FL}$,* and $\lambda = 2\pi c/\omega$ is the vacuum wavelength of the light. Then,

$$\text{if } N_F \geq 1, \quad R = R4\text{MAX} = 1.5 \cdot R\text{MAXA} \cdot \left| 1 - \frac{\text{ZBD}}{\text{FL}} \right|$$

and

$$\text{if } N_F < 1, \quad R = R4\text{MAX} = 2.44 \cdot \lambda \cdot \frac{\text{FL}}{R\text{MAXA}}$$

The default value of $\ell_{\text{max}} = \text{NI}$ is 200 for this Lens 4.

If the medium is identified as a Lens 5, the numbers in fields 3-6 of this data card are interpreted differently. ZMIN (defaulted at 0.9 FL) is the minimum distance from the plane of the lens to the first axial point at which the focused intensity is calculated; ZMAX (defaulted at 1.1 FL) is the maximum distance from the plane of the lens to the last axial point at which the focused intensity is calculated; the number of axial points between ZMIN and ZMAX at which the focused intensity is calculated is given by NZ (defaulted at 200) while the lens focal length is given by FL (defaulted at $100 \cdot R\text{MAXA}$).

If the medium is identified as a Lens 6 and if SMFA is defaulted to 0.0, the propagated field starts at $1/2 \Delta r$, as with the usual numerical scheme; if $\text{SMFA} \neq 0.0$, the first field point starts at $r = 0$ (this is done to allow for use of the Kirchhoff-Fresnel integral as the method of propagation throughout a whole problem; if $\text{SMFA} = 0$, the propagated field is computed on the same radial mesh $r_K = (K - 1/2) \Delta r$ ($K = 1, 2, \dots, K\text{MAXA}$) as the tridiagonal numerical method has used up to that point). The default values should never be used for the Lens 6 routine. One should instead choose $R4\text{MAX} = K\text{MAXA} \cdot \Delta r$ and $\text{NI} = K\text{MAXA}$ ($K\text{MAXA}$ here is that value used for the previous medium). The propagation distance ZBD and focal length FL may then be chosen as desired. Lens 6 is frequently used to propagate the pulse over long air gaps, although it can be selected, by setting $\text{SMFA} \neq 0$, to be the primary method of propagation in pure diffraction problems (such as an iterative Fox-Li resonator calculation).

This completes a discussion of the four data cards required for each medium in a problem. Again, up to 100 different media can be entered at the

* These quantities are defaulted at $\text{ZBD} = \text{FL}$ and $\text{FL} = 100 \cdot R\text{MAXA}$; either these defaults or other specific values on the data card are used in the computation being described.

start of a problem. The first medium is the medium in which the pulse is initialized. This first medium should have the parameters of vacuum for obvious reasons; the code will run otherwise, but it will produce wrong answers if the first medium has any response at all - nonlinear index, two-photon absorption, or two- or four-level gain - since the initialization causes the pulse to "materialize" within the medium rather than propagate into it. Since the different media are not numbered on the data cards, the user must be certain that the cards in the deck are in the same sequence as the media in the actual problem. Normally, the media cards follow the general input data cards of Section III.A above.

C. END Card

The last card of the data deck must be a card with +1000 in field 1 and all other fields blank. This is a required card which must appear at the end of any data card deck.

D. Restart information

LAPU2 has been coded to allow for periodic information dumps onto tape at intervals specified by the parameter IRSDMP of CARD 7, field 5. The information dumped onto tape is sufficient to allow continuation of the problem at a later time. This is useful if a problem requires more computer time than the user can obtain in one chunk, or as a means of protection against unpredictable computer hardware failures occurring after a significant amount of computation time has already been used up in solving a large problem. In order to restart a problem from a dump tape, two data cards are needed:

CARD 1:	<u>Field</u>	<u>Content</u>
	1	RESTARTbbb
	2	+xxxxxx+EE

The word RESTART, followed by three blanks, appears in the first 10 columns, and the cycle number of the restart (in an E10.5 format, +123456+04 = +1.23456 x 10⁴) in the next 10 columns.

CARD 2:	<u>Field</u>	<u>Content</u>
	1	+1000

The RESTART card must be followed by an end card with +1000 (format E12.7) in field 1 and blanks thereafter.

The code has another option: on restarting from a dump, the initial data can be significantly modified so that in effect almost a whole new problem can be specified to the code. To specify modifications of previous input data on

restarting from a dump, CARD 1 is slightly changed to:

CARD 1:	<u>Field</u>	<u>Content</u>
	1	RESTARTbbb
	2	+xxxxxx+EE
	3	MODIFYbbbb

The word MODIFY is added in field 3. Following this card, almost any of the 11 general input cards can be placed with new values of whatever parameters the user wants to change. There are some prohibitions: the pulse must be entirely in one medium or wrong results will occur; CARDS 3, 5, and 6 are ignored (the data on these cards cannot be changed), a new header card is not allowed, and no change is allowed (but the code does not check this) in the values of OMEGA and EZZ (ω and ϵ_{00} of CARD 4). Changing ZBI (CARD 2, field 5) affects only the bar graph window position plot on each two-dimensional plot.

The field values and response functions of the medium that the pulse is in at the dump can be modified by a RESCALE CARD (EI2.7 format):

RESCALE CARD:	<u>Field</u>	<u>Content</u>
	1	+1.0
	2	E_R and E_I multiplier
	3	X_R and X_I multiplier

Note in particular that Δr can be changed on CARD 2 (the medium type SWMED should not be changed, but the code does not check for this). If Δr is changed, $E(r,z,t)$ and $X(r,z,t)$ must be inversely changed to keep the total pulse and medium energy constant. For example, if Δr is doubled, $E(r,z,t)$ must be halved to keep $\frac{\Delta z}{8\pi} \int 2\pi r dr |E(r,z,t)|^2$ constant. X_R and X_I are the real and imaginary parts of the general medium response function $X(r,z,t)$ of Eq. (18).

Up to 99 new media can also be read in at a RESTART. The data for each medium are on four data cards as described in Section III.B above. If new media are entered, the medium that the pulse is in at the RESTART becomes the new first medium; all succeeding media (if any) of the original data deck are dropped, and the new media cards are read as medium 2, 3, ..., . Notice that this means that the parameter NM (CARD 2, field 4) must be changed if the new number of media ($N + 1$ if N new media are read in at the RESTART) is not equal to the previous total number. NM is changed by including CARD 2 in the new data deck. It is obvious that this method allows LAPU2 in principle to handle the propagation of a pulse through an arbitrary number of successive media as part of the same single problem.

We note that a blank (or zero) in any field is interpreted by the MODIFY routine as no change in the value of the corresponding parameter (from either its initial value or value at the previous MODIFY), not as the initial data default value. The last card of the data deck of a RESTART and MODIFY must be the END card (+1000 in field 1).

This completes a discussion of all input-output of LAPU2. The user is cautioned to observe the many qualifications specified in various places or suffer the consequences: the code will run but will produce wrong answers. Time and storage limitations have prevented automatic checks of all of these quantities to be included in the code.

IV. EXAMPLES

In this section we give examples of various features of the code. These examples are intended to serve a didactic function with respect to the code; a complete summary of the results of calculations that have been done using LAPU2 (and earlier versions) on actual CO_2 and Nd:glass laser systems is the subject of another report¹² now in preparation.

