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BY

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# Coalescence Instability of Magnetic Islands

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## ABSTRACT

We have investigated the stability of a periodic magnetic island structure using the ideal MHD equations. An instability is found which describes the tendency toward coalescence of parallel currents in the neighboring islands. It is expected that this instability will proceed at a fast MHD rate as long as the forces driving the instability can overcome the stabilizing forces due to the compression of the magnetic field between the islands. Beyond that phase, resistivity is expected to dominate the tendency toward island coalescence. Island coalescence of this kind can explain why in the observation of tearing mode instabilities in tokamaks, only the modes with minimum values of m and n are seen.

## I. INTRODUCTION

It is well-known that a current layer separating two regions of oppositely directed magnetic fields in a resistive plasma is unstable to the tearing mode instability which produces two-dimensional magnetic configurations, sometimes called magnetic islands. The linear and nonlinear development of such configurations have been of great interest in astrophysics and plasma physics for more than a decade now.[1,2]

Recent observations of large amplitude helical distortions around rational magnetic surfaces[3,4] in tokamaks have led to a great deal of renewed interest in the properties of magnetic islands. Magnetic islands in tokamaks are basically two-dimensional configurations with helical symmetry which confine the plasma in localized r-θ domains in and around the original rational magnetic surface. In this paper we investigate the stability of a periodic array of magnetic islands to those ideal MHD perturbations which have a tendency to coalesce smaller islands into larger ones. Such a coalescence tendency is known to be energetically favorable and has been conjectured upon previously.[5]

A particularly simple island configuration is the infinite array in slab geometry shown in Fig. 1. This configuration is especially suited to space plasma and astrophysicial applications and is also a reasonable approximation to a very high m number mode in a tokamak; the ignorable direction (z coordinate) is the direction of helical symmetry. Also shown as inserts in Fig. 1

are plots of the plasma current $j_z$ along the symmetry direction as a function of the coordinate x of monotonic variation and the coordinate y of periodic variation. Note that a magnetic island structure can also be interpreted as an array of current filaments superposed on a basic minimum current $j_{zm}$. The O points are the points with the peak current in the filament and the X points are the points with minimum current. The coalescence tendency is now easy to understand on physical grounds since it simply means that parallel currents (and hence the O points) attract each other. Now it is well known that in ideal MHD a moving fluid carries flux with it. Since the flux cannot move through X points (where the resultant electric fields would drive infinite current in the zero-resistivity limit), the coalescence tendency of two O points produces a piling up of flux on both sides of the X point. This produces a stabilizing magnetic pressure which opposes the coalescence tendency. The present investigation is thus aimed at determining whether conditions can be found in the limit of linear MHD theory where the destabilizing tendency due to current attraction can overcome the stabilization due to magnetic field buildup near X points.

We have used the energy principle [6] for studying the stability of magnetic islands. We restrict our attention to two-dimensional perturbations i.e., perturbations with the same symmetry as the island itself. Furthermore, we consider only incompressible perturbations in the x-y plane. In space plasma applications this can be justified as a high β approximation whereas in the low β tokamak plasma this is justified because of the large toroidal magnetic field which cannot be compressed easily and therefore prevents compression of the plasma in ideal MHD. This puts further constraints on the choice of perturbations. Using the results of an approximate normal mode analysis we choose the trial function in such a manner that the coalescence velocity (i.e., the y velocity) is localized near the O points in the y direction and also reverses sign in the neighborhood of the edge of the island in the x direction. Our energy principle calculation shows that a coalescence instability can occur in an island configuration with zeroth-order current fluctuation within the island as small as a few percent. The calculations also indicate that a threshold amplitude of the current fluctuation in the island must be exceeded in order that the coalescence instability may occur. In addition, to make the investigation more realistic for tokamaks, we have also done the calculations in cylindrical geometry, with very similar results.

Let us speculate a little on the consequences of this instability. In ideal MHD, the X points are fixed relative to the fluid. Thus the instability will result in a distortion on the current profiles in a fixed grid of X points, the distortion being such as to bring the current maxima towards each other [Fig. 2a]. Inclusion of resistivity should permit the merging process of two islands to proceed smoothly to completion with two O points and one X point disappearing and a new O point emerging [Fig. 2b]. The whole merging process should proceed on some intermediate time scale between resistive and MHD time scales and can therefore be quite rapid. To speculate further, it is possible that the merging rate would be governed by Petschek's mechanism[2], and would therefore approach the MHD rate. In toroidal configurations, the consequence of this merging would be that high m, n modes continue to coalesce until the minimum values of m and n consistent with the helicity requirement of the particular rational surface is reached. Observationally, there is good evidence that high m, n values are observed early in tokamak discharge and disappear later leaving only low m, n values behind[3]. There is also definite computer simulation evidence for this kind of coalescence of magnetic islands[7].

It is interesting to note that the instability requires $\frac{\partial}{\partial\psi} J_z \neq 0$ within the island. Thus it appears that the saturated state of the nonlinear tearing mode in some of the recent theories [8] may not be unstable to these perturbations. However, these processes could be taking place during the nonlinear growth phase when $\frac{\partial J_z}{\partial\psi} \neq 0$ and the temporal growth is algebraic[9] and hence slow.

