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Richard C. Arnold

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High Energy Hadron Dynamics Based on
a Stochastic-Field Multi-Eikonal Theory*

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ABSTRACT:

Multi-Eikonal theory, using a stochastic-field representation for collective long range rapidity correlations, is developed and applied to the calculation of Regge pole parameters, high transverse momentum enhancements, and fluctuation patterns in rapidity densities. If a short-range-order model, such as the one-dimensional planar bootstrap, with only leading t-channel meson poles, is utilized as input to the multi-eikonal method, the pole spectrum is modified in three ways; promotion and renormalization of leading trajectories (suggesting an effective pomeron above unity at intermediate energies), and a proliferation of dynamical secondary trajectories, reminiscent of dual models. When transverse dimensions are included, the collective effects produce a growth with energy of large- P_t inclusive cross-sections. Typical-event rapidity distributions, at energies of a few TeV, can be estimated by suitable approximations; the fluctuations give rise to "domain" patterns, which have the appearance of clusters separated by rapidity gaps. The relations between this approach to strong-interaction dynamics and a possible unification of weak, electromagnetic, and strong interactions are outlined.

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I. INTRODUCTION

The simultaneous requirements of quantum mechanics and special relativity place severe restrictions on dynamical models in particle physics. Investigations of dispersion relations and analyticity, known as S-matrix theory, begun two decades ago, led to the "bootstrap" suggestion⁽¹⁾ that any such model (which must properly incorporate unitarity, analyticity, and crossing relations) would be characterized by a very small number of free parameters.

The rationale underlying a lagrangian local field theory can also be stated as the expression of all particle observables in terms of a few parameters, such as coupling constants and bare masses of fundamental fields. Stated in this way, the goals of S-matrix theory and local field theory are closely analogous.

Incorporation of weak and electromagnetic interactions together with strong interactions has been an advantage of the field theory formalism, when the nonstrong interactions can be treated perturbatively. However, if there are conditions under which weak and electromagnetic interactions can result in nonperturbative (strong) effects, this advantage is dissipated. Such circumstances arise in both quark-confinement theories and in theories of universality between the classes of interactions at very high energies.

A bootstrap theory incorporating weak and electromagnetic interactions appeared impossible, at first glance, since the forces which bind particles together appear too weak to be self-sustaining. However, this reasoning is based on a dynamical picture including two (or a few) particles in the s-channel. If we admit important collective correlations in intermediate states of very high multiplicity we may find this problem disappears, as the combinatorics can overcome the small couplings. In this way it is possible to envisage leptons and photons as low-lying bound states of enormous numbers of hadrons.

An important role for collective effects in multiparticle unitarity requires a theory, for n -particle production matrix elements, which is of richer structure than a multiperipheral model (MPM) containing only near-neighbor corrections. (We define a theory for multiparticle production through an algorithm for constructing high-multiplicity matrix elements given a small, finite number of low-multiplicity matrix elements). We have investigated suitable extensions of MPM, incorporating Regge cut effects in the form of long range exclusive correlations between all pairs of particles.⁽²⁾ Our original motivation for the work described here was an improved strong-interaction phenomenology, allowing the analogs of critical phenomena in the Feynman-Wilson fluid analogy.⁽³⁾ However, we have found⁽⁴⁾ that collective phenomena could also manifest themselves in many other observables (e.g. large P_T cross-sections), and, in their theoretical role at astronomical energies, provide a mechanism for bootstrapping the nonstrong interactions.⁽⁵⁾

In a rough sense, we propose to identify collective hadronic phenomena (e.g. clustering in hadron production near zero impact parameter) with the basic "point-like" fields (partons) in a field theory of hadrons. It is conventional to interpret hadrons as composites of parton constituents; but conversely, it is possible instead to interpret such partons themselves as composites, whose constituents are ordinary hadrons. In both viewpoints, a literal picture can be only approximate and incomplete; any description of strongly interacting systems in terms of observable constituents and identifiable composite structure is necessarily oversimplified, in accordance with general principles of quantum mechanics and relativity.

We have previously⁵ described our basic hypothesis concerning the connection between weak and strong interactions. In the present work, we investigate particle spectra (through Regge poles), large- P_T behaviour, and rapidity-density fluctuation patterns, in strong processes. We show how to find and develop suitable mathematical tools to extract the behaviour at high energies of inclusives and the dominant contributions to s -channel unitarity sums.

These results provide us with a phenomenological hadron production theory which can be tested for consistency. It contains severe constraints between low-multiplicity and high-multiplicity cross-sections at high energies.

We will develop only the strong interaction sector in this paper. We begin by reviewing the general Physics viewpoint, generating functional formalism, and the integral equations of AT⁽⁴⁾ appropriate for weak long-range exclusive correlations, in sections 2-4. The appropriate theory in the presence of additional strong short-range correlations, in the form of a functional integral representation⁵, is then derived, in section 5. We then explore various aspects of the structure of the theory which are substantially different from NPM. Section 6 introduces characteristic parametrizations of the input amplitudes, including Regge cut effects. Section 7 discusses approximation methods for the functional integrals. In Section 8, the Output Regge Pole spectrum is discussed. Large $-P_T$ behavior is examined in Section 9, and rapidity-density fluctuations in Section 10. A brief overview is given in Section 11.

II. MULTI-EIKONAL THEORY

For theories of collective phenomena in systems with large numbers of particles the combinatoric factors, including internal symmetries, are very important.

The ranges and signs of interactions are next in importance ; next, their magnitudes and qualitative shapes ; and least of all, the specific functional forms.

-With these guides in mind, we propose to investigate a factorization structure for multiparticle S-matrix elements whose combinatoric structure is correct in at least two complementary limits : (a) nearest-neighbor correlation models, such as simplified NPM's, and (b) semiclassical emission-plus-Glauber-rescattering models. We conjecture also that a multiparticle Eikonal approximation of some field theories (e.g. ϕ^3) would contain the postulated structure, at least for large sub-energies, based on experience with two-body scattering amplitudes. This conjecture is embodied in our title, "multi-eikonal", for the theory.

We believe that the incorporation of a semiclassical-limit structure in the algorithm for constructing high-multiplicity matrix elements, given those of low-multiplicity (which are more directly related to crossing properties), is an essential and important step in developing a theory of hadrons based on S-matrix principles. The collective properties implied by this structure appear vital to the incorporation of weaker interactions into the bootstrap framework.⁽⁵⁾

Internal quantum numbers, spin, etc. will not be explicitly included in this paper, to simplify the presentation of other basic ideas. Such complications are probably essential; we have discussed quantum numbers in connection with the weaker interactions⁵.

The specific structure we study is a pair-wise factorizability in impact-parameter and rapidity variables.⁽⁴⁾

Let $A_n(\vec{R} | \vec{r}_1 \dots \vec{r}_n)$ be the matrix element for producing n particles in the central rapidity region, in a collision of hadrons a and a' . Here $\vec{r}_j = (y_j, \vec{b}_j)$, and $\vec{R} = (Y, \vec{B})$, where Y and $\vec{B}$ are the (relative) rapidity and (relative) impact parameters of the initial-state particles (a and a'). We implicitly adopt high-energy longitudinally-dominated kinematics, with two outgoing leading particles (c, c') approximately saturating energy and beam-direction momentum conservation laws.⁽⁴⁾ Then our assumption is the structure:

$$A_n(\vec{R} | \vec{r}_1 \dots \vec{r}_n) = A_0(\vec{R}) \prod_j G(\vec{r}_j | \vec{R}) \prod_k K(\vec{r}_j, \vec{r}_k | \vec{R}) \quad (2.1)$$

where A_0 , G , and K are independent of multiplicity, and thus determined by a knowledge of $n = 0, 1, 2$ amplitudes, e.g. from experiment. The combinatoric structure of A_n is represented in Figure 1.

A simple physical interpretation of such a structure is independent emission of individual particles, with "bare" probabilities $|G|^2$, followed by pair-wise rescattering among the outgoing particles with scattering phase shifts given by

$$\exp[i\delta(\vec{r}_j, \vec{r}_k)] = K(\vec{r}_j, \vec{r}_k) \quad (2.2)$$

which combine in a multiple-scattering series. Additivity of phase-shifts corresponds to our pair-wise product structure of the K 's, and is a natural assumption in the Eikonal picture. We would also expect K to depend only on differences of impact parameters and rapidities, in such a case. For notational convenience we introduce a function V defined by $\exp(V/2) = K$.

Although the above is a simple picture, we must be careful to avoid being too physically specific, since many types of models have such a decomposition, but with very different physical interpretations. For example, we also wish to include the simplified MPM structures, where V is nonzero only when $|y_j - y_k|$ is very small, but the magnitude of V may be arbitrarily large in such a region. G then incorporates Toller-angle dependences (damping in P_T) in Reggeon-particle-reggeon vertices associated with the leading-singularity exchanges.

We will henceforth assume G is independent of y , and that V is translation-invariant (not a function of $y_j + y_k$), in the central rapidity region, and that

$$V \rightarrow 0 \text{ as } |y_j - y_k| \rightarrow \infty \quad (2.3)$$

These conditions assure that the usual Regge t -channel factorization properties of leading singularities are preserved at the exclusive level.² On the basis of current data, we expect A_0 , the "skeleton" Reggeon exchange predominantly contributing to most of the production amplitudes, to have an energy dependence characteristic of charge-exchange.

Analytic computation at high energies of inclusives, overlap functions (including t -channel Regge poles), or high-multiplicity semi-inclusive observables, from our amplitudes is considerably more complex than in the MPM. The simplicity of the MPM lies in the one-term recursion relations

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between A_{n+1} and A_n , which enables a closed form solution to be obtained through solution of a linear integral equation. To apply our theory we must first develop analytic tools which provide a comparable facility. For this purpose we introduce a generating functional formalism, which allows us to obtain closed integral equations for inclusive quantities when V is small but of long range in rapidity (Regge cuts). When a strong short-range contribution is combined with such a weak long-range term, the problem can be formulated in a stochastic field-theory, to which well-known field-theoretic approximation methods can be applied.