Figure 1(a) shows a three-dimensional plot of a Gaussian pulse. Electric field amplitude is plotted vertically (axis labeled A) vs radius (horizontal

Fig. 1(a).
Electric field amplitude vs R and Z
for a Gaussian pulse

Fig. 1(b).
Energy density, enclosed energy, and
phase of central z-slice vs R.

axis labeled R) and longitudinal position (axis labeled Z). The scale of amplitude values is indicated on the left side while the maximum plotted radius RMAX and the total window length (pulse length) are given in centimeters at the top. Figure 1(b) is the corresponding two-dimensional plot. It shows the total energy density (solid line), phase of the central z-slice (convex dotted line), and total enclosed energy density (dotted line starting at origin which increases monotonically) vs radius. The maximum scale values for the energy density (in ergs per square centimeter) and enclosed energy (in ergs) are given above the plots by E/cm^2 and WMAX, respectively. The phase is of course a continuous function of radius; the discontinuity in the plot occurs because the numerical arc tangent function returns values only between $(-\pi)$ and $(+\pi)$. Notice that the convex shape of the phase corresponds to a diverging wave. The time in seconds is also noted in each of the two plots, and a schematic layout of the optical system of the problem and the location of the window along that system is given in the two-dimensional plot.

Figure 2(a) shows the pulse passing through an aperture (Lens 1). Figure 2(b) shows the pulse some distance beyond the aperture and Fig. 2(c) is the corresponding two-dimensional plot. Notice that the total energy WMAX of the pulse has diminished, and that the phase shows no curvature near $R = 0$ but it shows a curvature indicating divergence near the aperture clipping point. The

Fig. 2(a).
Aperturing of a Gaussian pulse.

Fig. 2(b).
Diffraction of pulse beyond aperture.

Fig. 2(c).
2-D plot of pulse of Fig. 2(b).

Lens 1 was also a converging thin lens as well as an aperture.

Figure 3(a) shows the pulse traversing a Lens 2 while Fig. 3(b) shows the pulse after passage through that "burn pattern" lens. Figure 3(c) is the corresponding two-dimensional plot while Fig. 3(d) is the result of the Lens 2 calculation. Notice that Fig. 3(d) exhibits somewhat sharper features than does Fig. 3(c). This is because the plot of Fig. 3(c) includes z-slices at many distances (Fresnel numbers) from the aperture and thus averages over the diffraction ripples. The plot of Fig. 3(d) would correspond to the darkening of film if a strip of film (of a type which is sensitive only to total energy density) were placed at that longitudinal position. The horizontal dotted line of Fig. 3(d) is extraneous as phase information is not retained by the Lens 2 routine.

Fig. 3(a).
Pulse traversing Lens 2.

Fig. 3(b).
Pulse after traversing Lens 2.

Fig. 3(c).
2-D plot of pulse of Fig. 3(b).

Fig. 3(d).
Lens 2 output.

Fig. 4(a).
180 J CO₂ output pulse.

Fig. 4(b).
2-D plot of pulse of Fig. 4(a).

Figure 4(a) shows a 180 joule pulse which resulted from amplification of a 1 millijoule initial pulse by a four-stage CO₂ amplifier system. Figure 4(b) is the corresponding two-dimensional plot of this pulse. Figure 4(c) shows the result of a Lens 3 calculation on this pulse. The power vs time is the peaked curve (with the scale stated above the graph) while the monotonic curve is the

Fig. 4(c).
 Lens 3 output.

total energy integral vs time (in joules with the scale stated above the graph). The total temporal duration of the window is given below the plots.

Figure 5(a) shows the same 180 joule pulse and Fig. 5(b) is the associated two-dimensional plot. The pulse has traversed a Lens 5 and the output plots of this routine for three particular z-slices of the pulse (numbered 10, 47, and 66) are shown in Figs. 5(c), 5(d), and 5(e). Note that information about the focal length of the lens which produced these on-axis energy density plots, the scale of the energy density plots and the total energy WMAX of each z-slice, and the longitudinal distance, in centimeters, beyond the plane of the lens are among the data given on each graph. The focused axial energy density summed over the entire pulse is plotted in Fig. 5(f). The quantity ZBD on this last plot is the location of the peak axial energy density, a possible choice of the location of best focus; notice that ZBD is not equal to the focal length of the lens.

Fig. 5(a).
 180 J CO₂ output pulse.

Fig. 5(b).
 2-D plot of pulse of Fig. 5(a).

Fig. 5(c).
Lens 5 output for z-slice 10.

Fig. 5(d).
Lens 5 output for z-slice 47.

Fig. 5(e).
Lens 5 output for z-slice 66.

Fig. 5(f).
Lens 5 output for entire pulse.

Figures 6(a) and 6(b) show the same 180 joule pulse as before. This pulse is passed through a Lens 4 to calculate radial focal patterns. The radial focal patterns of z-slices numbered 10, 47, and 66 are shown in Figs. 6(c), 6(d), and 6(e). The Fresnel number N_F corresponds to the quantity specified on MEDIUM CARD 4 (Section III.B). These patterns are for the same focal length

Fig. 6(a).
180 J CO₂ output pulse.

Fig. 6(b).
2-D plot of pulse of Fig. 6(a).

Fig. 6(c).
Lens 4 output for z-slice 10.

Fig. 6(d).
Lens 4 output for z-slice 47.

lens as in Fig. 5 and they are calculated at a distance ZBD in front of the lens, the location of the peak axial energy density of Fig. 5(f). Figure 6(f) shows the focused radial energy density integrated over the entire pulse. The break in the horizontal dotted line is a computer glitch.

Fig. 6(e).
Lens 4 output for z-slice 66.

Fig. 6(f).
Lens 4 output for entire pulse.

V. FLOW CHART

In this section we briefly outline the main computing routes of the code. A complete, cross-referenced FORTRAN listing of the code (LASL Identification No. LP-0780) can be found on a microfiche card enclosed in a pocket on the inside back cover of this report.

We refer to Fig. 7 which diagrammatically illustrates the information flow in the code. After all arrays are set to zero at the beginning of a run, the code decides whether it has to deal with a brand new problem or a continuation of a previous calculation by reading the first data card. In the latter case, RESDMP is called to read in the dump file. In the former case, the routines READIN, PINIT, and ICPRT are called to read the input data, initialize all variables, and print out the initial conditions.

Next, a call is made to WAVE, which controls the main calculation. The diffraction calculation is done in PULSE while CKII and CKIR compute, respectively, the imaginary and real parts of the material response function. After doing these calculations for every z-slice of the pulse, the window is advanced by ADVW and the time is incremented by one time step. The code then checks whether output information is required at this computational cycle. If the answer is yes, the appropriate prints and plots are produced or an information dump is made on a disc file.

Fig. 7.
Computer-generated flow chart of LAPU2.

The code next checks to see if the problem has run to completion on this cycle. If it has not, a call is made to WAVE and the above process is repeated again to propagate the pulse one more cycle. If the code recognizes that the current cycle coincides with the specified end of the problem, a brief series of prints and plots is made before the program calls EXIT.

Since a complete cross-referenced FORTRAN listing is included with this report, it is hoped that this brief outline will be adequate. All necessary further details are contained in the listing.

