OPTICAL THEOREM FOR HEAVY-ION SCATTERING*

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Integrated elastic and total cross-sections cannot be defined in the presence of an unscreened Coulomb field. The problem appears as a familiar singularity in the differential elastic cross-section. Nevertheless an equivalent of the optical theorem can be stated in the charged situation¹:

$$\int \left[\left(\frac{d\sigma}{d\Omega} \right)_{el}(\theta) - \left(\frac{d\sigma}{d\Omega} \right)_{Coul}(\theta) \right] d\Omega + \sigma_{reaction} = \frac{4\pi}{k} \text{Im} f_n(0) \quad (1)$$

with the remainder or nuclear elastic scattering amplitude $f_n(0)$ defined as a difference of elastic and Coulomb amplitudes

$$f_n(\theta) = f(\theta) - f_{Coul}(\theta) \quad (2)$$

Care must be taken in deriving (1), with perhaps a screening parameter or cutoff in angular momentum introduced first and the proper limit then taken. In certain situations in heavy-ion reactions, at surprisingly high energies relative to the Coulomb barrier and for large Coulomb fields, the amplitude $f_n(0)$ is especially small, a result which may be understood semi-classically. A direct implication is that in these cases the reaction cross-section is determined by the measured elastic scattering differential cross-section and by the presumably known charge distributions of target and projectile. Thus optical potentials yielding equivalent fits to elastic scattering must predict identical $\sigma_{reaction}$'s. Calculations have been made to explore the dependence of $f_n(0)$ on potential parameters, mass, charge and energy, indicating in particular where $\text{Im} f_n(0)$ becomes large.

A heuristic derivation of (1) follows from a consideration of the partial wave expansion of the elastic amplitude $f_n(\theta)$ and of the total reaction cross-section σ_r . Introducing the elastic s-matrix element in the l^{th} partial wave

$$\eta_l = \eta_l^c \eta_l^n = e^{2i\sigma_l} \eta_l^n \quad (3)$$

where σ_l is the Coulomb phase shift and η_l^n would be the nuclear S-matrix in the absence of Coulomb forces, these expansions are

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$$\sigma_r = \pi \lambda^2 \sum_{\ell} (2\ell+1) (1-|\eta_{\ell}|^2) = \pi \lambda^2 \sum_{\ell} (2\ell+1) (1-|\eta_{\ell}^n|^2) \quad (4)$$

and

$$f(\theta) = f_c(\theta) + \frac{1}{2ik} \sum_{\ell} (2\ell+1) (\eta_{\ell} - \eta_{\ell}^c) P_{\ell}(\cos\theta) \quad (5)$$

The Coulomb amplitude in (5) is often taken to be the point charge Rutherford value²

$$f_c(\theta) = \frac{\eta}{2k \sin^2 \theta/2} e^{-i\eta \ln(\sin^2 \theta/2) + i\pi + 2i\sigma_0} \quad (6)$$

the effects of a finite size charge distribution being incorporated in $f_n(\theta)$. We note

$$|1-\eta_{\ell}|^2 - |1-\eta_{\ell}^c|^2 = -(1-|\eta_{\ell}^n|^2) - [(\eta_{\ell} - \eta_{\ell}^c) + (\eta_{\ell} - \eta_{\ell}^c)^*] \quad (7)$$

With either a cutoff in partial waves ($\ell \leq \ell_{\max}$) or a screening in the Coulomb field this latter equation may be multiplied by $\pi \lambda^2 (2\ell+1)$ and summed over ℓ to yield

$$(\sigma_{el} - \sigma_{Coul}) = -\pi \lambda^2 \sum_{\ell} (2\ell+1) (1-|\eta_{\ell}^n|^2) - \frac{4\pi}{k} \left(\frac{1}{4k} \sum_{\ell} (2\ell+1) [(\eta_{\ell} - \eta_{\ell}^c) + c.c.] \right) \quad (8)$$

The right hand side of Eq. (8) is defined, and finite, even in the limit of no screening (or $\ell_{\max} \rightarrow 0$), since for large ℓ , $\eta_{\ell} \rightarrow \eta_{\ell}^c$ (and hence $\eta_{\ell}^n \rightarrow 1$) sufficiently rapidly.