We define (for each R) a generating function^{5,6} $F[\eta]$, where η is any complex-valued test function with arguments r , as follows:

$$F[\eta] = \frac{1}{|A_0|^2} \sum_{n=0}^{\infty} \frac{1}{n!} \int d^3r_1 \dots d^3r_n |A_n(r_1 \dots r_n)|^2 \eta(r_1) \dots \eta(r_n) \quad (2.4)$$

Note this definition does not depend upon our specific assumptions on the structure of amplitudes. Inclusive b -space sums can be computed as functional derivatives of $F[\eta]$. Thus, an analytic expression for F in terms of G and K will solve the problems alluded to above. To accomplish this when V is small, but collective phenomena are still important,⁴ we introduce an inclusive cluster expansion for F . The general form is:

$$\begin{aligned} \ln F[\eta] &= \ln F[1] \\ &+ \int d^3r g(r) [\eta(r)-1] + \frac{1}{2!} \iint d^3r_1 d^3r_2 u(r_1, r_2) [\eta(r_1)-1] [\eta(r_2)-1] \\ &+ \sum_{k \geq 2} \frac{1}{k!} \int \prod_{j=1}^k d^3r_j [\eta(r_j)-1] u_k(r_1 \dots r_k) \end{aligned} \quad (2.5)$$

The u_k 's are thus expansion coefficients in a functional Taylor series around $\eta = 1$. These coefficients may be determined in our theory through identification of the expression for F from our Ansatz (2.1). By definition

$$g(r) = \frac{1}{F[1]} \left. \frac{\delta F[1]}{\delta \eta(r)} \right|_{\eta=1} \quad (2.6)$$

Using (2.1) and (2.4), we find⁴ (after rearrangements) that this functional derivative can be expressed in terms of the functional value:

$$g(r) = |G(r)|^2 \frac{F[|K(r, \cdot)|^2]}{F[1]} \quad (2.7)$$

where the dot notation (.) refers to the dummy integration variable in the definition (2.4) of the functional; explicitly, in expression (2.7), we have $K(r, r^1)$ as an argument $\eta(r^1)$ with parameter r .

Similarly, for the second order term, by definition

$$g(r_1)g(r_2) + u(r_1, r_2) = \frac{1}{F[1]} \left. \frac{\delta^2 F}{\delta \eta(r_1) \delta \eta(r_2)} \right|_{\eta=1} \quad ; \quad (2.8)$$

and after some additional rearrangements of terms we find, using (2.1) and (2.4),

$$u(r_1, r_2) + g(r_1)g(r_2) = |G(r_1)|^2 |G(r_2)|^2 |K(r_1, r_2)|^2 \quad (2.9)$$

$$\cdot \frac{F[|K(r_1, \cdot)|^2 |K(r_2, \cdot)|^2]}{F[1]}$$

Relations (2.7) and (2.9) are identities within our theory as defined by (2.1), and do not depend upon other properties of K . They do not yet provide a convenient expression for F , as the higher coefficients $u_k (k > 2)$ are not determined in closed form. However, if we restrict our attention to small V , we can develop a useful approximation method⁴, and we now discuss this case.

III. INTEGRAL EQUATIONS VALID FOR WEAK, LONG-RANGE V

If $V(r_1, r_2)$ is small for all r_1 and r_2 , we see the righthand sides of (2.7) and (2.9) involve integrations only over functions near unity. Thus the expansion (2.5) can be truncated at second order when it is used in expressions (2.7) and (2.9). In this way we obtain a closed pair of integral equations for g and u in terms of G and V , by substituting (2.5) into (2.7) and (2.9). Assuming a real V for simplicity, we obtain:

$$g(r) = |G(r)|^2 \exp\left(\int d^3r^1 g(r^1) V(r, r^1)\right) \quad (3.1)$$

and

$$\begin{aligned} \frac{u(r_1, r_2)}{g(r_1) g(r_2)} &= 2V(r_1, r_2) + 4 \int d^3r g(r) V(r, r_1) V(r, r_2) \\ &+ 2 \iint d^3r d^3r^1 u(r, r^1) [V(r_1, r) V(r_2, r^1) + V(r_1, r^1) V(r_2, r)] \end{aligned} \quad (3.2)$$

The specific assumptions here are:

$$|V| \ll 1 \quad (3.3)$$

but

$$|gV| \text{ of order unity or larger is allowed.} \quad (3.4)$$

Thus, nontrivial (nonperturbative) results can emerge, if single-particle densities are sufficiently large in some regions of impact parameter.

The predictions of (3.1) for large- P_T behaviour were briefly examined in AT⁽⁴⁾. The effect of a positive V , a sign phenomenologically indicated by small- P_T inclusive correlations, is to enhance the inclusive large- P_T behaviour at sufficiently high Y , compared to an independent emission model. A somewhat more detailed discussion is given later in this paper.

IV. CROSS-SECTIONS IN TERMS OF G AND V

Experimental data is obtained at fixed P_T , not fixed b . Thus it is necessary to express observable cross-sections in terms of our functional F , which is defined in b -space. These relationships have been derived in AT, in general form, and we will quote them here in the special cases where G and V are independent of $\vec{R}$.

The single-particle central-region-exclusive production cross-section is given by:

$$\frac{1}{\sigma_0} \frac{d\sigma}{dy d^2P_T} = \iint d^2b d^2b^1 e^{iP_T \cdot (b-b^1)} G(y,b) G^*(y,b^1) \quad (4.1)$$

where

$$\sigma_0 = \int |A_0(\vec{b})|^2 d^2b \quad (4.2)$$

is the zero-particle central-region-exclusive (no particles produced in central region) cross-section.

The inclusive sums can be expressed in terms of the generating functional. The single-particle inclusive cross-section is:

$$\rho(y, P_T) = \frac{\sigma_0}{\sigma} \iint d^2b d^2b^1 e^{iP_T \cdot (b-b^1)} G(y,b) G^*(y,b^1) \quad (4.3)$$

$$\cdot F [K(y,b; \cdot) K^*(y,b^1; \cdot)]$$

where the notation used for the arguments of F was explained in Section II. Here, σ is the total cross-section.

The analytic expression (2.5) can be used for F , after the coefficients (g, u , etc.) are determined.

Similar, but more complicated, expressions can be obtained for two-particle cross-sections.

Simplification, particularly of inclusions, can be achieved first by introducing the Fourier transforms of G and $g^{\frac{1}{2}}$, as in Section V of AT, and then by adhering to the restrictions $|V| \ll 1$ and V real as in Section III.

Let

$$\bar{G}(y, P_T) = \int d^2 b e^{i b \cdot P_T} G(y, b) \quad (4.4)$$

and

$$f(y, P_T) = \int d^2 b e^{i b \cdot P_T} g^{\frac{1}{2}}(y, b) \quad (4.5)$$

Then (4.1) can be written

$$\frac{1}{\sigma_0} \frac{d^3 \sigma_1}{dy d^2 P_T} = |\bar{G}(y, P_T)|^2 \quad (4.6)$$

and (4.3) becomes simply

$$\rho(y, P_T) = |f(y, P_T)|^2 \quad (4.7)$$

Now if $|V| \ll 1$, and V is real, the two-particle exclusive cross-section can be expressed compactly. We introduce the two-particle exclusive correlation function

$$D(y_1, P_{T1}; y_2, P_{T2}) \equiv \frac{1}{\sigma_0} \cdot \frac{d^6 \sigma_2}{dy_1 d^2 P_{T1} dy_2 d^2 P_{T2}} \quad (4.8)$$

$$= \frac{1}{\sigma_0^2} \frac{d^3 \sigma_1}{dy_1 d^2 P_{T1}} \cdot \frac{d^3 \sigma_1}{dy_2 d^2 P_{T2}}$$

Note D must be translation-invariant in y_1 and y_2 (in the central region).

Define the rapidity Fourier transform of D by

$$\bar{D}(\omega; P_{T_1}, P_{T_2}) = \int_{-\infty}^{\infty} dy e^{i \omega(y_1 - y_2)} D(y_1, P_{T_1}; y_2, P_{T_2}) \quad (4.9)$$

where $y = y_1 - y_2$ (D is independent of $y_1 + y_2$).

Then we find under the above restrictions,

$$\bar{D}(\omega; P_{T_1}, P_{T_2}) = 2 \bar{G}(P_{T_1}) \bar{G}(P_{T_2}) \int \frac{d^2 q}{(2\pi)^2} v(\omega, q) \bar{G}(P_{T_1} - q) \bar{G}(P_{T_2} + q) \quad (4.10)$$

where v is the Fourier transform of V with respect to relative rapidity and relative impact parameter.

The two-particle inclusive correlation function is defined by

$$C_2(y_1, P_{T_1}; y_2, P_{T_2}) = \rho_2(y_1, P_{T_1}; y_2, P_{T_2}) - \rho(y_1, P_{T_1}) \rho(y_2, P_{T_2}) \quad (4.11)$$

This function must also be translation-invariant in rapidity in the central region. Let its Fourier transform with respect to $(y_1 - y_2)$ be denoted by $\bar{C}_2$.

A correspondingly compact expression for $\bar{C}_2$ can be obtained (again, using (3.1), valid for V real, in the limit $|V| \ll 1$), if we can neglect the integral terms in (3.2), as follows:

$$\bar{C}_2(\omega; P_{T_1}, P_{T_2}) = 2 f(P_{T_1}) f(P_{T_2}) \int \frac{d^2 q}{(2\pi)^2} f(P_{T_1} - q) f(P_{T_2} + q) v(\omega, q) \quad (4.12)$$

These expressions complete the connections between g , v , and observables.

V. FUNCTIONAL INTEGRAL FORMULATION TO INCLUDE STRONG SHORT-RANGE V'S

Although instructive, the approximations leading to (3.1) and (3.2) are not sufficiently realistic to provide an adequate theory, either for detailed phenomenology⁽³⁾ or for theoretical questions involving Pomeron intercept degeneracies and a weak interaction bootstrap.⁽⁵⁾ The existence both theoretically and experimentally of strong short-range correlations must be taken into account.

We will now develop an approximation scheme based on a separation between strong short-range correlations and a weak long-range (cut) effects. This scheme is modeled after similar mathematical methods⁷ employed for condensation and critical phenomena in a Van der Waals fluid.