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APPENDIX A.

NUMERICAL SOLUTION

We repeat here the numerical solution of Eq. (9) as presented in Ref. 1. First we rewrite Eq. (9) in the following form, which is consistent with the general medium response function $X(r,z,t)$ of Eq. (18).

$$\frac{\partial}{\partial z} E + i\kappa YE = -i\kappa \frac{1}{r} \frac{\partial}{\partial z} \left(r \frac{\partial E}{\partial r} \right), \quad (\text{A.1})$$

where

$$\kappa \equiv \frac{v}{2\omega}. \quad (\text{A.2})$$

The partial derivatives in Eq. (A.1) are with τ held constant. Thus, integration of Eq. (A.1) involves different z values (as well as r values of course). But the code is written in (r,z,t) , not (r,z,τ) ; thus, advancing by $\Delta z(N)$ means also advancing by $\Delta t = \Delta z(N)/v$. If we recall that the window also advances by $\Delta z(N)$, we see that we shall deal with $E_{K,J}^{n+1}$ and $E_{K,J}^n$. Since the J index does not change, we shall leave it out of all of the subsequent formulas to somewhat simplify them.

Suydam's differencing scheme for Eq. (A.1) leads to the following equation, which is tridiagonal in the radial index K .

$$-A_K E_{K+1}^{n+1} + B_K E_K^{n+1} - C_K E_{K-1}^{n+1} = D_K. \quad (\text{A.3})$$

The solution is given by

$$E_{K-1}^{n+1} = F_{K-1} E_K^{n+1} + G_{K-1}, \quad (\text{A.4})$$

where

$$F_K = \frac{A_K}{B_K - C_K F_{K-1}} \quad (A.5)$$

and

$$G_K = \frac{D_K + C_K G_{K-1}}{B_K - C_K F_{K-1}} \quad (A.6)$$

From the boundary condition at $r = 0$ one concludes that the appropriate initial values at the auxiliary point $r_0 = -\frac{1}{2}\Delta r$ are

$$F_0 = 1 \quad \text{and} \quad G_0 = 0 \quad (A.7)$$

The recursions Eqs. (A.5) and (A.6) can now be done. To do Eq. (A.4), the boundary condition for a perfectly conducting wall is used:

$$E_{K_{\max}}^{n+1} = 0 \quad (A.8)$$

The result of these manipulations in Eq. (A.1) gives the following explicit solution: breaking

$$E_K^{n+1} = U_K^{n+1} + iV_K^{n+1} \quad (A.9)$$

The solution, where $K = K_{\max}, K_{\max} - 1, K_{\max} - 2, \dots, 1$, is

$$U_{K-1}^{n+1} = P_{K-1} U_K^{n+1} - Q_{K-1} V_K^{n+1} + M_{K-1} \quad (A.10a)$$

$$V_{K-1}^{n+1} = P_{K-1} V_K^{n+1} + Q_{K-1} U_K^{n+1} + N_{K-1} \quad (A.10b)$$

and

$$U_{K_{\max}}^{n+1} = V_{K_{\max}}^{n+1} = 0 \quad (A.10c)$$

The quantities M_K and N_K are computed from

$$M_K = \left[(\bar{\alpha}_K^2 + \beta_K^2) \Delta \right]^{-1} \left\{ (\beta_K \Delta_q - \bar{\alpha}_K \Delta_p) (R_K + \rho \beta_K N_{K-1} - \rho \bar{\alpha}_K M_{K-1}) \right. \\ \left. - (\bar{\alpha}_K \Delta_q + \beta_K \Delta_p) (S_K - \rho \bar{\alpha}_K N_{K-1} - \rho \beta_K M_{K-1}) \right\} , \quad (A.11a)$$

$$N_K = \left[(\bar{\alpha}_K^2 + \beta_K^2) \Delta \right]^{-1} \left\{ (\bar{\alpha}_K \Delta_q + \beta_K \Delta_p) (R_K + \rho \beta_K N_{K-1} - \rho \bar{\alpha}_K M_{K-1}) \right. \\ \left. + (\beta_K \Delta_q - \bar{\alpha}_K \Delta_p) (S_K - \rho \bar{\alpha}_K N_{K-1} - \rho \beta_K M_{K-1}) \right\} , \quad (A.11b)$$

and

$$M_0 = N_0 = 0 \quad . \quad (A.11c)$$

The R_K and S_K in Eq. (A.11) are computed from

$$R_K = \frac{2 - \theta}{\theta} \left\{ -\alpha_K \left[U_{K+1}^n - (1 + \rho) U_K^n + \rho U_{K-1}^n \right] \right. \\ \left. + \beta_K \left[V_{K+1}^n - (1 + \rho) V_K^n + \rho V_{K-1}^n \right] \right\} + \tilde{R}_K \quad (A.12a)$$

and

$$S_K = \frac{2 - \theta}{\theta} \left\{ -\alpha_K \left[V_{K+1}^n - (1 + \rho) V_K^n + \rho V_{K-1}^n \right] \right. \\ \left. - \beta_K \left[U_{K+1}^n - (1 + \rho) U_K^n + \rho U_{K-1}^n \right] \right\} + \tilde{S}_K \quad . \quad (A.12b)$$

The $\tilde{R}_K$ and $\tilde{S}_K$ in Eq. (A.12) are given by

$$R_K = e^{\gamma_K} \left((\cos \phi_K) \left\{ \eta U_{K+1}^n + \left[1 - (1 + \rho) \eta \right] U_K^n + \rho \eta U_{K-1}^n \right\} \right. \\ \left. + (\sin \phi_K) \left\{ \eta V_{K+1}^n + \left[1 - (1 + \rho) \eta \right] V_K^n + \rho \eta V_{K-1}^n \right\} \right) \quad (A.13a)$$

and

$$\begin{aligned} \tilde{S}_K = e^{\gamma_K} & \left\{ (\cos \phi_K) \left\{ \eta v_{K+1}^n + \left[1 - (1 + \rho)\eta \right] v_K^n + \rho \eta v_{K-1}^n \right\} \right. \\ & \left. - (\sin \phi_K) \left\{ \eta u_{K+1}^n + \left[1 - (1 + \rho)\eta \right] u_K^n + \rho \eta u_{K-1}^n \right\} \right\} . \end{aligned} \quad (A.13b)$$

The P_K and Q_K of Eq. (A.10) are given by

$$P_K = \frac{\Delta p}{\Delta} , \quad (A.14a)$$

$$Q_K = \frac{\Delta q}{\Delta} , \quad (A.14b)$$

and

$$P_0 = 1 \quad \text{and} \quad Q_0 = 0 . \quad (A.14c)$$

The Δp , Δq , and Δ appearing in Eqs. (A.11) and (A.14) are given by

$$\Delta p = 1 + \rho - \frac{\bar{\alpha}_K}{[\bar{\alpha}_K^2 + \beta_K^2]} - \rho P_{K-1} , \quad (A.15a)$$

$$\Delta q = \frac{\beta_K}{[\bar{\alpha}_K^2 + \beta_K^2]} - \rho Q_{K-1} , \quad (A.15b)$$

and

$$\Delta = (\Delta p)^2 + (\Delta q)^2 . \quad (A.15c)$$

In Eq. (A.15), as well as in Eq. (A.11), we have suppressed the index K on Δp , Δq , and Δ ; that is, $\Delta p = (\Delta p)_K$, etc.