## II. EQUILIBRIA AND BASIC EQUATIONS

We study the phenomenon of coalescence of the currents within magnetic islands by considering perturbations on an exact MHD equilibrium possessing islands. Such an equilibrium is characterized by a flux function $\psi_o$ which satisfies the equation

$$\nabla^2\psi_o = f(\psi_o) \tag{1}$$

for any arbitrary functional form f. The simplest choice for f would be a linear function. However, the solutions to the linear equation

$$\nabla^2\psi_o + k^2\psi_o = 0$$

in slab geometry ,

$$\psi_o = \cos kx + \varepsilon \cos [\alpha(1 + \alpha^2)^{-1/2} kx] \cos [(1 + \alpha^{2-1/2} ky] \tag{2a}$$

and in cylindrical (i.e., helical) geometry [10].

$$\psi_o = J_o(kr) + \varepsilon J_m (kr) \cos (m\theta\ k_z z) \tag{2b}$$

are unacceptable because they lead to island chains either very close to each other or to the walls. Fortunately, there is a well known exact equilibrium in slab geometry [11], which can be readily extended to the cylindrical case (as shown below) and does not suffer from these defects. It satisfies the equation

$$\nabla^2\psi_o = 4\pi j_z(\psi_z) = (1 - \varepsilon^2)\ k^2 \exp(-2\psi_o)$$

and gives

$$\psi_o = \ln (\cosh kx + \varepsilon \cos ky) \tag{3}$$

This is a generalization to two dimensions of the tanh profile $B_y \sim \tanh x$. We may thus write the equilibrium field as

$$\underset{\sim}{B}_o = B_p\, \underset{\sim}{e}_z \times \nabla\psi_o\ . \tag{4}$$

As already mentioned, we assume the perturbations to have the same symmetry as the equilibrium. We can thus introduce a perturbation flux function $\psi_1(x,y)$ satisfying $\underset{\sim}{B}_1 = \underset{\sim}{e}_z \times \nabla\psi_1$.

Furthermore, as discussed in the introduction, we assume incompressibility of the velocity pertubations in the x-y plane; hence we may introduce the velocity potential $\phi(x,y)$ satisfying $v_\perp = \underset{\sim}{e}_z \times \nabla\phi$. In a tokamak plasma, this is justified by the large toroidal field ($B_T >> B_p$) in the z-direction. For space plasma applications, incompressibility is justified by a high $\beta$ approximation. (For $\beta >> 1$ all phenomena considered are so slow compared to sound wave time scales that density perturbations get washed out). In terms of the functions $\phi$ and $\psi$, the linearized ideal MHD equations for perturbations around the island equilibrium read

$$\frac{\partial\psi_1}{\partial t} = - v\cdot\nabla\psi_o = - \underset{\sim}{e}_z\cdot\left(\nabla\phi \times \nabla\psi_o\right),$$

$$\nabla^2\psi = 4\pi j_z/B_p, \tag{5}$$

$$\rho \frac{\partial}{\partial t} \nabla^2\phi = \underset{\sim}{B}_0\cdot\nabla j_{z1} + \underset{\sim}{B}_1\cdot\nabla j_{z0}$$

$$= B_p\underset{\sim}{e}_z\cdot\left(\nabla\psi_o \times \nabla j_{z1}\right) + B_p\underset{\sim}{e}_z\cdot\left(\nabla\psi_1 \times \nabla j_{z0}\right).$$

In the third equation we have assumed that the density $\rho$ is constant. This is consistent with the assumption that most of the motion occurs near the islands, which are assumed to be narrow. Assuming that $\phi$ and $\psi$ vary in time as $\exp(\gamma t)$, and expressing the equations in terms of Poisson brackets defined by $[f,g] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$, we find

$$\gamma\,\psi_1 = [\psi_0,\phi],$$

$$\frac{4\pi\rho\gamma}{B_p}\nabla^2\phi = [\psi_o,\nabla^2\psi_1] - \frac{d}{d\psi_o}(\nabla^2\psi_0)[\psi_0,\psi_1] \ . \tag{6}$$

Our slab model assumes that the plasma extends to infinity. Thus the energy principle for this equilibrium involves no boundary terms, and can be expressed in the form

$$\delta W = \int |\nabla\psi_1|^2 + \psi_1^2 \frac{d}{d\psi_0} \nabla^2\psi_0 \ , \tag{7}$$

where the integral extends over the total plasma volume. Since we are concerned with incompressible perturbations, we also have the constraint

$$\oint \frac{d\ell}{|\nabla\psi_0|} \ \psi_1 = 0 \ , \tag{8}$$

where the integral is along a flux surface of $\psi_0$. The first term in (7), which is stabilizing, is due to the term $\underset{\sim}{j}_1 \times \underset{\sim}{B}_0$ in the equation of motion [and thus the first term on the right hand side of 6a)], and is associated with the force encountered in trying to squeeze the flux surfaces up against the X points. The second term, which is destabilizing in our example, is due to the term $\underset{\sim}{j}_0 \times \underset{\sim}{B}_1$ in the equation of motion [and hence the second term on the right hand side of (6a)], and represents the attraction of the current filaments in adjacent islands.