We will assume that the summations over n -particle states can be analytically performed if no long-range contributions to V are present, but only very short-range contributions. The archetype of such a sum is the solution of the MPM, wherein the total absorptive part of a $2 \rightarrow 2$ amplitude is obtained by inverting a linear operator diagonal in the (t -channel) l -plane representation. In such a short-range theory, computation of the generating functional with arguments of constant functions $\eta = z$ is reduced to a determination of the leading pole trajectory, as a function of "fugacity" (coupling constant) z , which is obtained by locating the zeros of a denominator $[1 - z K(z)]$. In a more sophisticated approach,⁵ the short-range theory can be taken as the planar bootstrap.⁸⁻¹¹ In that case, the long range part of V is to be identified with topologically complex terms in the large- N expansion.⁸

To include the effects of long range pairwise terms in V , we first consider a functional integral representation for $F[\eta]$. First we must assume a separation $V = V_L + V_S$ where V_L has no zeros, and V_S has only a short range in rapidity, so that F can be evaluated explicitly when $V_L = 0$.

To clarify the presentation of the functional integral methods we ignore P_T variables for the present and consider only rapidity (one-dimensional approximation). The complete F , before any approximations, is:

$$F[\eta] = \sum_n \frac{1}{n!} \int dy_1 \dots dy_n \eta(y_1) \dots \eta(y_n)$$

$$\cdot \exp\left\{\frac{1}{2} \sum_{j,k} V_S(y_j, y_k)\right\} \exp\left\{\frac{1}{2} \sum_{j,k} V_L(y_j, y_k)\right\}$$

We will introduce a Gaussian functional-integral representation for the last factors, based on the identity (7,12)

$$\exp\left\{\frac{1}{2} \sum_{j,k=1}^n V(y_j, y_k)\right\} \equiv \frac{(\det B)^{\frac{1}{2}}}{(2\pi)^{n/2}} \int_{-\infty}^{+\infty} d\phi(y_1) \dots d\phi(y_n) \quad (5.2)$$

$$\cdot \exp\left\{\sum_j \phi(y_j)\right\} \cdot \exp\left\{-\frac{1}{2} \sum_{j,k} \phi(y_j) B(y_j, y_k) \phi(y_k)\right\}$$

valid for any positive definite V , where $(B)_{jk} = B(y_j, y_k)$ is the matrix inverse of $V(y_j, y_k)$, and $\phi(y)$ is an independent dummy (field) integration variable at each y .

We can represent the factor $(\det B)^{\frac{1}{2}}/(2\pi)^{n/2}$ as the inverse of the functional integral on the righthand side of (5.2) without the factor $\exp\left[\sum_j \phi(y_j)\right]$.

Introducing (5.2) into (5.1), we obtain [taking a continuum limit for $\phi(y)$, appropriate in the $Y \rightarrow \infty$, $\bar{u} \rightarrow \infty$ limit] the sum of a series of functional integrals, which can then be re-arranged as a single functional integral over F_S with argument ne^ϕ ;

$$F[\eta] = N^{-1} \int d\phi \exp\left[\frac{-1}{2} \iint \phi(y_1) B(y_1, y_2) \phi(y_2) dy_1 dy_2\right] F_S[ne^\phi] \quad (5.3)$$

where F_S is F with $V_L = 0$, N is F with F_S replaced by unity, and B is the Fredholm-operator inverse of V_L . Expression (5.3) is then a formal closed form for the complete F , given F_S for arbitrary argument.

We can proceed further,⁽⁷⁾ using the assumption that the range of V_S is much shorter than the characteristic range of V_L . The functional integral contributions in (5.3) are mainly $\phi(y)$'s which are slowly varying in y , if V_L is weak and long range. This implies F_S can be evaluated as if ϕ were constant in the integrand, since the correlation lengths in F_S are much shorter than the characteristic ranges in y over which the dominant ϕ 's vary.

If we specialize to constant $n=2$, we can express explicitly the result for the multiplicity-generating function

$$F(z) = N^{-1} \int \delta\phi \exp \left[\frac{-1}{2} \iint \phi B \phi \right] \exp \left[\int_0^Y \alpha_S(z e^{\phi(y)}) dy \right] \quad (5.4)$$

where $\alpha_S(z)$ is the short-range ensemble "pressure" function³, defined by

$$\alpha_S(z) = \frac{1}{Y} \ln F_S(z) . \quad (5.5)$$

We will now examine (for one-dimensional theory) some specific parameterizations of V_L and α_S , suggested by Regge cuts and MPM's respectively.

VI. PARAMETERIZATIONS FOR V_L AND α_S

The qualitative behaviour of V should follow from general features of 2 Reggeon-2 particle amplitudes, when compared to simple Reggeon exchange for such processes, in the low-multiplicity reactions $a + a^1 \rightarrow c + c^1 + h_1 + h_2$. Here h_1 and h_2 are in the central region, and (c, c^1) are the leading particles. Figure 2 shows the Reggeon-exchange interpretation of the factors A_0 , G , and $K = \exp(\frac{1}{2}V)$.

At small rapidity separations, $|y_1 - y_2|$ of order unity, $V \sim V_S$ will be dominated by short-range phenomena such as thresholds and resonances. We return to a parameterization of these later.

At large rapidity separations, $|y_1 - y_2| \rightarrow \infty$, V_L is determined by Reggeon-Pomeron (RP) cuts, as normalized to Reggeon exchange.

A suitable parameterization of the asymptotic rapidity behaviour of RP cuts is given by

$$V_L(y) \sim y^{-1-\sigma} \text{ with } \sigma > 0, \quad (6.1)$$

where $y = |y_1 - y_2|$.

This form represents a cut whose discontinuity in the z -plane vanishes at the tip of the cut, as required by t -channel unitarity.⁽¹³⁾ We expect $\sigma = \frac{1}{2}$ if elastic t -channel unitarity near threshold is a reasonable guide.

We may reasonably parameterize $V_L(y)$ by a two-parameter form¹⁴ which is regular near $y = 0$ and matches the asymptotic behaviour;

$$V_L(y) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} e^{i\omega y} \frac{1}{\lambda^{-\sigma} \omega^{\sigma} + \mu^2} \quad (6.2)$$

This form is chosen to provide mathematical convenience when obtaining the inverse operator, B . We believe the specific functional form of V_L is not essential for our investigations.

For $\sigma = 2$, we would obtain simple Regge pole behaviour in V_L ; this case is useful for obtaining analytic solutions which can be valuable in comparisons with certain hadron production models, such as that of Scalapino and Sugar.⁽¹⁵⁾ We will return to this case, for a discussion of long-range rapidity density fluctuations at "intermediate" energies, later.

The inverse operator $B = (V_L)^{-1}$ is now given in rapidity-Fourier-transform variables as $(\lambda^{-\sigma} \omega^{\sigma} + \mu^2)$, or, in a symbolic y -space pseudo-differential-operator form, as

$$B(y) = \lambda^{-\sigma} \nabla_y^{\sigma} + \mu^2$$

where $y = y_1 - y_2$.

Note our parameterization implies $V > 0$, i.e. positive Regge-Pomeron cut contributions in $RR \rightarrow hh$ amplitudes [cf. Figure 2(c)]. This sign, although opposite to that in $hh \rightarrow hh$, is natural in the Gribov Reggeon calculus¹⁶.

Returning to the question of parameterizing $\alpha_s(z)$, we wish to include a minimum number of essential dynamical features in the short range component, V_s . One feature certainly expected is that " $t_{\min}$ -effect", a small-rapidity-separation damping of the amplitude for two particles produced (at small P_T) in the central region.^{17,18,19} This effect is expected to be a relatively universal feature whose existence is not strongly dependent on quantum numbers.

In addition to these threshold "hard core" contributions to V_s , we expect resonances (or secondary Reggeons) in non-exotic channels. These will be determined by low energy dynamics in the $RR \rightarrow hh$ amplitude [cf Figure 2(c)] where crossing (or duality) plays a major role.

We may parameterize $\alpha_s(\xi)$, in general, through a power series (cluster) expansion in ξ , since there can be no phase-transition phenomena (non-analyticity in ξ) in a one-dimensional short-range system.

For small $|V_L|$, we will see that (for inclusives) the dominant contributions in the functional integral (5.4) will involve small $|\phi|$. Thus an expansion of $\alpha_s(\xi)$ in a neighbourhood of $\xi = 1$ (inclusive short-range expansion) is appropriate. We can then write the desired parameterization, for z near unity, in terms of an expansion of $\alpha_s(ze^\phi)$ in powers of ϕ around $\phi = 0$;

$$\alpha_s(ze^\phi) = \alpha_s(z) + \sum_{k=1}^{\infty} \frac{\phi^k}{k!} C_k(z) \quad (6.4)$$

where

$$C_k(z) = \left(\frac{d}{d \ln z} \right)^k \alpha_s(z) \quad (6.5)$$

We illustrate such a parameterization at this point using a specific model for the $t_{\min}$ -hard core in MPM's, originally proposed by Chew and Snider,⁽⁷⁾ and further discussed by Dash,⁽¹⁸⁾ and by Pinsky and Thomas.⁽¹⁹⁾

In MPM, $\alpha_s(z)$ is determined from the zeros of

$$D(z) = 1 - z \tilde{K}(z) \quad (6.6)$$

where $\tilde{K}(z)$ is the Laplace transform of $K(y)$, the one-dimensional MPM two-particle kernel.

With only $t_{\min}$ damping included, (17-19) in a zero-mass pion-exchange model

$$\tilde{K}(z) = \frac{g^2 e^{-bz}}{z} \quad (6.7)$$

where g is a coupling constant and b is a logarithm of ratios of masses involved in the exchange kinematics.

Then $\alpha_s(z)$ satisfies the transcendental equation

$$\alpha_s e^{b\alpha_s} = zg^2 \quad (6.8)$$

Expanding around $z = 1$, we can obtain the coefficients $C_k(z)$ explicitly in terms of the values of $\alpha_s(z)$. The first two terms are:

$$C_1(z) = \alpha_s / (1 + b\alpha_s) \quad (6.9a)$$

and

$$C_2(z) = \alpha_s^3 / (1 + b\alpha_s)^3 \quad (6.9b)$$

The functional integral for $F(z)$, when V_s and V_L are parameterized by (6.3) and (6.4), then takes the explicit form:

$$N F(z) = \int \delta\phi \exp \left\{ - \int_0^Y dy \left[L_0[\phi(y)] + L_1[\phi(y)] \right] \right\} \quad (6.10)$$

where

$$L_0[\phi] = \frac{1}{2} \left[\lambda^{-\sigma} (V^{\sigma/2} \phi)^2 + \mu^2 \phi^2 \right] \quad (6.11)$$

and

$$L_1[\phi] = - \sum_k C_k(z) (\phi)^{k/k!} \quad (6.12)$$

The leading t-channel "output" Regge intercepts in $2 \rightarrow 2$ scattering can be obtained through the leading $Y \rightarrow \infty$ asymptotic behaviour of $F(1)$, which is proportional to the $2 \rightarrow 2$ overlap function in a one-dimensional approximation. To find these intercepts we must evaluate the functional integral (6.10), which requires approximation techniques.⁽⁷⁾ In the case $\sigma = 2$, or V_L arising from secondary poles rather than cuts, such integrals have previously been studied, as they are identical to those of a one-dimensional field theory with polynomial self-interactions. The leading output Regge intercepts are determined by the ground state of such a theory. A special technique, applicable for $\sigma = 2$ (in one dimension), is the transfer-operator theory,⁽⁷⁾ used by Scalapino and Sugar.⁽¹⁵⁾

Truncation of the sum (6.12) at $k = 4$ yields a conventional ϕ^4 field theory for the fluctuation (stochastic) field ϕ , of the type which has been considered in Ref.(15).