The constants α_K , $\bar{\alpha}_K$, β_K , and ρ are given by

$$r_K = (K - \frac{1}{2})\Delta r , \quad (A.16)$$

$$\alpha_K = \frac{\theta_{r_{K+\frac{1}{2}}}}{r_K(\Delta r)^2[Y_R^2 + Y_I^2]} \left[Y_R(1 - e^{\gamma_K} \cos \phi_K) + Y_I e^{\gamma_K} \sin \phi_K \right] , \quad (A.17)$$

$$\beta_K = \frac{\theta r_{K+\frac{1}{2}}}{r_K (\Delta r)^2 [Y_R^2 + Y_I^2]} \left[Y_R e^{Y_K} \sin \phi_K - Y_I (1 - e^{Y_K} \cos \phi_K) \right] , \quad (A.18)$$

$$\rho = \frac{r_{K-\frac{1}{2}}}{r_{K+\frac{1}{2}}} , \quad (A.19)$$

and

$$\bar{\alpha}_K = \alpha_K + \eta . \quad (A.20)$$

Here we have split Y of Eq. (A.1) into real and imaginary parts

$$Y_K^n = (Y_R)_K^n + i(Y_I)_K^n , \quad (A.21)$$

$$\phi_K = \lambda \Delta z(N) (Y_R)_K^n , \quad (A.22)$$

and

$$Y_K = \lambda \Delta z(N) (Y_I)_K^n . \quad (A.23)$$

Here $\Delta z(N)$ is the propagation step size for medium N . We have omitted the n, K indices on Y_R and Y_I in Eqs. (A.17) and (A.18). Finally, θ is a stability parameter of the difference scheme ($\theta = \frac{1}{2}$ usually; $\theta \geq \frac{1}{2}$ gives unconditional stability while $\theta < \frac{1}{2}$ unconditional instability; $\theta > \frac{1}{2}$ leads to damping of the high spatial frequency transverse ripples of the diffraction pattern) and η is related to the numerical "group velocity" [see Eq. (17)]. The code also checks that the denominators of Eqs. (A.5) and (A.6) are bounded away from zero; if they become too small, the propagation step Δz is halved.

APPENDIX B.

SOLUTION OF TWO-LEVEL RATE EQUATION MODEL

This solution was originally furnished by B. R. Suydam. We first define a dimensionless variable $\bar{\tau}$ by

$$\bar{\tau} = \frac{\tau}{\Gamma} \quad , \quad (B.1)$$

so that Eq. (21) becomes

$$\frac{\partial}{\partial \bar{\tau}} g_A + \left(1 + q|E|^2\right) g_A = g_A^0 \quad . \quad (B.2)$$

Direct integration gives

$$\begin{aligned} g_A(\bar{\tau} + \Delta\bar{\tau}) &= g_A(\bar{\tau}) \exp \left[- \int_{\bar{\tau}}^{\bar{\tau} + \Delta\bar{\tau}} dx \left(1 + q|E|^2\right) \right] \\ &+ \int_{\bar{\tau}}^{\bar{\tau} + \Delta\bar{\tau}} d\bar{\tau} \exp \left[- \int_{\bar{\tau}}^{\bar{\tau} + \Delta\bar{\tau}} dy \left(1 + q|E|^2\right) \right] g_A^0 \quad . \end{aligned} \quad (B.3)$$

Approximating the integrals we get

$$\begin{aligned} g_A(\bar{\tau} + \Delta\bar{\tau}) &\simeq g_A(\bar{\tau}) \exp \left[-\Delta\bar{\tau} \left(1 + q|E_1|^2\right) \right] \\ &+ \int_{\bar{\tau}}^{\tau} d\tau' g_A^0 \exp \left[-\Delta\bar{\tau} \left(1 + q|E_2|^2\right) \right] \exp \left[-(\tau' - \tau) \left(1 + q|E_2|^2\right) \right] \quad , \end{aligned} \quad (B.4)$$

where

$$\left(1 + q|E_1|^2\right) \Delta\bar{\tau} = \int_{\bar{\tau}}^{\bar{\tau} + \Delta\bar{\tau}} dx \left(1 + q|E|^2\right) \quad (B.5)$$

and

$$\left(1 + q|E_2|^2\right) (\bar{\tau} + \Delta\bar{\tau} - \tau') = \int_{\tau'}^{\bar{\tau} + \Delta\bar{\tau}} dy \left(1 + q|E|^2\right) \quad . \quad (B.6)$$

Integrating Eq. (B.4) gives

$$g_A(\bar{\tau} + \Delta\bar{\tau}) = g_A(\bar{\tau}) \exp\left[-\Delta\bar{\tau}(1 + q|E_1|^2)\right] \\ + \exp\left[-\Delta\bar{\tau}(1 + q|E_2|^2)\right] g_A^0 \left\{ \frac{\exp\left[\Delta\bar{\tau}(1 + q|E_2|^2)\right] - 1}{1 + q|E_2|^2} \right\} . \quad (B.7)$$

If we now use the approximation that

$$|E_1|^2 = |E_2|^2 = |E^*|^2 , \quad (B.8)$$

Eq. (B.7) becomes

$$g_A(\bar{\tau} + \Delta\bar{\tau}) = \frac{g_A^0}{1 + q|E^*|^2} + \exp\left[-\Delta\bar{\tau}(1 + q|E^*|^2)\right] \left[g_A(\bar{\tau}) - \frac{g_A^0}{1 + q|E^*|^2} \right] . \quad (B.9)$$

For fixed z , $\Delta\tau = \Delta t$; however, since the window moves by $\Delta z(N)$ in Δt , the z location specified by the window index $J + 1$ at time $t - \Delta t$ becomes indexed by J at time t . Thus, the differenced form of Eq. (B.9) is

$$(g_A)_{K,J}^n = \frac{g_A^0}{1 + q|E^*|^2} + \exp\left[-\frac{\Delta z(N)}{vT}(1 + q|E^*|^2)\right] \\ \times \left[(g_A)_{K,J+1}^{n-1} - \frac{g_A^0}{1 + q|E^*|^2} \right] \quad (B.10a)$$

and

$$|E^*|^2 = \frac{1}{2} \left(|E_{K,J}^n|^2 + |E_{K,J+1}^{n-1}|^2 \right) . \quad (B.10b)$$

If $\Delta z(N) < \Delta z$, linear interpolation is used for the appropriate values at time step $n - 1$. That is,

$$(g_A)_{K,J+1}^{n-1} \xrightarrow[\text{by}]{\text{replaced}} \left[1 - \frac{\Delta z(N)}{\Delta z}\right] (g_A)_{K,J}^{n-1} + \frac{\Delta z(N)}{\Delta z} (g_A)_{K,J+1}^{n-1} \quad (\text{B.10c})$$

and

$$|E_K^*|^2 = \frac{1}{2} |E_{K,J}^n|^2 + \frac{1}{2} \left| \left[1 - \frac{\Delta z(N)}{\Delta z}\right] E_{K,J}^{n-1} + \frac{\Delta z(N)}{\Delta z} E_{K,J+1}^{n-1} \right|^2 \quad (\text{B.10d})$$

We recall that $\Delta z(N)$ in a two-level medium must always be less than or equal to the pulse z-slice thickness Δz :

$$\Delta z(N) \leq \Delta z \quad , \quad (\text{B.11})$$

where N is a two-level medium.