So far we have discussed the slab model. For a tokamak plasma this is a good approximation to the actual situation only for large m. For smaller m, geometrical effects can play a role and so it is more pertinent to use a cylindrical (i.e., helical) model. One can readily extend the exact slab equilibrium given by Eq. (3) to the cylindrical case. By making the substitution $x = \ln(r/r_0)$ we note that the function

$$\psi_0 = \ln[\cosh mx + \varepsilon \cos (m\theta + kz)] + x$$

or

$$\psi_0 = \ln\left[\frac{1}{2}(r/r_0)^{m+1} + \frac{1}{2}(\gamma_o/\gamma)^{m-1} + \varepsilon (r/r_0) \cos(m\theta + k_z z)\right] \quad (9)$$

satisfies the equation

$$\nabla_2^2\psi_0 = 2k_z B_z + 4\pi m j_z(\psi_0) = m^2(1 - \varepsilon^2) e^{-2\psi_0} .$$

The operator

$$\nabla_2^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{m^2}{r^2}\frac{\partial^2}{\partial \tau^2}$$

with $\tau \equiv m\theta + k_z z$ is the usual Laplacian in cylindrical geometry with helical symmetry and incorporates the large aspect ratio approximation $k_z^2 r^2 << m^2$. Eq. (9) defines the flux function for our cylindrical equilibrium.

Eqs. (6) for the perturbations retain the same form as before provided we use the fact that in terms of the variables $(r,\tau)$ the Poisson bracket becomes

$$[f,g] = \frac{1}{r}\frac{\partial f}{\partial r}\frac{\partial g}{\partial \tau} - \frac{1}{r}\frac{\partial f}{\partial \tau}\frac{\partial g}{\partial r} ,$$

and also substitute the operator $\nabla_2^2$ for operator $\nabla^2$.

As far as the energy principle calculation goes, in the cylindrical model we would like to apply the boundary condition of a conducting wall at the plasma edge. However, the equilibrium (9) has no surface r = constant where $B_r$ is zero. Nevertheless, we may require the displacement to be zero on some surface $r = b$. We then expect the integral over the volume $r \leq b$ to be nearly independent of b if b is large enough. The expression for $\delta W$ in the helical case is, except for an unimportant overall factor of $m^{-2}$, identical to (5) if we replace $|\nabla\psi_1|^2$ by $|\nabla_2\psi_1|^2$, where

$$|\nabla_2 f(r,t)|^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{m^2}{r^2}\left(\frac{\partial f}{\partial \tau}\right)^2$$

and, as before, replace $\nabla^2$ by $\nabla_2^2$. The same is true of the constraints (8).

Since the mode we envision is driven by the zeroth order current fluctuations, which are largest inside and near the islands, we expect it to be localized near the rational surface However, the localization cannot be too great if the mode is to be unstable, since $|\nabla\psi_1|^2$ is stabilizing. In particular, the rational surface cannot be too near the conducting wall (in the cylindrical case). Also, if the equilibrium model has more than one set of islands, we must require that they be sufficiently far apart. This is because we do not want the interaction of two distinct chains of islands to interfere with our study of interactions within one chain. It is for precisely these reasons that we must **reject** the equilibria (2a) and (2b).

We expect the perturbations to cause two neighboring islands to approach each other. Fig. 2a shows the expected mode structure superimposed upon the equilibrium flux surfaces for the case of slab geometry. The y-component of velocity should vanish at the X-points ($y = 0, 2\pi, \ldots$) and therefore the velocity potential $\phi$ should be of the form

$$\phi(x,y) = \sum_{\ell=1}^{\infty} \phi_\ell(x)\sin(2\ell - 1)ky/2 \ . \qquad (10)$$

The functions $\phi_\ell(x)$ are odd in x and localized near the island, i.e., near $|kx| \lesssim \varepsilon^{1/2}$ .

## III. NORMAL MODE EQUATIONS FOR SLAB MODEL

We now proceed to study the normal mode equations (6) using the equilibrium (3). We shall assume $\varepsilon << 1$. Furthermore, we approximate the velocity potential $\phi$ in Eq. (10) by $\phi \sim \phi_1(x)\sin\frac{ky}{2}$. This is obviously a crude approximation because the equilibrium couples all the harmonics $\phi_\ell$ with each other. In fact, our calculation will show that in this approximation, the normal mode equations do not exhibit any instability. The coupling to higher harmonics is therefore essential for an instability calculation. However, the normal mode analysis with the inclusion of one or more harmonics becomes completely intractable. That is why we have to resort to an energy principle calculation in the next section. The calculations of the present section should therefore be primarily treated as a guide for the choice of trial functions in the $\delta W$ calculation.

Substituting (6a) into (6b), thereby eliminating $\psi_1$, we find

$$\Lambda^2\nabla^2\phi = \left[\psi_0, \ [\psi_0, \ \nabla^2\phi]\right] + 2\left[\psi_0, \left(\frac{\partial^2\psi_0}{\partial x^2} - \frac{\partial^2\psi_0}{\partial y^2}\right)\frac{\partial^2\phi}{\partial x\partial y}\right] - 2\left[\psi_0, \frac{\partial^2\psi_0}{\partial x\partial y}\left(\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2}\right)\right], \qquad (11)$$

where $\Lambda^2 = 4\pi\rho\gamma^2 B_p^{-2}$. It is interesting to note that the term

$$d/d\psi_0\left(\nabla^2\psi_0\right)\left[\psi_0, [\psi_0, \ \phi]\right]$$

exactly cancels. Now, if we approximate $\phi$ by $\phi_1(x)\sin\frac{1}{2}y$, (at this point we express x and y in terms of the dimensionless variables kx and ky), Eq. (11) reduces, after some algebra, to

$$\Lambda^2\left(\phi_1'''' - \frac{1}{4}\phi\right) = \frac{1}{16}\left(\tanh^2 x - \frac{\varepsilon}{\cosh x}\right)\phi_1$$
$$- \frac{\sinh x}{2\cosh^3 x}\phi_1'$$
$$- \frac{1}{4}\left(\tanh^2 x - \frac{\varepsilon}{\cosh x} - \frac{2\varepsilon}{\cosh^3 x}\right)\phi_1''$$
$$- \frac{\varepsilon^2 \sinh x}{\cosh^3 x}\phi_1''' + \frac{\varepsilon^2}{2\cosh^2 x}\phi_1'''' . \qquad (12)$$

(Primes denote derivatives with respect to x.)
We have assumed $\varepsilon \ll 1$, and have neglected higher harmonics (sin, cos of $\frac{3}{2}y$, $\frac{5}{2}y$.)