For $\sigma < 2$, a more realistic Regge cut behaviour, we must resort to less-well-tested approximation methods. One of these is the direct application of functional saddle-point technique, which we discuss in the next section.

VII. SADDLE-POINT APPROXIMATIONS

For $Y \rightarrow \infty$ we may attempt a steepest-descent or saddle-point evaluation of the functional integral (6.10).

The dominant contributions will be obtained from functions ϕ for which the integrand is stationary, i.e.

$$\frac{\delta(L_0 + L_1)}{\delta\phi} = 0 \quad (7.1)$$

The leading contribution, if non-degenerate, will come from functions which are constant in y , since the derivative terms in L_0 are negative definite. However, finding non-leading singularities (e.g., secondary trajectories which are absent in the input short-range theory) must involve the derivative terms.

Equation (7.1) leads (for each z) to a differential equation for the stationary functions ϕ :

$$\lambda^{-\sigma} \nabla^{\sigma} \phi + \mu^2 \phi = \sum_k \{(\phi)^{k-1}/(k-1)!\} c_k \quad (7.2)$$

whose solutions must be found subject to appropriate boundary conditions at $y = 0$ and $y = Y$, which depend on the external-particle couplings. The least restrictive boundary conditions would be periodic b.c.'s.

For σ non-integral, Eq(7.2) is not a true differential equation; the fractional derivative must be interpreted through its operation on the Fourier transform of ϕ . However, its linearity and boundedness properties are correctly exhibited by the ∇^{σ} notation.

Each solution of (7.2) leads to a distinct z -plane singularity appearing in $F(z)$. In this way we see that an infinite series of poles (daughters) appear, for each pole α_s generated by the short-range forces alone. In addition, the intercept of the leading pole is shifted compared to α_s .

When leading singularities become degenerate, as expected for the Pomeron which we do not expect to be a simple pole, the evaluation of the functional integral requires more subtlety. The mathematical methods required are those appropriate for phase transitions and critical phenomena; the expressions (6.10), (6.11) and (6.12) can be interpreted

as a version of Landau-Ginzburg-Wilson theory with long range forces in one dimension.⁽⁵⁾ In field theory language, we must deal with degenerate ground states, with accompanying complications from strong long range fluctuations. Transitions between ground states are possible as Y becomes very large; this is a form of "instability", and its possible implications for a bootstrap of weak and electromagnetic interactions have been studied in reference (5).

For a non-degenerate leading singularity, we may seek a constant solution of (7.2), which may be written in that case

$$\mu^2 \phi = \rho_s (ze^{\phi}) \quad (7.3)$$

where

$$\rho_s(\xi) \equiv \xi \frac{d\alpha_s(\xi)}{d\xi}$$

The solutions of (7.3) are mean-field approximations. If we use (6.8) for α_s , we obtain exactly the Van der Waals equation of state generating function $F(z)$. As discussed in an earlier work,⁽³⁾ we find such an ensemble is a reasonable phenomenological treatment of multiplicity distributions at high energies.

VIII. OUTPUT REGGE POLES

In this section we examine in more detail some properties of the t-channel poles found in the complete overlap function, as determined through our generating functional, assuming simple models for short-range input poles.

For any given t channel, we define an overlap generating functional

$$F[\eta] = \sum_n \frac{1}{n!} \sum_{\{v_j\}} \int \prod_{j=1}^n d^3 r_j \eta(\vec{r}_j, v_j) |G(\vec{r}_j, v_j)|^2 \prod_{k \neq j} \exp[V(\vec{r}_j, \vec{r}_k; v_j, v_k)] \quad (2.1)$$

where we have written V here for the function which would be the real part of V, if V were complex. (2.2)

The limits of y integrations in the above expressions are $y = \pm Y/2$; the b integrals are cut off by $|G|^2$. As explained earlier, we will assume V can be divided into $V_L + V_S$, where V_S is short range in rapidity, and V_L is small in magnitude compared to unity, but long range in rapidity (dominated by Regge cuts) and of only one sign (presumably positive). Then we can express F as a functional integral over F_S , the short-range-only ensemble generating functional, defined as is F but with no V_L :

$$F_S[\eta] = \sum_n \frac{1}{n!} \sum_{\{v_j\}} \int \prod_{j=1}^n d^3 r_j \eta(\vec{r}_j, v_j) |G(\vec{r}_j, v_j)|^2 \prod_k \exp[V_S(\vec{r}_j, \vec{r}_k; v_j, v_k)] \quad (3.3)$$

We can represent F_S at high energies as a sum over the leading singularities in the Laplace-transform with respect to Y. We may define corresponding short-range functionals $\alpha_m[\eta]$ and $\beta_m[\eta]$ by:

$$F_S[\eta] = \sum_{m=0}^{\infty} \beta_m[\eta] \exp(Y \alpha_m[\eta]) \quad (3.4)$$

where the summation runs over the leading short-range-system poles (at a given η), with ordering $\alpha_0 > \alpha_1 > \alpha_2 \dots$

The one-dimensional parametrization, long-range in rapidity, adopted earlier for V_L can be generalized to include b . If we assume an exponential decrease in $|\vec{b}|$, an appropriate Ansatz is:

$$V_L(\vec{r}) = \frac{1}{(2\pi)^3} \int d^3k \frac{e^{i\vec{k} \cdot \vec{r}}}{\lambda_y^{-\sigma} k_1^\sigma + \lambda_b^{-2} k_2^2 + \mu^2} \quad (8.5)$$

where $\vec{k} = (k_1, \vec{k}_2)$ is Fourier conjugate to $\vec{r} = (y, \vec{b})$. We assume here that $V_L(r_1, r_2)$ depends only on $\vec{r} = \vec{r}_1 - \vec{r}_2$, and also does not depend upon the internal quantum numbers v_i , as appropriate for a Pomeron-Reggeon cut (divided by the Reggeon). A cutoff at large $\vec{k}$ is implied in the Ansatz for V_L .

Using the approximation that short range correlation lengths are much smaller than the characteristic range of V_L , we obtain the generating functional in the form:

$$F[\eta_1, \dots, \eta_N] = \frac{\sum \int \delta^N \phi \exp \left\{ - \int_{-Y/2}^{+Y/2} dy (L_0[\vec{\phi}(y)] + L_1^{(m)}[\vec{\phi}(y)]) \right\}}{\int \delta^N \phi \exp \left\{ - \int_{-Y/2}^{+Y/2} dy L_0[\vec{\phi}(y)] \right\}} \quad (8.6)$$

where, if we ignore the ultraviolet cutoff in V_L (and the necessity for renormalizing),⁵

$$L_0[\vec{\phi}] = \int \frac{1}{2} \left[\lambda_y^{-\sigma} (v_y^{\sigma/2} \vec{\phi})^2 + \mu^2 \vec{\phi}^2 + \lambda_b^{-2} (v_b^2 \vec{\phi})^2 \right] d^2b \quad (8.7)$$

where we will take $\sigma = 1/2$, and

$$L_1^{(m)}[\vec{\phi}] = - \int \left\{ \alpha_m [\eta_1 e^{\vec{\phi}_1}, \dots, \eta_N e^{\vec{\phi}_N}] + (1/Y) \ln \beta_m [\eta_1 e^{\vec{\phi}_1}, \dots, \eta_N e^{\vec{\phi}_N}] \right\} d^2b \quad (8.8)$$

From the formal expression (8.6) all observables can be calculated, as shown in section 4. The multiplicity generating function, and the Regge-trajectory intercepts, are obtained by specializing to constant $\eta_v = z_v$ ($v = 1, \dots, N$).

We wish to investigate here the relation between the intercepts and residues of the short-range-only ensemble, defined by (8.3), and the final output intercepts and residues calculated from (8.6).

For this purpose it is sufficient to ignore the impact parameter variables and consider a one-dimensional system. Here we seek to evaluate the behavior, as $Y \rightarrow \infty$, of the multiplicity generating function given by (8.6) with L_0 as in (8.7), without the $(\gamma_b \phi)^2$ term, and L_1 as in (8.8) with $(z_1, \dots, z_N)$ replacing $(\eta_1, \dots, \eta_N)$. We drop the subscript on λ_y henceforth. The high-lying output intercepts will be determined in general by saddle points of the functional integrals in the numerator of (8.6). We will return to $\vec{b}$ dependences in the next section.

We consider each m separately. The saddle-point functions ϕ will satisfy $\frac{\delta(L_0 + L_1)}{\delta \phi} = 0$, which gives the set of coupled nonlinear (formally differential) equations:

$$\lambda^{-\sigma} \nabla_y^\sigma \phi_j(y) + \mu^2 \phi_j(y) = p_j^{(m)}(z_1 e^{\phi_1(y)}, \dots, z_N e^{\phi_N(y)}) \quad (8.7)$$

with $j=1, \dots, N$; $m=0, 1, 2, \dots$

Here

$$p_j^{(m)}(z_1, \dots, z_N) \equiv z_j \frac{\partial}{\partial z_j} \left[\alpha_m(z_1, \dots, z_N) + \frac{1}{Y} \ln \beta_m(z_1, \dots, z_N) \right]$$

For each solution $\phi^{(m)}$ of (8.7), the corresponding saddle-point approximation for an associated output intercept and residue is:

$$-\frac{1}{Y} \cdot \int_{-Y/2}^{+Y/2} dy \left[L_0[\hat{\phi}] + L_1^{(m)}[\hat{\phi}] \right] = \hat{\alpha}^{(m)} + \frac{1}{Y} \ln \hat{\beta}^{(m)} \quad (8.9)$$

The leading output intercept (at this level of approximation) will be the largest value of $\hat{\alpha}$ when all possible ϕ 's are used in (8.9). For large Y , the term involving β_m in (8.8) plays no role in determining ρ , and we may drop it during calculation of the ϕ 's. The leading $\hat{\alpha}$, for each m , will be expected to arise from a constant ϕ , since contributions from "kinetic" (derivative) terms are negative in $\hat{\alpha}$.