Note that the code does not work directly with the amplitude gain g_A but rather with $(2\omega/v)g_A = (\omega/v)g_I$. This was done historically because of the form of the basic equations [Eq. (7) or (9)].

APPENDIX C.

SOLUTION OF THE FOUR-LEVEL RATE EQUATION MODEL

This solution was constructed by B. R. Suydam. First, define new variables:

$$\chi = \frac{\sigma\omega}{v} (n_a - n_b) \quad , \quad (\text{C.1a})$$

$$\psi = \frac{\sigma\omega}{v} (n_a + n_b) \quad , \quad (\text{C.1b})$$

$$S = \frac{\sigma\omega}{v} \left(\frac{1}{N_1\tau_1} n_0 - \frac{1}{N_2\tau_2} n_1 \right) \quad , \quad (\text{C.1c})$$

and

$$\Sigma = \frac{\sigma\omega}{v} \left(\frac{1}{N_1\tau_1} n_0 + \frac{1}{N_2\tau_2} n_1 \right) \quad . \quad (\text{C.1d})$$

Equation (22) becomes

$$\dot{\psi} + \lambda\psi + \mu\chi = \Sigma \quad , \quad (\text{C.2a})$$

$$\dot{\chi} + [\lambda + 2\alpha\sigma I]\chi + \mu\psi = S \quad , \quad (\text{C.2b})$$

$$\begin{aligned} \dot{S} + \frac{1}{2} \left(\frac{1}{N_1\tau_1} + \frac{1}{N_2\tau_2} \right) S + \frac{1}{2} \left(\frac{1}{N_1\tau_1} - \frac{1}{N_2\tau_2} \right) \Sigma \\ = \frac{1}{2} \left(\frac{1}{N_1\tau_1^2} + \frac{1}{N_2\tau_2^2} \right) \chi + \frac{1}{2} \left(\frac{1}{N_1\tau_1^2} - \frac{1}{N_2\tau_2^2} \right) \psi \quad , \end{aligned} \quad (\text{C.2c})$$

and

$$\begin{aligned} \dot{\Sigma} + \frac{1}{2} \left(\frac{1}{N_1\tau_1} + \frac{1}{N_2\tau_2} \right) \Sigma + \frac{1}{2} \left(\frac{1}{N_1\tau_1} - \frac{1}{N_2\tau_2} \right) S \\ = \frac{1}{2} \left(\frac{1}{N_1\tau_1^2} + \frac{1}{N_2\tau_2^2} \right) \psi + \frac{1}{2} \left(\frac{1}{N_1\tau_1^2} - \frac{1}{N_2\tau_2^2} \right) \chi \quad , \end{aligned} \quad (\text{C.2d})$$

where

$$\lambda = \frac{1}{2} \left(\frac{1}{\tau_1} + \frac{1}{\tau_2} \right) \quad (\text{C.3})$$

and

$$\mu = \frac{1}{2} \left(\frac{1}{\tau_1} - \frac{1}{\tau_2} \right) \quad . \quad (\text{C.4})$$

The method involves solving Eqs. (C.2c) and (C.2d) first and then substituting the results into Eqs. (C.2a) and (C.2b) to solve for ψ and χ . We shall present only the final results.

The initial values are computed from the small signal intensity gain g , the ratios N_1 and N_2 of Eq. (23), and one additional ratio called f :

$$f = \frac{n_b(t=0)}{n_a(t=0)} \quad . \quad (\text{C.5})$$

We then find the following initial values:

$$\chi^0 = \frac{\omega\sqrt{\epsilon_0}}{c} g \quad , \quad (C.6a)$$

$$\psi^0 = \frac{1+f}{1-f} \chi^0 \quad , \quad (C.6b)$$

$$S^0 = \lambda\chi^0 + \mu\psi^0 \quad , \quad (C.6c)$$

and

$$\Sigma^0 = \lambda\psi^0 + \mu\chi^0 \quad ; \quad (C.6d)$$

and we also define the following quantities:

$$q = \frac{\sigma c \sqrt{\epsilon_0}}{4\hbar\omega} \quad (C.7)$$

and

$$|E^*|^2 = \frac{1}{2} \left(|E_{K,J}^n|^2 + |E_{K,J+1}^{n-1}|^2 \right) \quad . \quad (C.8)$$

The quantity $|E^*|^2$ is the same as in Eq. (B.10b). Because the radial index K does not enter into the solution of these time differential equations, we suppress it on all quantities below.

The difference equation solution of Eq. (C.2) is as follows:

$$\chi_J^n = \frac{1}{\Delta} \left[R_1 (1 - I_2 Q_2 - K_2 Q_1) + R_2 (I_1 Q_2 + K_1 Q_1) \right] \quad , \quad (C.9a)$$

$$\psi_J^n = \frac{1}{\Delta} \left[R_1 (I_2 Q_1 + K_2 Q_2) + R_2 (1 - I_1 Q_1 - K_1 Q_2) \right] \quad , \quad (C.9b)$$

$$S_J^n = C + D + Q_1 \chi_J^n + Q_2 \psi_J^n \quad , \quad (C.9c)$$

and

$$\Sigma_J^n = C - D + Q_2 \chi_J^n + Q_1 \psi_J^n \quad . \quad (C.9d)$$

The various terms in Eq. (C.9) are

$$\Delta = 1 - (I_1 + K_2)Q_1 - (I_2 + K_1)Q_2 + (I_1K_2 - I_2K_1)(Q_1^2 - Q_2^2) \quad , \quad (C.10)$$

$$R_1 = A_+ + B_+ + A_- + B_- + I_1(C + D) + K_1(C - D) \quad , \quad (C.11)$$

and

$$R_2 = \frac{m_+ + \lambda}{\mu} (A_- + B_-) + \frac{m_- + \lambda}{\mu} (A_+ + B_+) + I_2(C + D) + K_2(C - D) \quad . \quad (C.12)$$

Because μ is a divisor in Eq. (C.12), $\mu \neq 0$; that is, this solution holds only for $\tau_1 \neq \tau_2$. However, $|\tau_1 - \tau_2|$ can be an arbitrarily small number.

$$Q_1 = \frac{1}{4\tau_1} \left(1 - e^{-\Delta t/N_1\tau_1} \right) + \frac{1}{4\tau_2} \left(1 - e^{-\Delta t/N_2\tau_2} \right) \quad (C.13)$$

and

$$Q_2 = \frac{1}{4\tau_1} \left(1 - e^{-\Delta t/N_1\tau_1} \right) - \frac{1}{4\tau_2} \left(1 - e^{-\Delta t/N_2\tau_2} \right) \quad . \quad (C.14)$$

Next we define

$$m_+ = -\lambda - q|E^*|^2 + \sqrt{\mu^2 + (q|E^*|^2)^2} \quad (C.15a)$$

and

$$m_- = -\lambda - q|E^*|^2 - \sqrt{\mu^2 + (q|E^*|^2)^2} \quad . \quad (C.15b)$$