For the purpose of solving (12), the plane splits naturally into two regions, the inner region($|x| \ll \varepsilon^{1/2}$) and the outer region($|x| \gg \varepsilon^{1/2}$). If $\Lambda^2$ is ordered to be $\sim\varepsilon$, the outer region effectively has $\varepsilon,\Lambda^2 = 0$ and hence (12) reduces to

$$\tanh x\ \phi_1'' + \frac{2}{\cosh^2 x}\phi_1' - \frac{1}{4}\tanh x\ \phi_1 = 0 .$$

The two linearly independent solutions of this equation are $\phi_+(x)$ and $\phi_-(x)$, defined by

$$\phi_\pm(x) = e^{\pm x/2}\left(\frac{1}{\tanh x} \mp 2\right). \qquad (13)$$

Since $\phi_1$ must be odd in x, and must vanish as $x \to \pm \infty$, the outer solution is $\phi_-(x)$ for $x > \varepsilon^{1/2}$.

In the inner solution we expect d/dx to scale as $\varepsilon^{-1/2}$, and so we define $\xi = 2^{1/4}\varepsilon^{-1/2}x$. Using $\varepsilon << 1$ and remembering $\Lambda^2 \sim \varepsilon$, (12) reduces to the fourth order equation

$$\frac{d^4\phi_1}{d\xi^4} - \left(a + \frac{1}{4}\,\xi^2\right)\frac{d^2\phi_1}{d\xi^2} - \frac{1}{2}\,\zeta\,\frac{d\phi_1}{d\xi} = 0 \tag{14}$$

where $a = 2^{1/2}\left(\Lambda^2/\varepsilon - 3/4\right)$. Expressing this equation in terms of $W = d\phi_1/d\xi$ and integrating once, we find that W satisfies the inhomogeneous Weber equation

$$\frac{d^2W}{d\xi^2} - \left(a + \frac{1}{4}\,\xi^2\right) W = K, \tag{15}$$

where K is, at present, an arbitrary constant. Upon Fourier transforming with respect to the variable k', (15) becomes

$$\frac{d^2\tilde{W}}{dk'^2} - 4\,(k'^2 + a)\,\tilde{W} = 8\pi K\delta(k'). \tag{16}$$

The solution which is even in k' and vanishes as $k' \to \infty$ is

$$\tilde{W}(k') = 2\pi K U'(a,0)^{-1}\, U(a, 2|k'|), \tag{17}$$

where U is the parabolic cylinder function which vanishes as $k' \to \infty$. [12] The constant K is determined by matching $W(\xi)$ as $\xi \to \infty$ with (13) as $x \to 0$. The result is

$$W(\xi) = \frac{1}{2^{3/4}\varepsilon^{1/2}U'(a,0)}\int_0^\infty U(a,2k')\cos k'\xi \, d\xi' \tag{18}$$

To obtain an eigenvalue condition we now match $\partial\psi_1/\partial x$ between the inner and outer regions. Equation (6a) shows that, if we again ignore higher harmonics, $\psi_1$ takes the form $\psi_1(x)\cos\frac{1}{2}y$, where

$$\psi_1 = (2\gamma)^{-1}[\tanh x\phi_1(x) + \varepsilon\phi_1'(x)/\cosh x] \ . \tag{19}$$

Also, (6b) shows

$$\frac{4\pi\rho\gamma}{B_p^2}\phi_1'' = \frac{\varepsilon}{2\cosh x}\left(\psi''' - \frac{1}{4}\psi_1'\right) - \frac{1}{2}\tanh x \ (\psi_1'' - \frac{1}{4}\psi_1) \ . \tag{20}$$

In the inner region, these become

$$\psi_1 = (2\gamma)^{-1}2^{-1/4}\varepsilon^{1/2}(\xi\phi_1 + 2^{1/2}d\phi_1/d\xi) \tag{21}$$

and

$$2^{5/4}\varepsilon^{-1/2} \ 4\pi\rho\gamma B_p^{-2}\frac{d^2\phi_1}{d\xi^2} = 2^{1/2}\frac{d^3\psi_1}{d\xi^3} - \xi\frac{d^2\psi_1}{d\xi^2} \tag{22}$$

Substituting (21) in the first term on the right of (22), dividing by $\xi$ and integrating, we find

$$\Lambda' \equiv [2\psi(0)]^{-1}[\psi_1'(\infty) - \psi_1'(-\infty)]$$

$$= -2^{3/2}(\Lambda^2/\varepsilon - 3/4)\int_{-\infty}^{\infty}\frac{d\xi}{\xi}\frac{d^2\phi_1}{d\xi^2} + \int_{-\infty}^{\infty}\frac{d\xi}{\xi}\frac{d^4\phi_1}{d\xi^4} \ . \tag{23}$$

Therefore, using (18) we find

$$\Delta' = \frac{2^{1/4}\pi}{\varepsilon^{1/2} U'(a,0)} \int_0^\infty \left(ak' + \frac{1}{2} k'^3\right) U(a,2k')dk \tag{24}$$

To complete the matching, (6a) and (13) show that $\psi_1$ in the outer region is given by

$$\psi_1 = (2\gamma)^{-1} e^{-x/2} \quad (1+2\tanh x) .$$

This leads to

$$\Delta' = 3/2 \tag{25}$$

The eigenvalue condition is given by (24) and (25). An instability exists if there is a solution with $a > a_c = -3\times 2^{1/2}/4$. However, it can be shown that the right-hand side of (24) is negative for $a > a_c$ [in fact both the integral and the denominator $U'(a,0)$ vanish at $a = -1/2$] and therefore no instability occurs.