Thus, for the leading intercept, we expect $\hat{\phi}$ to satisfy (8.7) without the derivative terms. These form N coupled "Mean-Field" equations (for each m):

$$\mu^2 \phi_j = \rho_j(z_1 e^{\phi_1}, \dots, z_N e^{\phi_N}) \quad (8.10)$$

Note the residues of the output poles are obtained in this approximation, by evaluating the short-range β at $(z_1 e^{\hat{\phi}_1}, \dots, z_N e^{\hat{\phi}_N})$.

When α_0 is taken from a "hard-core" short-range model,¹⁷⁻¹⁹ with short-range correlations arising only from kinematic ($t_{\min}$) effects, we obtain from (8.10) exactly the Van der Waals mixture theory studied (at criticality) by Thomas⁽³⁾, a very successful description (with essentially no free parameters) of high energy multiplicity distributions.

Corrections to the saddle-point $\hat{\alpha}$'s can be computed by expanding $(L_0 + L_1)$ in a Taylor series around $\phi = \hat{\phi}$, as explained earlier. We can write the saddle-point expression for (8.6) keeping only the dominant

(constant) $\hat{\phi}$ for each m , as:

$$F(z_1, \dots, z_N) = D^{-1} \left\{ \sum_m \beta_m (z_1 e^{\phi_1}, \dots, z_N e^{\phi_N}) \cdot \exp \left[Y \left[\alpha_m (z_1 e^{\phi_1}, \dots, z_N e^{\phi_N}) - \frac{\mu^2}{2} (\phi_1^2 + \dots + \phi_N^2) \right] \right] \cdot F_m \right\} \quad (8.11)$$

with

$$F_m = \int \delta \tilde{\psi} \exp \left\{ - \int_{-Y/2}^{+Y/2} dy L_2^{(m)} [\tilde{\psi}(y)] \right\} \quad (8.12)$$

We have written $\psi = \phi - \hat{\phi}$ for each m ; to 4th order in ψ ,

$$L_2^{(m)}[\psi] = 1/2 \lambda^{-\sigma} (v^{\sigma/2} \psi)^2 + 1/2 \mu^2 \psi^2 - 1/2 \sum_{ij} \psi_i (C_2^m)_{ij} \psi_j - (1/3!) \sum_{ijk} (C_3^m)_{ijk} \psi_i \psi_j \psi_k - (1/4!) \sum_{ijkl} (C_4^m)_{ijkl} \psi_i \psi_j \psi_k \psi_l \quad (8.13)$$

where the C_p^m 's are defined by:

$$[C_p^m(z_1, \dots, z_N)]_{ijk \dots} \equiv \frac{\partial}{\partial \ln z_i} \frac{\partial}{\partial \ln z_j} \frac{\partial}{\partial \ln z_k} \dots \alpha_m(z_1 e^{\phi_1}, \dots, z_N e^{\phi_N}) \quad (8.14)$$

The factor D is independent of the short-range dynamics and is given by the denominator of (8.6). Since L_0 is only quadratic in ϕ , we can analytically obtain D if desired, but the expression is not required here.

If the output intercepts are not in the vicinity of a critical point, we may drop terms higher than quadratic in (8.13), the functional integral in (8.12) then becomes a quadratic form in ψ , and can be expressed in closed form;

$$F_m = \exp \left\{ \frac{-Y}{2\pi} \int dk \text{Tr} \ln \left[\lambda^{-\sigma} k^{\sigma} + \mu^2 - C_2^m \right] \right\} \quad (8.15)$$

where the trace operation refers to the matrix indices on C_2 .

For small λ , the correction to the mean-field result for the intercept from F will be of order λ , unless ϕ is in the vicinity of criticality, where μ^2 nearly cancels C_2 . We will call (8.15) the "quadratic" approximation to F_m . A cutoff on the k integral in (8.15) is implied, as in (8.5).

We discuss the shift in trajectory-intercepts, first according to (8.10) then the role of nonconstant solutions to (8.7), and finally the influence of fluctuations at extremely high energies, as determined by the cubic and quartic terms in (8.13).

We can distinguish three new characteristic features present in our theory, compared to the simple short-range ensemble:

(1) If the long-range correlations were reduced indefinitely in magnitude, we would find $\mu^2 \rightarrow \infty$, thus from (8.10) all $\hat{\phi} \rightarrow 0$, and the output intercepts $\hat{\alpha}$ would coincide with those of the short-range system alone. As we increase $|V_L|$, μ^2 decreases, and the output intercepts become larger, as determined by $\hat{\phi}$ from (14) and $\hat{\alpha}$ from (8.9). We may call this a "promotion" of the short-range poles.

(2) For nonzero V_L we encounter also a second phenomenon, visible in the quadratic approximation; a family of secondary (i.e. nonleading) output poles, for each single input pole α_m . For any reasonable short range system there will be an infinite number of these, forming a rich spectrum of secondary poles, which are not present in the short-range ensemble. We will call this a "proliferation" of the high-lying poles. With $\sigma = 2$, and small λ , this is seen explicitly in the model calculation of Scalapino and Sugar.¹⁵

(3) At very large Y , the long range fluctuations will become important, and as in critical phenomena, the true asymptotic behavior associated with the leading intercept will be corrected, compared to the mean-field approximation (8.10), by a substantial amount. It is only at this stage that the Froissart bound is relevant. We expect generally that this final correction is negative, a downward shift in α . This can be called a "renormalization" of the leading intercept. Thus we expect the effective intercept (at FNAL-ISR energies and

below) is above unity, but for $Y \rightarrow \infty$ only a singularity exactly at unity can appear, with logarithmic corrections.

All three phenomena rely quantitatively on the functional dependence $\alpha_0(z)$ of the short-range ensemble. Thus, the amount of promotion, the characteristics of proliferation, and the final renormalization, will all depend on the quantum numbers of the short-range poles, through the "conventional" MPM dynamics. These dependences may be seen, for example, in the prototype 3-pole MPM, discussed by Pinsky and Thomas⁽²⁰⁾ in the context of inclusive phenomenology, which includes charge symmetry (but not higher symmetries).

The relative magnitudes of the leading-trajectory $\hat{\phi}$'s are dependent, to first order in $\hat{\phi}$, on the magnitudes of the short-range densities ρ_j evaluated at $z=1$. This gives us a systematic way to study the relative shifts in perturbation theory, when $\mu^{-2} \ll 1$, provided the perturbation is not carried out close to a critical point. Unfortunately, this is just the case when the Pomeron is considered. However, to gain some insight, we can examine the relative signs of shifts, when $\hat{\phi}$ is unphysically small; the poles then remain well below unity, away from criticality.

Expanding (8.10) around $\hat{\phi} = 0$ we obtain, to 1st order in $\hat{\phi}$,

$$\sum_j [\delta_{ij} \mu^2 - (C_2)_{ij}] \hat{\phi}_j = \rho_i \quad (8.16)$$

where

$$(C_2)_{ij} = z_i \frac{\partial \rho_j}{\partial z_i} ; \quad (8.17)$$

ρ_i and its derivatives are evaluated at $z_1 = \dots = z_N = 1$ here. The signs of the shifts $\hat{\phi}_j$ are then determined by the matrix

$$M_{ij} = \mu^2 \delta_{ij} - (C_2)_{ij}. \quad (8.18)$$

For $\mu^{-2} \ll 1$, the matrix M is positive definite, so all shifts are positive.

It is instructive to examine the intercept shifts and residue shifts in more detail, and compare our results with those of Chew and Rosenzweig⁽¹¹⁾ and Schmidt and Sorenson⁽¹⁰⁾, using the quark-number expansion. For this purpose we need a specific MPM, with quantum number dependent dynamics; the PT model⁽²⁰⁾ would suffice, but an even simpler model first reveals some essential qualitative features, as we now show.

Consider first a single-channel Chew-Pignotti (CP) model⁽¹⁰⁾ with $\alpha(z) = Gz + \text{constant}$. Then we can investigate the dependence of the shift $\hat{\phi}$ upon the value of G . We find $\rho = C_2 = G$ in this case; from (8.16) $\hat{\phi}$ is given by:

$$\hat{\phi} = \frac{G}{\mu^2 - G} \quad (8.19)$$

Thus the shift is an increasing function of G , and increases very rapidly as the input intercept (G) is increased toward μ^2 .

In the critical Van der Waals model⁽³⁾ we find $\mu^2 = \frac{1}{2a} = \frac{32}{27}$. Thus we expect in realistic models the shift $\hat{\phi}$ will be very sensitive to $\alpha(0)$ for $\alpha(0)$ of order $1/2$ or larger. If the short-range pole representing non-quantum-number-exchange is slightly higher than quantum-number-exchange poles, the former will be promoted higher than the latter, even if no further subtleties are involved.

The CP model for the short-range ensemble is inadequate to account for criticality, however, which requires a degeneracy of leading singularities after promotion. It is necessary to include (short range) correlations in $\alpha(z)$.

A second feature of short-range dynamics which qualitatively affects the promotion appears when we include N internal degrees of freedom, still within the CP framework¹⁰;

$$\alpha(z_1, \dots, z_N) = z_1 G_1 + \dots + z_N G_N \quad (8.20)$$

With N -fold internal symmetry, we have $G_1 = G_2 = \dots = G_N$; then we can write

$$\alpha(z_1, \dots, z_N) = (z_1 + \dots + z_N) \frac{G}{N} \quad (8.21)$$

with normalization chosen such that the leading intercept is $\alpha(1, \dots, 1) = G$ to correspond to the simpler case discussed above.