Alternatively defining the elastic differential cross-section by

$$\frac{d\sigma_{el}}{d\Omega} = |f(\theta)|^2 = |f_c(\theta)|^2 + |f_n(\theta)|^2 + 2\text{Re} f_c(\theta) f_n^*(\theta) \quad (9)$$

we see the difficulty in the left hand side of Eq. (1) occurs for the integral of the Coulomb-nuclear interference near $\theta = 0$. The integrated cross section difference

$$I(\theta_0) = \sigma_{el}(\theta_0) - \sigma_{Coul}(\theta_0) = 2\pi \int_{\theta_0}^{\pi} |f_n(\theta)|^2 \sin\theta d\theta + 2\pi \int_{\theta_0}^{\pi} 2\text{Re} f_c^* f_n \sin\theta d\theta$$

(where $\sigma(\theta_0) = 2\pi \int_{\theta_0}^{\pi} \frac{d\sigma}{d\Omega} \sin\theta d\theta$)

(10)

is finite but possesses a rapid oscillation for small θ_0 . This situation is admirably discussed by Cooper and Johnson¹ who give for the integral from 0 to θ_0

$$2\pi \int_0^{\theta_0} 2\text{Ref}_c^* f_n \sin\theta d\theta = \frac{4\pi}{k} f_n(0) - \frac{4\pi}{k} \text{Im}f_n(0) \cos W - \frac{4\pi}{k} \text{Ref}_n(0) \sin W \quad (11)$$

+ terms which vanish as $\theta_0 \rightarrow 0$,

with $W = \eta \ln \sin^2 \theta / 2 - 2\sigma_0$.

The combination

$$I(\theta_0) + 2\pi \int_0^{\theta_0} 2\text{Ref}_c^* f_n \sin\theta d\theta$$

is finite and well behaved but an attempt to calculate $I(\theta_0)$ for $\theta_0 \rightarrow 0$ from say an optical model will lead to problems. Indeed further use of the partial wave expansions for f_c , f_n results in

$$2\pi \int_0^{\pi} 2\text{Ref}_c^* f_n \sin\theta d\theta = \frac{4\pi}{k} (f_n(0) - \hat{f}_n(0)) \quad (12)$$

where yet another reference nuclear amplitude is introduced

$$\hat{f}_n(\theta) = \frac{1}{2ik} \sum (2\ell+1) (\eta_\ell^n - 1) P_\ell(\cos\theta) \quad (13)$$

leading to a perhaps more normal form of optical theorem

$$\int d\Omega |f_n(\theta)|^2 + \sigma_r = \frac{4\pi}{k} \text{Im}\hat{f}_n(0) \quad (14)$$

Having now established the finiteness of Eq. (1) and having indicated the difficulties in handling this equation we may turn to applications in purely heavy-ion situations. In Fig. 1 is displayed a typical deflection function for heavy-ion induced elastic scattering. At an observation angle θ the elastic amplitude may be evaluated semiclassically from a steepest descents treatment as

$$f(\theta) = f_1(\theta) + f_2(\theta) + f_3(\theta) \quad (15)$$

Contributions at smaller impact parameters are greatly suppressed by absorption. The most exterior contribution at the impact parameter s_1 may be very

nearly identified with the purely Coulomb amplitude

$$f_1(\theta) \approx f_c(\theta) \quad (16)$$

and hence the remainder or nuclear amplitude is

$$f_n(\theta) = f_2(\theta) + f_3(\theta) \quad (17)$$

Now, a major effect of the strong absorption is to turn the deflection function away from its purely nuclear form, from which even negative deflection angles may result, back to a nearly Coulomb value. For the deflection function in Fig. 1 it would appear $f_n(0) = 0$, i.e., no glory scattering occurs. In actual fact scattering at $\theta < \theta_{\min}$ will obtain at complex impact parameter, implying damping of such scattering. In particular $\text{Im}f_n(0)$ will be small in situations where a strong Coulomb field and absorption operate together. The effect of absorption enhances the action of the Coulomb force, eliminating paths at non-zero impact parameter which might have passed, on the average, undeflected through the overall nuclear + Coulomb potential. The Coulomb field sweeps flux away from the "back" of the target nucleus.

Assuming indeed $\text{Im}f_n(0)$ is small ($|\frac{4\pi}{k} \text{Im}f_n(0)| \ll \sigma_r$), a clear application of our heavy ion optical theorem is to situations in which the measured elastic scattering is fitted by two or more distinct optical potentials including however identical Coulomb potentials. Closely similar reaction cross sections $\sigma_r^{(1)} = \sigma_r^{(2)}$ must be predicted in the optical calculations, probably well within the limits set by experimental errors on the individual elastic points. More generally

$$\sigma_r^{(1)} - \sigma_r^{(2)} = \sigma_{\text{Coul}}^{(1)} - \sigma_{\text{Coul}}^{(2)} + \frac{4\pi}{k} [\text{Im}f_n^{(1)}(0) - \text{Im}f_n^{(2)}(0)] \quad (18)$$

perhaps implying that an actual measurement of σ_{reaction} may test deviations at short distances from the Coulomb potential generated by unpolarized spherical charge distributions. Care must be taken in discussing such constraints on the reaction cross-section because of the interdependences implicit in Eq. (18) even when $|\text{Im}f_n(0)|$ is small.