The remaining terms are

$$A_+ = \frac{e^{m_+\Delta t}}{(m_+ - m_-)} \left[(m_+ + \lambda)\chi_{J+1}^{n-1} - \mu\psi_{J+1}^{n-1} \right] \quad , \quad (C.16a)$$

$$A_- = \frac{e^{m_+\Delta t}}{(m_+ - m_-)} \left[-(m_- + \lambda)\chi_{J+1}^{n-1} + \mu\psi_{J+1}^{n-1} \right] \quad , \quad (C.16b)$$

$$B_+ = \frac{1 - e^{\frac{m_+ \Delta t}{2m_+ (m_+ - m_-)}}}{2m_+ (m_+ - m_-)} \left[-(m_+ + \lambda) S_{J+1}^{n-1} + \mu \Sigma_{J+1}^{n-1} \right] , \quad (C.17a)$$

$$B_- = \frac{1 - e^{\frac{m_- \Delta t}{2m_- (m_+ - m_-)}}}{2m_- (m_+ - m_-)} \left[(m_- + \lambda) S_{J+1}^{n-1} - \mu \Sigma_{J+1}^{n-1} \right] , \quad (C.17b)$$

$$C = \frac{1}{2} \left(S_{J+1}^{n-1} + \Sigma_{J+1}^{n-1} \right) e^{-\Delta t / N_1 \tau_1} + \frac{1}{4\tau_1} \left(\chi_{J+1}^{n-1} + \psi_{J+1}^{n-1} \right) \left(1 - e^{-\Delta t / N_1 \tau_1} \right) , \quad (C.18)$$

$$D = \frac{1}{2} \left(S_{J+1}^{n-1} - \Sigma_{J+1}^{n-1} \right) e^{-\Delta t / N_2 \tau_2} + \frac{1}{4\tau_2} \left(\chi_{J+1}^{n-1} - \psi_{J+1}^{n-1} \right) \left(1 - e^{-\Delta t / N_2 \tau_2} \right) , \quad (C.19)$$

$$I_1 = \frac{1}{2(m_+ - m_-)} \left[\frac{m_- + \lambda}{m_-} \left(1 - e^{\frac{m_- \Delta t}{m_-}} \right) - \frac{m_+ + \lambda}{m_+} \left(1 - e^{\frac{m_+ \Delta t}{m_+}} \right) \right] , \quad (C.20)$$

$$I_2 = K_1 = \frac{1}{2(m_+ - m_-)} \left[\frac{1}{m_+} \left(1 - e^{\frac{m_+ \Delta t}{m_+}} \right) - \frac{1}{m_-} \left(1 - e^{\frac{m_- \Delta t}{m_-}} \right) \right] , \quad (C.21)$$

$$K_2 = \frac{1}{2(m_+ - m_-)} \left[-\frac{m_+ + \lambda}{m_-} \left(1 - e^{\frac{m_- \Delta t}{m_-}} \right) + \frac{m_- + \lambda}{m_+} \left(1 - e^{\frac{m_+ \Delta t}{m_+}} \right) \right] , \quad (C.22)$$

and, of course, in these equations $\Delta t = \Delta z(N)/v$. Note that the propagation step $\Delta z(N)$ must always be equal to the pulse z-slice thickness Δz in a four-level medium:

$$\Delta z(N) = \Delta z , \quad (C.23)$$

where N is a four-level medium.

APPENDIX D.

SATURATION OF FOUR-LEVEL SYSTEM

We consider the four-level model of Eq. (22). For simplicity, we assume that the upper and lower levels have the same kinetic properties; that is,

$$\tau_1 = \tau_2 = \tau \quad (D.1)$$

and

$$N_1 = \frac{n_0(0)}{n_a(0)} = \frac{n_1(0)}{n_b(0)} = N_2 = \gamma^{-1} \quad (D.2)$$

Then Eq. (22) can be written as

$$\dot{n}_0 = -\frac{\gamma}{\tau} n_0 + \frac{1}{\tau} n_1 \quad , \quad (D.3)$$

$$\dot{n}_a = -\frac{1}{\tau} n_a + \frac{\gamma}{\tau} n_0 - \sigma I(n_a - n_b) \quad , \quad (D.4)$$

$$\dot{n}_b = -\frac{1}{\tau} n_b + \frac{\gamma}{\tau} n_1 + \sigma I(n_a - n_b) \quad , \quad (D.5)$$

$$\dot{n}_1 = -\frac{\gamma}{\tau} n_1 + \frac{1}{\tau} n_a \quad , \quad (D.6)$$

and

$$I = \frac{1}{\hbar\omega} \frac{c\sqrt{\epsilon_0}}{8\pi} |E(r,z,t)|^2 \quad (D.7)$$

We have also set the parameter α of Eqs. (22b) and (22c) to unity. All population densities refer to a fixed location z within the medium where no z argument is specifically stated. We ignore transverse, or radial, variations in what follows.

The energy per unit volume extracted by a pulse, ΔE , is given by

$$\Delta E = \hbar\omega [n_0(0) + n_a(0) - n_0(t_f) - n_a(t_f)] \quad (D.8)$$

Here, t_f is the time at which the pulse completes its traverse past the axial location z . It is evident by adding Eqs. (D.3) - (D.6) that the following conservation law holds.

$$n_0 + n_a + n_b + n_1 = \text{constant} = N \quad (D.9)$$

We change the variables as follows:

$$s = n_a + n_b \quad , \quad (D.10)$$

$$\delta = n_a - n_b \quad , \quad (D.11)$$

$$S = n_0 + n_1 \quad , \quad (D.12)$$

and

$$\Delta = n_0 - n_1 \quad . \quad (D.13)$$

The inverse of this transformation is clearly

$$n_a = \frac{1}{2} (s + \delta) \quad , \quad (D.14)$$

$$n_b = \frac{1}{2} (s - \delta) \quad , \quad (D.15)$$

$$n_0 = \frac{1}{2} (S + \Delta) \quad , \quad (D.16)$$

and

$$n_1 = \frac{1}{2} (S - \Delta) \quad . \quad (D.17)$$

The equations of motion [Eqs. (D.3) - (D.6)] become

$$\dot{s} = -\frac{1}{\tau} s + \frac{\gamma}{\tau} S \quad , \quad (D.18)$$

$$\dot{\delta} = -\frac{1}{\tau} \delta + \frac{\gamma}{\tau} \Delta - 2\sigma I \delta \quad , \quad (D.19)$$

$$\dot{S} = -\frac{\gamma}{\tau} S + \frac{1}{\tau} s \quad , \quad (D.20)$$

and

$$\dot{\Delta} = -\frac{\gamma}{\tau} \Delta + \frac{1}{\tau} \delta \quad , \quad (D.21)$$

and the conservation law, Eq. (D.9), becomes

$$s + S = N \quad . \quad (D.22)$$

Using Eq. (D.22) in Eq. (D.18), and integrating directly give:

$$s(t) = s(0)e^{-[(1+\gamma)/\tau]t} + N \frac{\gamma}{1+\gamma} \left\{ 1 - e^{-[(1+\gamma)/\tau]t} \right\} \quad . \quad (D.23)$$