Now

$$\rho_i = \frac{G}{N} \quad \text{and} \quad (C_2)_{ij} = \frac{G}{N} \delta_{ij}; \quad (8.22)$$

$$\hat{\phi}_i = \frac{G}{N\mu^2 - G} \quad (i=1, 2, \dots, N). \quad (8.23)$$

As N increases, for fixed μ^2 , the individual shifts $\hat{\phi}_i$ decrease. We find the leading output pole intercept becomes:

$$\begin{aligned} \hat{\alpha} &= \alpha(e^{\hat{\phi}_1}, \dots, e^{\hat{\phi}_N}) = \frac{\mu^2}{2} \sum_i \hat{\phi}_i^2 + G + \sum_i (\hat{\phi}_i \rho_i - \frac{\mu^2}{2} \hat{\phi}_i^2) \\ &= G + \frac{G^2}{N\mu^2 - G} - \frac{\mu^2}{2} \cdot \frac{NG^2}{(N\mu^2 - G)^2} \end{aligned} \quad (8.24)$$

In the limit $N \gg G/\mu^2$,

$$\hat{\alpha} \approx G + \frac{G^2}{2N\mu^2} \quad (8.25)$$

Thus, we see the net promotion of the leading pole decreases as N increases indefinitely. This suggests a reason for limited numbers of internal degrees of freedom in a bootstrap theory; but we will not go into that subject here.

If we wish to now to consider (for comparison of shifts) a quantum-number exchange trajectory, we must introduce dynamics which depend on the quantum numbers; this introduces a third feature, indicating a possibility of a greater shift for vacuum poles than nonvacuum poles.

Consider the relative shifts expected for charge-symmetric vs. charge-antisymmetric exchanges in PT model⁽²⁰⁾:

I. Leading antisymmetric ρ trajectory (eqn. 3.12 of PT):

$$\alpha^{(3)}(z_+, z_-, z_0) = z_0 C + \ell_2 \quad (8.26)$$

Associated densities:

$$\rho_+ = \rho_- = 0; \quad \rho_0 = C z_0 \quad (8.27)$$

C_2 coefficients: the only nonzero term is $(C_2)_{00} = C z_0$.

II. Leading symmetric P trajectory (eqn. 3.11 of PT):

$$\alpha^{(1)}(z_+, z_-, z_0) = \frac{1}{2}(z_0 A + \ell_1 + z_0 C + \ell_2) + \left\{ \left[\frac{1}{2}(z_0 A + \ell_1 - z_0 C - \ell_2) \right]^2 + 4z_+ z_- B^2 \right\}^{1/2} \quad (8.28)$$

Note all three components of $\vec{\phi}$ contribute to the upward shift of $\alpha^{(1)}$, while only one (ϕ_0) contributes to the shift of $\alpha^{(3)}$. Thus, even without a detailed estimate, we expect $\alpha^{(1)}$ will be promoted more than $\alpha^{(3)}$.

The magnitudes of $\hat{\phi}_+$ are determined by

$$\rho_+ = \frac{2z_+ z_- B^2}{\left\{ \left[\frac{1}{2}(z_0 A + \ell_1 - z_0 C - \ell_2) \right]^2 + 4z_+ z_- B^2 \right\}^{1/2}} \quad (8.29)$$

There can be, then, depending on the relative values of A, B, and C, a much larger shift upward in the symmetric channel than in the antisymmetric channel. Thus we see a reasonable mechanism exists for promoting the Pomeron up past unity, while leaving the ρ trajectory well below unity.

Exchange-degeneracy between output poles can only be achieved through the proliferation mechanism. A secondary pole (f) in the Pomeron sector can be degenerate with the leading singularity (ρ) in the antisymmetric sector. However, as the promotion mechanism depends on the same parameters as the proliferation mechanism, such a "conspiracy" need not be accidental.

Some consequences of such a mechanism for restoring exchange-degeneracy have been exploited in Ref.(5). We have argued there that the short-range input α_0 should be taken from the planar-plus-cylinder result of the dual-unitarization scheme⁸⁻¹¹; this differs from the planar amplitude only in the Pomeron sector. For the Pomeron intercept, we then expect $\alpha_0 = 1$ (at $t = 0$), applying the weak-correlation planar bootstrap idea¹⁰. Then the promoted Pomeron intercept (before renormalization) will lie above unity.

IX LARGE-TRANSVERSE-MOMENTUM INCLUSIVES

In this section, we will examine transverse momentum phenomena expected in the multi-eikonal theory, with emphasis on the implications of critical behavior for large-P inclusives.

As in reference (4), we can discuss only the limit $Y \rightarrow \infty$ for fixed P (Feynman-scaling limit), and the (slow Y-dependent) corrections to this limit. We consider $a + b \rightarrow a' + b' + h_1 + \dots + h_n$ amplitudes, using:

$$A_n(r_1, \dots, r_n | R) = A_0(R) \prod_{j=1}^n G(r_j | R) \prod_{k \neq j} K(r_j, r_k | R). \quad (9.1)$$

Here we will assume G and K independent of R; slow Y-dependences arise in inclusives, but only through phase-space integrals; G is independent of y; and K is translation-invariant. We make one-dimensional kinematic approximations as in AT, examining only $P_T \ll \sqrt{s}$, and ignore quantum numbers for simplicity.

Observables are calculated from the generating functional

$$F[\eta] = \sum_n \frac{1}{n!} \int \prod_j d^3 r_j \eta(r_j) \left[G(\vec{r}) \prod_k \exp[V(r_j, r_k)] \right]^n \quad (9.2)$$

where $|K|^2 = \exp V$, and where $\eta(r)$ is any function of $(y, \vec{b})$. The limits of y integrals are $\pm Y/2$; and the $\vec{b}$ integrals are damped by $|G|^2$.

As in the preceding section, we assume $V = V_S + V_L$ where V_S is short range in rapidity, while V_L is weak, slowly varying, translation-invariant in y and b , and positive definite.

A. PARAMETRIZATION of V

As a definite parametrization of V_L , we adopt the 3^- parameter form:

$$V_L(r) = \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i k \cdot \vec{r}}}{\lambda_y^{-\sigma} k_1^\sigma + \lambda_b^{-2} k_\tau^2 + \mu^2} \quad (9.3)$$

where $\vec{k}_\tau = (k_2, k_3)$ is Fourier-conjugate to $\vec{b}$, and k_1 is Fourier-conjugate to y .

This parametrization exhibits a power-law rapidity dependence characteristic of a branch point in the l - plane when $\sigma < 2$, and an exponential damping in $\vec{b}$.

In this section we will assume for greatest simplicity that $V_S = 0$; i.e., no short-range rapidity correlations are present. Thus the short-range ensemble obtained by deleting V_L has a (generalized) Chew-Pignotti (or independent emission) form:

$$F_S[\eta] = \sum_n \frac{1}{n!} \int \prod_j d^3 r_j n(r_j) |G(r_j)|^2 = \exp \left\{ \int d^3 r \eta(r) |G(r)|^2 \right\} \quad (9.4)$$

Since $|G|^2$ is independent of y , we obtain a simple Regge F_S for $\eta = \text{const.}$

To calculate the full $F[\eta]$ we apply the functional-integral representation as discussed earlier; here the expression is exact, for $Y \rightarrow \infty$, since $V_S \equiv 0$:

$$F[\eta] = \int \delta \phi^3 \exp \left\{ - \int d^3 r L[\phi(r); \eta(r)] \right\} / \int \delta \phi \exp \left\{ - \int d^3 r L_0[\phi(r)] \right\} \quad (9.5)$$

where

$$L_0[\phi] = \frac{1}{2} \lambda_y^{-\sigma} (\Delta_y^{-\sigma/2} \phi)^2 + \frac{1}{2} \lambda_b^{-2} (\vec{\nabla}_b \phi)^2 + \frac{\mu^2}{2} \phi^2 \quad (9.6)$$

and

$$L[\phi; \eta] = L_0[\phi] - \eta(r) |G(r)|^2 e^{\phi(r)} \quad (9.7)$$

We choose $\sigma = 1/2$, following the reasoning outlined earlier (Section 6).

To calculate single-particle inclusive densities, according to the formalism of sections 2 and 4, we must evaluate $F[\eta_F^2]$ with $\eta(r') = e^{V(r',r)}$, and divide by $F[1]$. Specifically, we compute the density g given by equation (2.7):

$$g(r) = |G(r)|^2 F[e^{V(\cdot, r)}] / F[1], \quad (9.8)$$

which should be independent of y as $Y \rightarrow \infty$.

The inclusive density in P_T is then given by the absolute square of the transverse Fourier transform of the square root of g , as explained in Section 4.

[We note $g > 0$, from the formula given above.]

For large P_T , the function f can be computed by steepest-descents, as sketched in Appendix C of AT, once we have a definite form for $g(\vec{b})$.

The parameters of our calculation are to be set such that $F[1]$ describes a critical ensemble; this sets the normalization of $\int |G|^2 d^2b$, in terms of the three parameters of $V = V_L$.

We wish to investigate the consequences of this critical behaviour for $\rho(\vec{P}_T)$ at large P_T . We note $|G(b)|^2$ is, in principle, determined by single particle exclusive production in the central region. We will determine in practice the b -dependence of $|G|^2$ from data on low-multiplicity, or moderate energy inclusive, cross-sections.

Experimentally one finds the shape of $\rho(\vec{P}_T)$ is very well approximated by e^{-6P_T} (where P_T is measured in GeV units) at laboratory collision energies below 100 GeV. Above that energy, an increase is seen at large P_T compared to lower energy data; this increasing trend continues at least through ISR energies. We associate the relative increase at large P_T with collective phenomena arising through V_L . Our goal here is to reach at least a qualitative understanding of this phenomenon, based on proximity of the ensemble to criticality. This was not possible in AT, as critical behaviour was excluded at an early stage through reliance on a truncated cluster expansion for F . Although we cannot give a complete solution to the problem as posed here, we can identify some characteristics, subject to experimental tests, following from our hypothesis that V_L is responsible for the large- P_T enhancements.

We note here that associated-multiplicity distributions are easily obtained, once the solutions for (9.5-9.8) are found; it is only necessary to introduce $\eta = Ze^V$ (where Z is a positive real variable); then Z plays the role of a generating variable for associated multiplicities.

Instead of eqn (9.8), we might use the alternative expression, equation (2.6).

This leads to a formally simple expression for $g/|G|^2$ in terms of the functional average of $\exp[\phi(r)]$. However, the features of g arising from V become somewhat more concealed in the formalism. We prefer eqn (9.8) for this reason; explicit V dependence appears in the "classical" saddle-point approximation, even without the "loop" corrections from L_0 . If we use eqn (2.6), the V -dependence arises entirely from "nonclassical" terms in the functional integral.