It is to be noted that optical theorem is acting here much as does the smile on the Cheshire Cat. We are not suggesting actual measurements at small angles to extract the "total cross-section" σ_n and the amplitude $f_n(0)$. Rather we are only using the likelihood $\text{Im}f_n(0)$ is small to impose

constraints on a theory extrapolated to all angles from measurements at some angles. It is interesting to note also that Eq. (1) does not guarantee $\sigma_n = \frac{4\pi}{k} \text{Im}f_n(0)$ is positive.

To test the detailed behaviour of $f_n(0)$ and σ_r , calculations have been performed for four elastic situations

(1)	$^{18}\text{O} + ^{58}\text{Ni}$	(Ref. 3)
(2)	$^{136}\text{Xe} + ^{209}\text{Bi}$	4)
(3)	$^{84}\text{Kr} + ^{208}\text{Pb}$	5)
(4)	$^{11}\text{B} + ^{26}\text{Mg}$	6)

using potential parameters varied about values obtained by fitting initially to data at 63.42, 1130, 700, and 114 MeV. Table 1 contains a summary of the results.

Some general observations: $\text{Im}f_n(0)$ remains small for higher energies relative to the barrier in the cases where the charge product Z_1Z_2 is largest. It would appear the Coulomb effects do not scale simply. In varying the parameters of the ^{16}O potential in Table 1 we did not fix the elastic scattering. It should be pointed out that when $\text{Im}f_n(0)$ is small different theories fitting the elastic scattering equally well must give nearly equal reaction cross-sections, but when $\text{Im}f_n(0)$ is not small such theories still need, by no means, diverge widely in predictions for σ_r . Also if in fitting to an optical potential some quasi-elastic cross-section is confused with elastic, then the resulting reaction cross-section is properly just a remainder.

REFERENCES

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TABLES: Calculations of σ_r , $\frac{4\pi}{k} \text{Im}f_n(0)$ for Changes in Mass Number, Potential Parameter and Energy.

Table 1a: Variation with $Z_1 Z_2$

	E_{Lab}	$E_{\text{c.m.}}$	E_{barrier}	$\frac{4\pi}{k} \text{Im}f_n(0)$ (barns)	σ_r (barns)
$^{18}\text{O} + ^{58}\text{Ni}$	63.4	48.4	32.6	0.035	1.18
$^{84}\text{Kr} + ^{208}\text{Pb}$	700	498.6	304	-0.003	2.78
$^{136}\text{Xe} + ^{209}\text{Bi}$	1130	684.6	430	-0.003	2.80
$^{11}\text{B} + ^{26}\text{Mg}$	120	84.3	12.3	-0.89	1.81

Potentials:

^{18}O :	$V = 100 \text{ MeV}$	$W = 20 \text{ MeV}$	$r_0 = 1.20 \text{ fm}$	$a = 0.55 \text{ fm}$
^{84}Kr :	$V = 100$	$W = 40$	$r_0 = 1.2$	$a = 0.55$
^{136}Xe :	$V = 8.47$	$W = 6.95$	$r_0 = 1.30$	$a = 0.44$
^{11}B :	$V = 55$	$W_{\text{vol}} = 20$	$r_0(\text{real}) = 1.08$	$a_R = a_S = 0.65$
		$W_S = 8.0$	$r_S = r_V = 1.17$	$a_V = 0.2$

Table 1b: Parameter Changes for $^{18}\text{O} + ^{58}\text{Ni}$

$E_{\text{Lab}} = 64.32$	$\frac{4\pi}{k} \text{Im}f_n(0)$	σ_r	
normal	0.035	1.18	
$r_0^I = 0.7$	0.079	1.10	
$W = 1.0$	0.057	0.83	
$V = 200$	-0.080	1.33	
$Z(180) = 4$		2.21	
0	6.433	3.227	$\sigma_{el} = 3.207$

Table 1c: Energy Dependence for $^{16}\text{O} + ^{58}\text{Ni}$, $E_{\text{barrier}} = 32.6$

E_{Lab}	E_{cm}	$\frac{4\pi}{k} \text{Im}f_n(0)$	σ_r
64.3	48.4	0.035	1.18
120	91.6	0.13	2.00
300	228.9	-0.59	2.38

Fig. 1 The deflection function $\Theta(s)$ (solid line) as a function of impact parameter (s) for typical heavy ion elastic scattering including absorption. The deflection function for pure Coulomb is shown as a dotted line and that in the real nuclear + Coulomb potential as a dashed line.