There is an obvious case for the extension of above normal mode calculation to a situation where higher harmonics in (10) are retained. However, as mentioned above, the calculations turn out to be intractable. The basic difficulty is that in the inner region all the harmonics couple to each other very strongly. The inner region equations are thus an infinite chain of coupled second order differential equations with no obvious small parameter. Since the normal mode calculation turned out to be so difficult, this motivated us to look at the stability using the energy principle. This is discussed in the next section. Results (13) and (18) derived above are used to construct appropriate trial functions for the $\delta W$ calculation in the next section.

## IV. ENERGY PRINCIPLE

In this section we present the results of numercial integration of Eq. (7), both for slab and helical geometry. We introduce a displacement potential $\eta = \gamma^{-1}\phi$ so that the perturbed flux is expressed in the form

$$\psi_1 = [\psi_0, \eta] \ , \tag{26}$$

where $\psi_0$ is given by (3) and (9). The constraint (8) is automatically satisfied if we express $\psi_1$ in this form.

We first treat the slab model. For a trial function we use the properties of $\phi$ (and hence of $\eta$) derived in the previous section. We approximate (10) by

$$\eta(x,y) = \eta_0(x)(\sin\tfrac{1}{2}\, y + \alpha \sin\tfrac{3}{2}\, y + \beta \sin \tfrac{5}{2}\, y + \ldots) \tag{27}$$

and use (18) to see that $\eta_0(x) \sim x/\varepsilon$ for $|x| << \varepsilon^{1/2}$, and that $\eta_0$ reaches a maximum $\eta_{om} \simeq \varepsilon^{-1/2}$ for $x \simeq \varepsilon^{1/2}$. ($\eta_o$ is, of course, assumed to be an odd function of x.) And from (13) we see that

$$\eta_0(x) \simeq e^{-x/2}(2 + \frac{1}{\tanh x}) \tag{28}$$

for $x >> \varepsilon^{-1/2}$. As a model we pick

$$\eta_0(x) = \tanh^2\left(\lambda\varepsilon^{-1/2}x\right)\ e^{-x/2}\left(2 + \frac{1}{\tanh x}\right), \tag{29}$$

so that there are three adjustable parameters, $\lambda$, $\alpha$ and $\beta$.

We perform the integration over y from 0 to $2\pi$ (i.e., over the length of one island) by Gauss-Legendre quadrature. The integration over x from 0 to $\infty$ is split into the inner and outer regions. We use a Gauss-Legendre quadrature in the inner region $0 \leq x \leq 2\varepsilon^{1/2}$ and a Gauss-Laguerre scheme in the outer region $2\varepsilon^{1/2} \leq x < \infty$.

The results are that if $\alpha = \beta = 0$, i.e., if no higher harmonics are present, $\delta W$ can be made negative only for $\varepsilon > .85$, (This does not contradict the results of the previous section, where we assumed $\varepsilon << 1$.) With $\alpha$ and $\beta$ nonzero, $\delta W$ is minimized at $\lambda \simeq 1$, $\alpha = -.32$, $\beta = .1$ and the instability occurs for a much smaller value of $\varepsilon$ namely $\varepsilon > .06$. (Incidentally, for $\varepsilon > .06$ the island full width divided by its length is 0.15.) Actually, $\lambda$, $\alpha$ and $\beta$ vary slightly with $\varepsilon$. These are typical values. Also, whereas the parameter $\alpha$ plays a crucial role in lowering $\varepsilon$, the parameter $\beta$ has little effect.

The fact that instability arises only when $\varepsilon > \varepsilon_c$ and the fact that $\varepsilon_c$ is rather insensitive to $\beta$ indicates to us that the instability has a threshold. This is far from a positive proof since, after all, (27) is certainly not an accurate quantitative description of the normal mode. However, our belief that a threshold exists for the instability is further strengthened by the following qualitative argument. The normal mode calculation of last section indicates that the mode localization is such that the stabilizing term $|\nabla\psi_1|^2$ in (7) should scale as $\varepsilon^{-1}$ compared to the destabilizing term. Thus the stabilizing term should always dominate as $\varepsilon \to 0$.

The conclusion $\alpha = -.32$ is interesting physically. Figure 3 is a sketch of the displacement $\xi_y = \partial\eta/\partial x$ as a function of y for x = 0. The displacement is very peaked around the O points and goes to zero rapidly near the X points. (In fact, for $\alpha = -1/3$ and $\beta \doteq 0$, both $\xi_y$ and $\partial\xi_y/\partial y$ are zero at the X points). This displacement is the one which minimizes the magnetic pressure near the X points without adversely affecting the attraction of the current filaments.