B. ANALYTIC APPROXIMATION METHODS

Proceeding as before we evaluate F by saddle-point methods. The saddle-point function $\hat{\phi}(y, \vec{b})$ will satisfy $\frac{\delta L}{\delta \phi} = 0$. Note that $\hat{\phi}$ will now be a functional of η , as will be the coefficients C_n (defined in the previous section) of the effective Lagrangian (expanded around $\hat{\phi}$).

Since we will need to evaluate $\hat{\phi}$ with only one choice of function $\eta(r) = e^{V(r - r_1)}$, we may write the dependence of ϕ on r_1 (as well as r) explicitly. Then $\hat{\phi}(r; r_1)$ satisfies:

$$\lambda_y^{-\sigma} \nabla_y^2 \phi + \lambda_b^{-2} \nabla_b^2 \phi + \mu^2 \phi = |G(r)|^2 \exp[\phi(r; r_1) + V(r - r_1)] \quad (9.9)$$

(with $\sigma = 1/2$).

Given a solution $\hat{\phi}(r; r_1)$, we then expand L around $\hat{\phi}$, to the fourth order. Defining $\psi = \phi - \hat{\phi}$, we change functional-integral variable to ψ , and write:

$$\begin{aligned}
L[\psi; e^{V(\cdot, r_1)}] = & \frac{1}{2} \lambda_y^{-\sigma} (\vec{v}_y^{1/2} \psi)^2 + \frac{1}{2} \lambda_b^{-2} (\vec{v}_b \psi)^2 + \frac{\mu}{2} \psi^2 \\
& - \left\{ \frac{C_2(r, r_1)}{2!} \psi^2 + \frac{C_3(r, r_1)}{3!} \psi^3 + \frac{C_4(r, r_1)}{4!} \psi^4 \right\} \\
& + \hat{L}(r, r_1)
\end{aligned} \tag{9.10}$$

$$\text{where} \quad \hat{L}(r, r_1) = L[\hat{\phi}(r, r_1); e^{V(r, r_1)}] \tag{9.11}$$

is the saddle-point value of L ; and

$$C_n(r, r_1) = |G(r)|^2 \exp [\hat{\phi}(r, r_1) + V(r - r_1)] \tag{9.12}$$

for all n , particularly simple because of our assumption $V_S \equiv 0$.

If we keep only quadratic terms in ψ , we can explicitly carry out the functional integrals over ψ , in terms of the eigenvalues of a linear differential equation for ψ . This will not lead to a completely satisfactory theory of behaviour for $V > 0$, as the fluctuations ψ^3 , ψ^4 , etc will become very large. However, we can identify qualitative phenomena and attempt to generalize. The mathematical problems are exactly those of a fluid at the critical point perturbed by a strong localized field, $V(r - r_1)$; we want to know the space-dependent density in the presence of this field.

Unlike the usual theories, e.g., of magnetic systems with external applied field, the dependence of all the C_n 's on V means that some "internal" effective parameters (e.g., local temperature) in the analog system will vary with r_1 , as well as the "external field" (analog) shift in ϕ . Thus the phase (i.e., ordering) of the analog system can change as a function of r_1 , and a very rapid change in inclusive density with $\vec{b}_1$ can appear, which enhances the large- T cross-sections.

We can illustrate the appearance of this phenomenon in the ψ^2 approximation; we write

$$L[\psi] \approx \hat{L} + L_2([\psi]; r, r_1) \quad (9.13)$$

where

$$L_2 = \frac{1}{2} \lambda_y^{-\sigma} (\bar{v}_y^{-\sigma/2} \psi)^2 + \frac{1}{2} \lambda_b^{-2} (\bar{v}_b \psi)^2 + \frac{1}{2} [\mu^2 - c_2(r, r_1)] \psi^2 \quad (9.14)$$

Then we evaluate

$$F[e^{V(\cdot, r_1)}] = f_0(r_1) \cdot \exp \left\{ - \int d^3r \hat{L}(r, r_1) \right\} \quad (9.15)$$

where $\hat{L}$ is given by eqn (9.11), and

$$f_0(r_1) = \int \delta \psi \exp \left\{ - \int d^3r L_2([\psi]; r, r_1) \right\} / \int \delta \phi \exp \left\{ - \int d^3r L_0[\phi] \right\}. \quad (9.16)$$

Since L_2 is quadratic in ψ we can express the functional integral in terms of eigenvalues appropriate to the diagonalization of L_2 . Define eigenvalues $E_k(r_1)$ and eigenfunctions $\psi_k(r, r_1)$ of the differential equation:

$$\left[\lambda_y^{-\sigma} \bar{v}_y^{\sigma} + \lambda_b^{-2} \bar{v}_b^2 + \mu^2 - c_2(r, r_1) \right] \psi_k(r, r_1) = E_k(r_1) \psi_k(r, r_1) \quad (9.17)$$

Expanding an arbitrary ψ in terms of this basis,

$$\psi(r, r_1) = \sum_k \xi_k \psi_k(r, r_1), \quad (9.18)$$

we can diagonalize L_2 ; and the functional integral in eqn (9.16) can be carried out over the coefficients ξ_k . The result⁽⁷⁾ is:

$$\frac{F(b_1)}{F(\infty)} = \exp \left[\sum_k \frac{1}{2} \ln E_k(b_1) + \sum_k \frac{1}{2} \ln E_k(\infty) \right] \quad (9.19)$$

where we have used the property $V(b_1) \xrightarrow{b_1 \rightarrow \infty} 0$ to eliminate the denominator

in eqn (9.16). The ratio (9.19) is that required in finding $g(b_1)$, which is independent of y_1 in the central region. (We may set $y_1 = 0$, therefore.) The

spectrum $\{E_k\}$ of eqn (9.17) will be continuous for $E_k > 0$; and only if all $E_k > 0$ is the quadratic approximation (9.13) possible.

We obtain critical rapidity-fluctuation behaviour in the densities at b_1 when

$$\lim_{k \rightarrow 0} E_k(b_1) = 0. \quad (9.20)$$

Note the eigenvalues E_k will, in general, decrease as V increases, from eqn (9.17). The form (9.3) assumed for V is monotonically increasing as b_1 decreases from ∞ , if $|G|^2$ is maximal at $b = 0$. Thus, if the densities are near criticality for large b_1 , we pass through a critical condition as b_1 decreases. If we could follow $F(b_1)$ further inward we would find the analog of a two-phase coexistence region; but this region is not accessible in the approximation (9.13). We consider the above discussion as sufficient to elucidate the character of the cooperative phenomena expected; a lowering of analog temperature as b_1 decreases, together with a b_1 -dependent external field.

The mathematical difficulties encountered by the quadratic approximation of a stochastic-field formulation of first-order phase transitions have been discussed by Langer.⁽²⁰⁾ His results show that including a quartic term in $\psi(\mathbf{r}, \hat{\mathbf{L}})$ would suffice to remove unphysical features from our functional integrals. There is at present no exact evaluation method for the integrals in that case, but simpler integrals of this type, with constant $|G|^2$ and V , have become the prototype for applications of renormalization-group methods in critical phenomena- cf. Wilson and Kogut.⁽²¹⁾

C. ANALOG-SYSTEM EXPECTATIONS AND A ROUGH APPROXIMATION

Since we have no analytic method at present adequate for extracting the consequences of equations (9.5-9.8), we may try to reason on the basis of the expected behaviour of a physical analog system. Using the Feynman-Wilson fluid analog,⁽²²⁾ the density g is regarded as a fluid density, critical at large b .

We have shown that for small b the system is then in a liquid-gas coexistence

region, as if the temperature were lower than critical. Evaluation of inclusive spectra at large P will sample only this small- b region, while at small P we will see critical behaviour. The structure of typical events in rapidity and b -space will be governed by functions $\phi(r)$ which reflect dominant ensemble configurations in the fluid, in y -space and b -space.

In the coexistence region of simple fluids we expect, instead of a homogeneous medium, small droplets of liquid in the gas and small bubbles of gas in the liquid.⁽²⁴⁾ These structures have, near criticality, a small size, limited by surface energies (in our theory by the long-range forces). Thus one expects a "grainy" $\phi(r)$ is typical of the coexistence region. In our analog, this means the densities at small b are rapidly varying in b on a small scale, leading to a Fourier transform which for large P is much larger than that expected for $V = 0$. This enhancement can only appear when Y is sufficiently large to allow many particles to participate in the correlations at a given b .

We can illustrate how the change in P_T slopes, as Y increases, arises in a qualitative way from our formalism, provided we remain at "moderate" energies where the collective effects are not at full strength, and there is no necessity to discuss a phase transition. Suppose V is small; then μ^2 is large, and we can solve eqn (9.9) approximately; we also drop the derivative terms here, to simplify the argument, assuming the dominant solution is constant in y and still slowly varying in b . Then

$$\hat{\phi}(r, r_1) = \mu^{-2} |G(r)|^2 \exp [V(r - r_1)]. \quad (9.21)$$

From eqn (9.8), using eqn (9.11) with a similar (nonderivative) approximation to $\hat{L}$ (and dropping $L - \hat{L}$), we obtain

$$g(r) = |G(r)|^2 \exp \left\{ \int dr' |G(r')|^2 [e^{V(r' - r)} - 1] \right\} \quad (9.22)$$

The integral over y' runs over the allowed phase space; $(-Y/2, +Y/2)$. Note G is independent of y . We should only keep terms linear in V in the exponent of g , consistent with $V \ll 1$. Defining

$$\bar{V}(\vec{b}; Y) = \int_{-Y/2}^{+Y/2} V(\vec{b}, y) dy, \quad (9.23)$$

we see $\bar{V}$ is an increasing function of Y , for each $\vec{b}$; and we obtain the approximation

$$g(\vec{b}) = |G(\vec{b})|^2 \exp \left\{ \int d^2b' |G(\vec{b}')|^2 \bar{V}(\vec{b} - \vec{b}', Y) \right\} \quad (9.24)$$

which shows g is an increasing function of Y for fixed $\vec{b}$, especially near $\vec{b} = 0$, where $\bar{V}$ will have its maximum effect. At large $\vec{b}$ there will be no energy dependence since $\bar{V}$ drops off, leaving $g \approx |G|^2$.

We recognize eqn (9.24) as an approximation to the g equation of AT, in the $Y \rightarrow \infty$ limit. At general b , eqn (9.24) yields a multiple-scattering series in which large P_T cross-sections are enhanced whenever $\bar{V} > 0$, as discussed in AT. We believe eqn (9.24) represents a reasonable extension of the formalism of AT to include the approach to the asymptotic limit.