Thus, Eq. (D.22) gives

$$S(t) = N \left\{ \frac{1}{1+\gamma} + \frac{\gamma}{1+\gamma} e^{-[(1+\gamma)/\tau]t} \right\} - s(0)e^{-[(1+\gamma)/\tau]t} \quad . \quad (D.24)$$

In the saturation regime, we assume that the stimulated emission rate is much larger than the rotational relaxation rate:

$$2\sigma I \gg \frac{1}{\tau} \quad , \quad \text{for } t_1 \leq t \leq t_2 \quad . \quad (D.25)$$

That is, the pulse intensity I is sufficiently large to make Eq. (D.25) hold over some portion of length $t_2 - t_1$ of the total pulse duration. Then Eq. (D.19) becomes approximately

$$\dot{\delta} \simeq 0 \simeq - \left(\frac{1}{\tau} + 2\sigma I \right) \delta + \frac{\gamma}{\tau} \Delta \quad , \quad (D.26)$$

so that over this time interval

$$\delta \simeq \frac{\gamma \Delta}{2\sigma \tau I} \quad . \quad (D.27)$$

Using Eq. (D.27) in Eq. (D.21) gives *

$$\dot{\Delta} = - \frac{\gamma}{\tau} \left(1 - \frac{1}{2\sigma \tau I} \right) \Delta \quad (D.28)$$

and

* Recall that the approximation holds starting at t_1 .

$$\Delta(t) = \Delta(t_1) \exp \left\{ -\frac{\gamma}{\tau} \int_{t_1}^t dt' \left[1 - \frac{1}{2\sigma\tau I(t')} \right] \right\} . \quad (D.29)$$

By virtue of Eq. (D.25), we can neglect the second term and write

$$\Delta(t_2) \approx \Delta(t_1) e^{-(\gamma/\tau)T_p} , \quad (D.30)$$

where

$$T_p = t_2 - t_1 . \quad (D.31)$$

Thus, Eq. (D.27) gives

$$\delta(t_2) \approx \frac{\gamma\Delta(t_1)}{2\sigma\tau I(t_2)} e^{-(\gamma/\tau)T_p} . \quad (D.32)$$

The approximation Eq. (D.25) implies that energy extraction by those portions of the pulse outside the interval $t_1 - t_2$ is much less than the energy extraction by the saturating portion of the pulse. Hence, the energy extraction Eq. (D.8), using Eqs. (D.14) and (D.16), becomes

$$\Delta E \approx \hbar\omega \left[n_0(t_1) + n_a(t_1) - n_0(t_2) - n_a(t_2) \right]$$

and

$$\Delta E \approx \frac{\hbar\omega}{2} \left[s(t_1) + \delta(t_1) + S(t_1) + \Delta(t_1) - S(t_2) - \delta(t_2) - S(t_2) - \Delta(t_2) \right] .$$

Recalling Eq. (D.22), this reduces to

$$\Delta E \approx \frac{\hbar\omega}{2} \left[\delta(t_1) + \Delta(t_1) - \delta(t_2) - \Delta(t_2) \right] .$$

Using Eqs. (D.27), (D.30), and (D.32), we get

$$\Delta E \approx \frac{\hbar\omega}{2} \Delta(t_1) \left[\gamma + 1 - \frac{\gamma}{2\sigma\tau I(t_2)} e^{-(\gamma/\tau)T_p} - e^{-(\gamma/\tau)T_p} \right] .$$

Since $\gamma < 1$, and in view of Eq. (D.25), we can ignore the third term of this expression and write the energy extraction per unit volume as

$$\Delta E \simeq \frac{\hbar \omega}{2\gamma\sigma} \left(\frac{\ln G}{L} \right) \left[\gamma + 1 - e^{-(\gamma/\tau)T_p} \right] . \quad (D.33)$$

Here, we have assumed that $\Delta(t_1) \approx \Delta(0)$, and that the amplifier has a length L and an initial small signal gain G given by

$$G = e^{\sigma\delta(0)L} = e^{\delta\gamma\Delta(0)L} .$$

The formula can be rewritten in terms of the energy extracted per square centimeter of active medium cross-sectional area, EN , for an amplifier of length L as

$$EN = \frac{\hbar \omega}{2\gamma\sigma} (\ln G) \left[1 + \gamma - e^{-(\gamma/\tau)T_p} \right] . \quad (D.34)$$

The energy extracted per unit volume of the active medium, Eq. (D.33), is thus primarily determined by T_p , the duration of the pulse during which the saturation condition of Eq. (D.25) is satisfied. This formula, and its alternative form in Eq. (D.34), are therefore expected to be accurate in circumstances where T_p does not change appreciably as the pulse traverses the active medium. For long, high-gain amplifiers, where T_p can change considerably while the pulse propagates through the amplifier, the estimate of Eq. (D.34) must be used with some caution. One can, however, be certain that, within the present rate equation treatment, in an amplifier T_p can only increase. This happens because severe modulation of the pulse envelope (pulse breakup), which can occur in a density matrix treatment of pulse propagation, cannot occur in a rate equation treatment.

APPENDIX E

GAIN REPUMPING

Current designs of large, short-pulse laser-amplifier systems frequently include optical components that guide the pulse back through the same gain medium several times. This increases the efficiency of the device because a greater fraction of the stored energy of an amplifier is ultimately transferred to the pulse than is possible with a single-pass system. Multiple passes through the same active medium also result in a more compact machine in which several amplification stages are combined effectively into one structure.

During the time between passage of the trailing edge of the pulse on its first pass through the medium and the incidence of the pulse's leading edge on its second pass through the medium, the active molecules in the gain volume evolve in time only by collisional relaxation processes. (We shall continue to ignore spontaneous emission.) Here we present the solution of one particular model of this process. This solution is not yet a part of LAPU2, but may become one of the future modifications.

We are interested only in the four-level model. In the absence of a radiation field, the upper levels n_0 and n_a are not coupled to the lower levels n_b and n_1 , so we consider only the upper levels. Since the original intent* of the four-level system was to model CO_2 , we shall continue to use the structure of this molecule as the descriptive framework of this section.

We begin not with n_a and n_0 , which represent populations of a single rotational level and the rest of the rotational levels of a single vibrational level, respectively, but with N_A and N_{001} , which represent, say, the total population of the higher excited states of the asymmetric stretch mode (ν_3) of CO_2 and the population of only the first excited state of this mode, respectively. We assume the following relaxation equations:

$$\dot{N}_A = \Gamma - \frac{1}{MT} N_A + \frac{1}{T} N_{001} \quad (\text{E.1})$$

and

$$\dot{N}_{001} = \gamma - \frac{1}{T} N_{001} + \frac{1}{MT} N_A \quad (\text{E.2})$$

*(and the intent of most subsequent applications)

Here, Γ is a rate ($\text{cm}^{-3} \text{s}^{-1}$) of pumping into the A-mode reservoir due possibly to electron excitation or collisions with energy transfer gases such as N_2 in a CO_2 laser, and γ is a rate of pumping into the 001 level only. The vibrational relaxation rate is $1/T$ and M is a number that characterizes the equilibrium ratio of N_A to N_{001} .