For the helical case, we concentrate on islands with m=4, n=2, (i.e., $k_z = -2/R$.) This structure is pictured as coalescing to a m=2, n=1 equilibrium which also has islands on the q=2 surface. As before, the normal mode equation in the outer region is obtained by setting $\varepsilon = \Lambda^2 = 0$. Then the outer region is obtained (with $r_0$ set equal to unity)

$$\frac{1}{r}\frac{d}{dr} r \frac{d\zeta}{dr} - \frac{m^2}{4r^2}\zeta + 8m^2(1-\varepsilon^2)\left(r^{m+1} + r^{1-m}\right)^{-2}\zeta = 0 \qquad (30)$$

where $\zeta(r) = \frac{1}{r}\frac{d\psi_0}{dr}\phi_1(r)$

For $r << 1$ or $r >> 1$ the third term is negligible and we obtain

$$\phi_1(r) = Ar^{\frac{m}{2}+2} + Br^{-\frac{m}{2}+2}$$

The rational surface is located at $r = r_s = [(1-m)/(1+m)]^{\frac{1}{2m}}$

Near $r = r_s$, we find

$$\phi_1(r) \sim (r^{-1} d\psi_0/dr)^{-1} \zeta(r_s)$$

$$\sim (m-1)^{-1+3/2m}(m+1)^{-1-3/2m}(r-r_s)^{-1}\zeta(r_s)$$

Using these relations we again construct a trial displacement potential function

$$\eta(r,\tau) = \eta_0(r)\ (\sin \tfrac{1}{2}\tau + \alpha \sin \tfrac{3}{2}\tau + \beta \sin \tfrac{5}{2}\tau + \ .\ .\ .) \qquad (31)$$

with $\eta_0$ given by

$$\eta_0(r) = \tanh^2[\lambda\varepsilon^{-1/2}(r-r_s)]\frac{r^4}{r_s^4}\left[\frac{A}{r_s-r} - \frac{A}{r_s} + r_s^4\right] \qquad r<r_s$$

$$= \tanh^2[\lambda\varepsilon^{-1/2}(r-r_s)]\ \frac{r^4-b^4}{r_s^4-b^4}\left[\frac{A}{r_s-r} - \frac{A}{r_s-b} + B(r_s^4 - b^4)\right] \quad r>r_s$$

(This equation has been specialized to m=4 for simplicity.) By numerical integration of (30) we determined that A = .0689. The quantities b, the radius of the conducting shell, and B were also determined, but we decided to treat them, as well as $\lambda$, $\alpha$ and $\beta$, as adjustable parameters.

This time the numerical integration was performed by Gauss-Legendre quadrature in both r and $\tau$. The results show that an instability develops for $\varepsilon > .12$, with $\lambda = 1.8$, $\alpha = -.2$. (This value of $\varepsilon$ corresponding to islands whose ratio of length to width is .11.)

The results show the same qualitative behavior as in the slab case. Therefore, for m=4 we conclude that geometric factors play a minor role. For large m values this is certainly true. We have not considered values of $m < 4$ because the $m=2$, $n=2$ case seems of less inherent interest; this is because the strong $m=1$ instability itself would preclude any observation of the coalescence we are describing.

## V. DISCUSSION AND CONCLUSIONS

By an energy principle calculation we have shown that a chain of magnetic islands is unstable to ideal MHD perturbations which correspond to a tendency towards coalescence of neighboring islands. This coalescence can proceed to completion only by relaxing the ideal MHD constraints near the X points by inclusion of resistivity or perhaps ergodicity of lines of force.

It is quite likely that this coalescence instability explains the absence of high m,n helical modes at later times in tokamak discharges[3]. (Alternatively, the absence of high n-number modes may be due to their reduced linear growth rate as the current channel peaks.) We should mention one defect of the equilibrium chosen by us, as far as application to tokamaks is concerned. Our equilibrium has a current profile which monotonically decreases on either side of the rational surface. In an actual experiment, of course, the rational surfaces are so placed that the current increases on one side and decreases on the other. We believe, however, that introduction of such an equilibrium will not significantly change any of our results because most of the dynamics of the instability is governed by processes highly localized around the rational surface (cf. our trial functions which disappear rapidly for large x).

Some recent observations on the magnetic field structure in the night side of the earth's magnetosphere are consistent with the generation of a single large island (bubble) by the tearing mode instability [13]. It is quite likely that any multi-island structures that may be generated by the linear tearing instability would ultimately go over into the single island state by the coalescence instability discussed above.

Biskamp et al. [14] have considered the stability of a two-dimensional configuration with neutral points. They find an instability due to finite electron-Larmor radius effects, related to particle orbits near the X points. We believe that the ideal MHD instability discussed by us above will have a much faster growth rate under most conditions.

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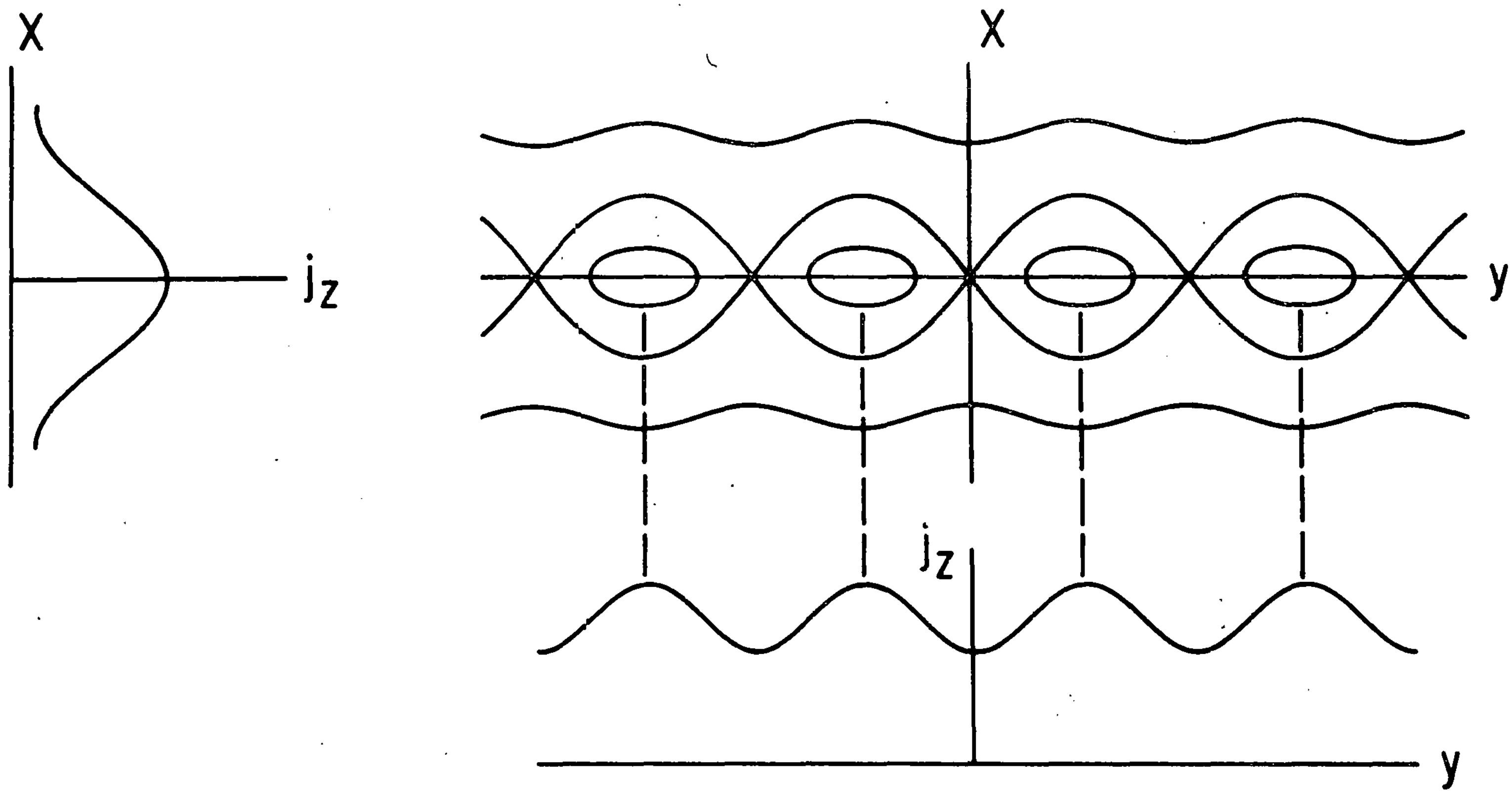