Adding Eq. (E.1) to Eq. (E.2) and integrating yields

$$N_A(t) + N_{001}(t) = N_A(0) + N_{001}(0) + (\gamma + \Gamma)t, \quad (\text{E.3})$$

where $t = 0$ corresponds to the passage time of the trailing edge of the pulse. Substituting for $N_A(t)$ in Eq. (E.2) using Eq. (E.3), then integrating give the following solution:

$$N_{001}(t) = N_{001}(0) e^{-\alpha t} + \left[\gamma + \frac{N_A(0) + N_{001}(0)}{MT} \right] \frac{1 - e^{-\alpha t}}{\alpha} + \frac{\gamma + \Gamma}{MT} \frac{1}{\alpha^2} \left[(\alpha t - 1) + e^{-\alpha t} \right], \quad (\text{E.4})$$

where

$$\alpha = \frac{1 + M}{MT}. \quad (\text{E.5})$$

Substitution of Eq. (E.5) into Eq. (E.3) immediately gives the solution for $N_A(t)$.

The populations n_a and n_0 are related to N_{001} by

$$n_a(t) + n_0(t) = N_{001}(t), \quad (\text{E.6})$$

and they satisfy the following equations:

$$\dot{n}_a = \gamma_a - \frac{1}{\tau} n_a + \frac{1}{N\tau} n_0 + \frac{f}{MT} N_A - \frac{1}{T} n_a, \quad (\text{E.7})$$

$$\dot{n}_0 = \gamma_0 - \frac{1}{N\tau} n_0 + \frac{1}{\tau} n_a + \frac{1-f}{MT} N_A - \frac{1}{T} n_0, \quad (\text{E.8})$$

and

$$f = \frac{1}{N + 1} \quad . \quad (E.9)$$

Here, N equals the equilibrium ratio of n_0 to n_a and $1/\tau$ is the rotational relaxation rate. Obviously, adding Eq. (E.7) to Eq. (E.8), using Eq. (E.6), gives Eq. (E.2), provided γ is given by

$$\gamma = \gamma_a + \gamma_0 \quad . \quad (E.10)$$

Eliminating $n_0(t)$ from Eq. (E.6), using Eq. (E.6), and using Eq. (E.3) to replace $N_A(t)$, then integrating directly gives

$$\begin{aligned} n_a(t) = & n_a(0) e^{-\beta t} + \left\{ \gamma_a + \frac{f}{MT} \left[N_A(0) + N_{001}(0) \right] \frac{1 - e^{-\beta t}}{\beta} \right\} \\ & + \frac{(\gamma + \Gamma)f}{MT} \frac{1}{\beta^2} \left[(\beta t - 1) + e^{-\beta t} \right] \\ & + \left(\frac{1}{N\tau} - \frac{f}{MT} \right) \left(N_{001}(0) \frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha} \right. \\ & + \frac{1}{\alpha} \left[\gamma + \frac{N_A(0) + N_{001}(0)}{MT} \right] \left(\frac{1 - e^{-\beta t}}{\beta} - \frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha} \right) \\ & \left. + \frac{\gamma + \Gamma}{MT} \frac{1}{\alpha^2} \left\{ \frac{\alpha}{\beta^2} \left[(\beta t - 1) + e^{-\beta t} \right] - \frac{1 - e^{-\beta t}}{\beta} + \frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha} \right\} \right) \quad . \end{aligned} \quad (E.11)$$

$$\beta = \frac{1}{\tau} + \frac{1}{N\tau} + \frac{1}{T} \quad . \quad (E.12)$$

The particular grouping of terms in Eq. (E.11) allows us to verify limiting cases easily. Obviously, $n_a(t) \rightarrow n_a(0)$ as $t \rightarrow 0$. Less obvious is the result that as $T \rightarrow \infty$ or $\alpha \rightarrow 0$, Eq. (E.11) reduces to

$$n_a(t) \xrightarrow{t \rightarrow \infty} n_a(0) e^{-\bar{\beta}t} + \left(\gamma_a + \frac{N_{001}(0)}{N\tau} \right) \frac{1 - e^{-\bar{\beta}t}}{\bar{\beta}} + \frac{\gamma}{N\tau} \frac{1}{\bar{\beta}^2} \left[(\bar{\beta}t - 1) + e^{-\bar{\beta}t} \right], \quad (E.13)$$

where

$$\bar{\beta} = \frac{N + 1}{N\tau}. \quad (E.14)$$

In this limit, N_A is decoupled from the repumping and we see that the form of Eq. (E.13) is the same as the form of Eq. (E.4), with obvious changes in the constants.

We obtain $n_0(t)$ directly from Eq. (E.6) using Eq. (E.11) and (E.4), and we can verify that as $t \rightarrow \infty$ all of these solutions asymptotically approach the physically expected results; that is,

$$N_A(t) \xrightarrow{t \rightarrow \infty} \left[N_A(0) + N_{001}(0) + (\gamma + \Gamma)t \right] \frac{M}{1 + M},$$

$$N_{001}(t) \xrightarrow{t \rightarrow \infty} \left[N_A(0) + N_{001}(0) + (\gamma + \Gamma)t \right] \frac{1}{(1 + M)},$$

$$n_a(t) \xrightarrow{t \rightarrow \infty} \left[N_A(0) + N_{001}(0) + (\gamma + \Gamma)t \right] \frac{1}{(1 + M)(1 + N)},$$

and

$$n_0(t) \xrightarrow{t \rightarrow \infty} \left[N_A(0) + N_{001}(0) + (\gamma + \Gamma)t \right] \frac{1}{1 + M} \frac{N}{1 + N}.$$

Obviously a formally identical solution applies for the relaxation of the lower level n_b . If we denote lower-state quantities N_{SB} , N_{100} , n_b , and n_1 , which correspond to the upper-state quantities N_A , N_{001} , n_a , and n_0 , respectively, the solutions allow us to compute "continuation conditions" for the second pass of a pulse through the same gain volume. The sequence of Fermi resonances between the symmetric (ν_1) and bending (ν_2) modes in CO_2 suggests that N_{SB} , the combined populations of those modes, might be the appropriate reservoir for the 100-level. The usual four-level scheme of Appendix C requires the following four quantities as "initial conditions" for evaluation of the second pass of the pulse: $g(t_1) = \sigma[n_a(t_1) - n_b(t_1)]$, $f = n_b(t_1)/n_a(t_1)$ [not to

be confused with f of Eq. (E.9) above], $N_1 = n_0(t_1)/n_a(t_1)$, and $N_2 = n_1(t_1)/n_b(t_1)$. Here t_1 is the time interval when the field-free relaxation occurs; obviously it will vary with position in the gain medium (due, for example, to the distance to a reflecting mirror placed at one end of an amplifier in a two-pass system). Note that kinetic rates such as $1/T$, $1/\tau$, γ , Γ , etc., may be different for upper and lower manifolds. (See footnote* and references therein for pumping rates in high-pressure electron-beam-controlled discharge CO_2 amplifiers.) Finally, we need features not yet in LAPU2 to treat multiple-pass configurations in which, because of either long pulse lengths or gain media bounded by mirrors, or both, part of a gain medium is acted upon simultaneously by forward and backward propagating portions of the same pulse near a reflecting surface.

*O. Judd, "Fundamental Kinetic Processes in the CO_2 Laser" in High Power Gas Lasers, 1975, E. R. Pike, Ed. (The Institute of Physics, London, 1976).