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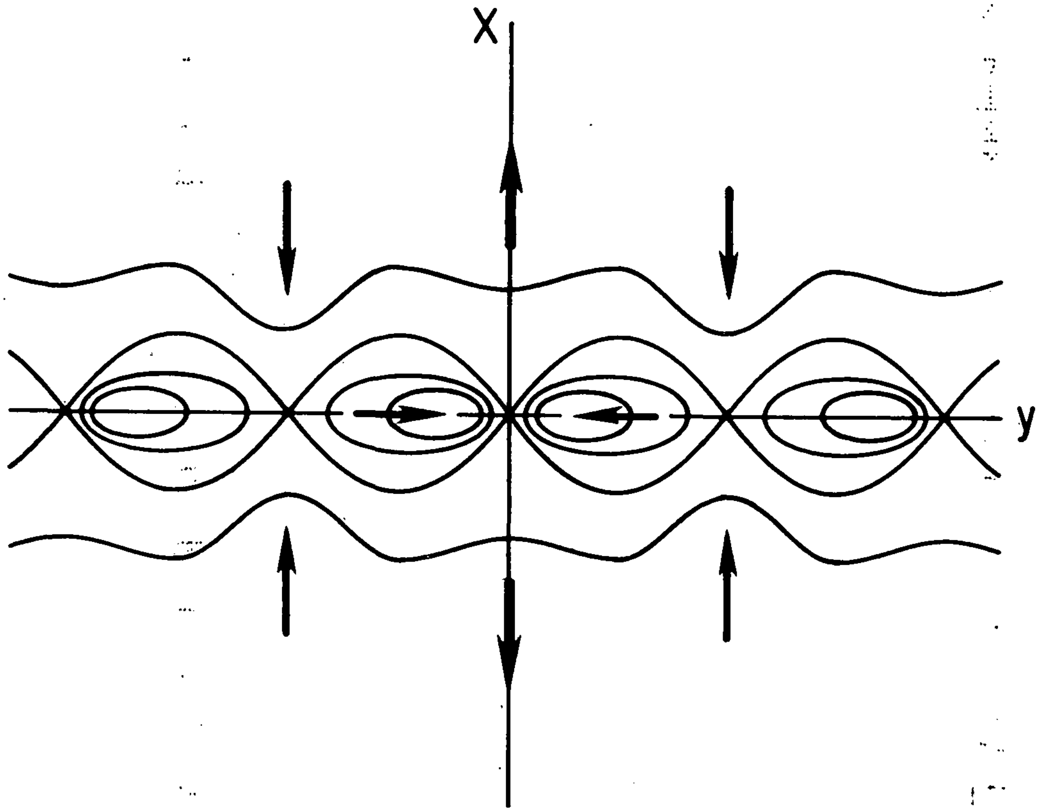
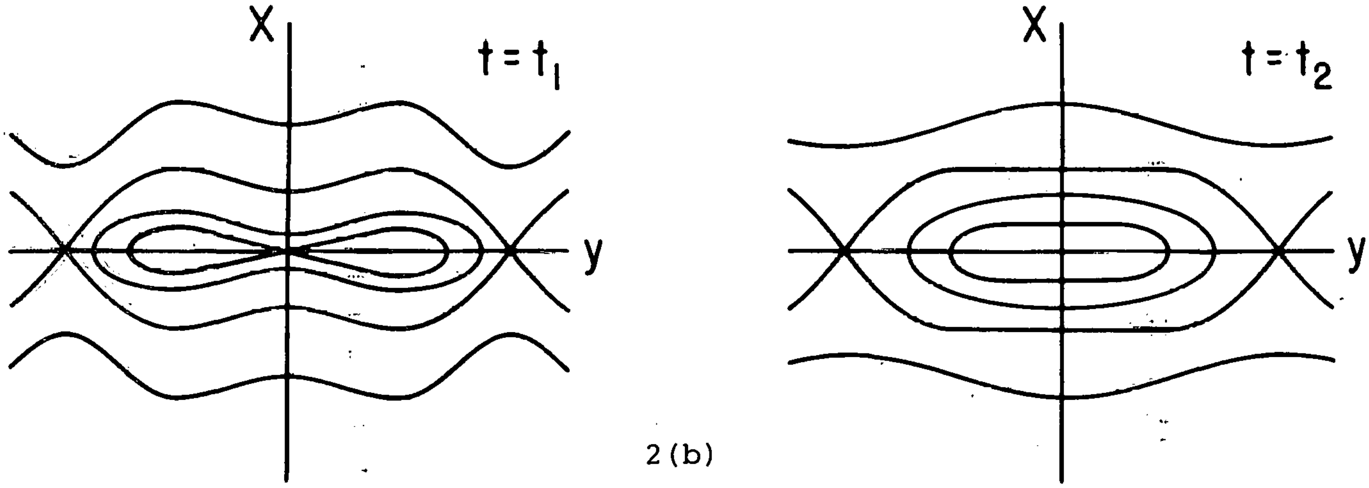
Fig. 1. Equilibrium flux surfaces and current distributions in slab geometry.




2(a)




2(b)

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Fig. 2(a). Perturbed flux surfaces with zero resistivity. (b). Conjectured merging of neighboring islands with finite resistivity at times $t_1$, $t_2 > t_1$.

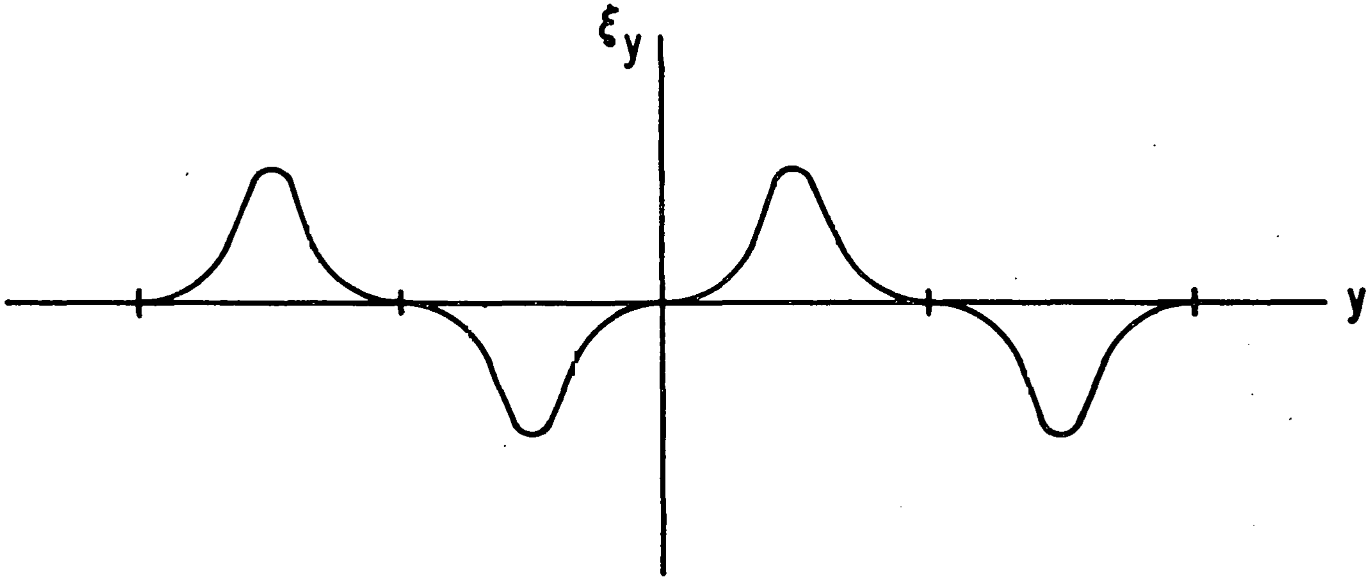


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Fig. 3. The displacement $\xi_y$ which minimizes $\delta W$. The X-points are marked on the y axis.