Some quantitative estimate of this enhancement can be reached if we examine the specific features of V , from eqn (9.3). At fixed b , as $Y \rightarrow \infty$, $\bar{V}$ approaches its limiting value from below with a correction term of order $Y^{-1/2}$. Thus, at fixed P_T , we expect the inclusive cross-section to approach its asymptotic limit in a similar way. The form (9.24) suggests $\ln \rho(P_T)$ should be plotted against $Y^{-1/2}$ at each fixed P_T . The intercepts at $Y^{-1/2} = 0$ should be approached linearly.

These (approximate) scaling expectations can thus be checked against hadron data. As a preliminary indication, note that in the region 100-2000 GeV, $Y^{1/2}$ varies about the same as $S^{1/8}$. From fits given for a scaling variable $(P_T / S^{1/8})$ in the survey of large- P_T cross-sections,²⁵ we see that a scaling form using the variable $P_T / Y^{1/2}$ gives a reasonable representation of the data, for $P_T > 3$ GeV. (In our model, as $Y \rightarrow \infty$ for fixed P_T , we expect to recover a function of P_T alone; however, the cross-section at fixed P_T should be a function of Y only through the scaling variable.)

We consider these results to support the general mechanism proposed, and encourage further work to elucidate directly the consequences of equations (9.5) - (9.8).

The associated-multiplicity distributions predicted by this approximate theory will be of Van der Waals type (with the Van der Waals b -parameter

equal to zero, with our specific assumption $V_S \equiv 0$), with $\frac{\bar{n}(P_T)}{Y}$ increasing with P_T . Since $V > 0$ at all b , we also find an increase with Y in the inclusive density at $P_T \equiv 0$, given by the integral of $g(b)$ over all b . In the model defined by eqn (3), in our "moderate-energy" approximation (9.24) the correction term to the asymptotic result decreases as $Y^{-1/2}$; the $Y = \infty$ limit is approached from below, as seen in ISR data. However, these functional dependences (as well as those in the preceeding paragraph) are not necessarily characteristic of the true asymptotic regime, including derivative terms in the complete solution, (9.5); true critical behaviour is more subtle, ⁽²²⁾ and we will have an even more complicated behaviour, as explained below.

A proper solution of the phase-transition problem posed in equations (9.5-9.8) may lead to very complex behaviour of g as $Y \rightarrow \infty$, for small b . We may try to guess some expected features, but we must be careful to keep certain essential difficulties in mind, as explained below.

In solving the ϕ -field theory represented in (9.5), the ground-state or "vacuum" ϕ in the presence of the "external" field V , will play a dominant role in determining $g(b)$. Equation (9.9) is an approximate equation for this ground state. The b -dependence of g is characterized by its response to a small change in b . If the vacuum state is stable and isolated from other states by an energy gap, as in a system at a temperature above its critical point, this response is smooth in b . However, if there is a degeneracy of the vacuum, a phase-coexistence region is present in the analog, and a small change in b , as a small perturbation in the external field, yields a large change in density g , analogous to moving along a horizontal isotherm in the phase diagram.

These transitions between different vacua are generally very difficult to treat analytically. (Classical theories of phase transitions, e.g. Van der Waals, have typically introduced "externally" the Maxwell construction to give flat isotherms.) Any detailed prediction from eqn (9.5) must be based on a theoretical method for phase transitions which gives analytic behaviour for $Y < \infty$, but flat isotherms in the $Y \rightarrow \infty$ limit. In this way we connect the theoretical problems of large- P_T cross-section with the ground-state instability properties, discussed in Ref (5), which may be responsible for the weak and electromagnetic interactions.

X. RAPIDITY-DENSITY FLUCTUATION PATTERNS

In this section we will discuss the characteristic features of event-structure in rapidity expected in such theories at "moderate" energies, $10^2 - 10^4$ GeV, where the long-range power-law decrease of rapidity correlations does not yet play a decisive role. In this energy regime, we will consider correlations which include negative short-range (kinematic) effects and positive moderate-range (1-2 units of rapidity) correlations. The general concepts of a functional description of rapidity fluctuations on an event-by-event basis have been discussed by Krzewicki²⁶ and, more crudely, by AKG¹⁶.

The purpose of our stochastic-field theory is to calculate properties of multiparticle events when the correlations are given (experimentally or theoretically) for exclusive 2-particle production (in the central region). The exclusive amplitude in our earlier sections was parameterized as a product containing short-range ($\exp V_S$) and long-range ($\exp V_L$) correlations, with V_L non-negative. The short-range (V_S) correlations can be summed up with conventional MFH methods, while collective effects from V_L are computed as a functional integral over the short-range ensemble with a weight function determined by V_L .

Parameterizing V_S as a Vander Waals repulsion, as suggested by multi-peripheral kinematics ($t_{\min}$ effect), and parameterizing V_L as a single Regge pole, we obtain exactly the KUH model²⁷ of a one-dimensional fluid, without the infinite-range limit discussed by KUH. The qualitative features of clustering in such a model have been discussed by Jalickee, Wiegel and Vezzetti²⁸. They showed it is sufficient to retain only 4th order in the expansion of the stochastic-field Lagrangian in powers of the field, to obtain qualitatively correct results, provided the fluid is not too close to criticality.

To obtain analytic results for clustering patterns, we note the functional saddle-points are determined by differential equations which have elliptic functions as their solutions. The specifics of this result are given by Krumhansl and Schrieffer,⁽²⁹⁾ who discuss a similar Lagrangian for describing structural phase transitions.

We determine characteristic parameters in the correlations by assuming a near-critical Vander Waals theory, as in the Arnold and Thomas discussions⁽³⁾ of multiplicity distributions. Exactly at criticality, the condition expected at asymptotic energy, the fluctuation theory is not valid. However, as shown in an earlier section, the behaviour of cross sections at moderate energies is determined by an effective Pomeron singularity above unity, corresponding to a two-phase coexistence analog fluid just below the critical temperature. We leave $\epsilon^2 = \alpha_{\text{eff}} - 1$ as a small parameter to be determined by data (of order 1/20). Writing $\bar{\phi} = \frac{\partial \phi}{\partial y}$, we obtain the stochastic-field Lagrangians appropriate to the problem posed here;

$$L_0[\phi] = \frac{1}{2} [(\bar{\phi})^2 + \mu^2 \phi^2] \quad (10.1)$$

and

$$L = L_0 + L_1, \quad (10.2)$$

$$L_1[\phi] = -\alpha_0 [e^{\phi}], \quad (10.3)$$

where $\alpha_0(z)$ satisfies the "hard-core" equation, (6.8):

$$zG = \alpha_0 e^{b\alpha_0}.$$

We will determine G and b from the criticality assumptions of Ref 3, and let μ^2 be a parameter slightly less than the critical value $(2a_c)^{-1}$, where a and b are the Van der Waals parameters. At Van der Waals criticality $b = 1/8$, $\mu^2 = 32/27$, according to Ref 3.

Expanding L_1 around $\hat{\phi}$, the (constant) mean-field value of ϕ , we obtain

$$L[\psi] = L_0 + L_1 = \frac{1}{2} (\bar{\psi})^2 + \frac{1}{2} \mu^2 \psi^2 - \frac{c_3}{3!} \psi^3 - \frac{c_4}{4!} \psi^4 + \hat{L} \quad (10.4)$$

where $\psi = \phi - \hat{\phi}$; $\hat{\phi}$ satisfies

$$\mu^2 \hat{\phi} = \frac{z \partial \alpha_0(z)}{\partial z} \Big|_{z=e^{\hat{\phi}}} ; \quad (10.5)$$

$$\hat{L} \equiv \frac{\mu^2}{2} \hat{\phi}^2 - \alpha_0(e^{\hat{\phi}}) ; \quad (10.6)$$

and $C_p \equiv (z \frac{\partial}{\partial z})^p \alpha_0(z) \Big|_{z=e^{\hat{\phi}}} \quad (10.7)$

As remarked in Ref (5), the "bare" parameters, in eqn (4), will be replaced by renormalized parameters when non-classical (loop) corrections are calculated. This renormalization is finite when $\sigma = 2$,^{as we have taken here,} but becomes infinite for $\sigma < 1$.

The event-distributions in rapidity are not dependent on $\hat{L}$; however, the C_p 's depend on the value of $\hat{\phi}$, and thus affect the y -distributions. All observables (functionals of ϕ) are obtained from the functional integral over ϕ , whose weight-function is proportional to $\exp \{ - \int_{-Y/2}^{+Y/2} L[\phi(y)] dy \}$. We expect the functions which lead to stationary values of the integrand to give a reasonable idea of typical event-configurations in rapidity, as these functions will represent the saddle-points of the functional integral. They satisfy:

$$\psi'' = (\mu^2 - C_2) \psi + \frac{C_3}{2} \psi^2 + \frac{C_4}{6} \psi^3 . \quad (10.8)$$

The solutions of (10.8) can be obtained in terms of elliptic functions.²³ We will consider solutions near criticality; let

$$\epsilon^2 = C_2 - \mu^2 > 0, \text{ but } \epsilon^2 \ll 1; \quad (10.9)$$

and we evaluate C_3 and C_4 near Vander Waals criticality,⁽³⁾ where

$$C_3 = 0; \quad (10.10)$$

while C_4 is a very sensitive function of the parameters near criticality. At the V.dW, critical point,

$$C_4 = -4 \left(\frac{2}{3} \right)^6 < 0; \quad (10.11)$$

Thus, if we try to develop the theory by assuming "classical" Vander Waals values for the critical parameters, we cannot truncate the functional $L[\phi]$ at fourth order in ϕ ; the sign of L for large ϕ becomes opposite to the correct sign [determined by $\alpha_0(e^\phi)$]. The alternative is to recognize that the correct critical point differs from the classical value, by renormalization⁵ corrections. This prevents us from making a simple estimate for $(C_4)_{\text{Ren}}$ based only on $\alpha_0(Z)$.

To obtain a crude estimate of the orders of magnitude of density fluctuation, we will assume that $(C_4)_{\text{Ren}} > 0$. With an order of magnitude characteristic of the classical critical-point value, eqn (10.11).

The amplitudes of oscillations in the solutions are determined by the roots of the right-hand side of (10.8);

$$\frac{C_4}{6} \psi_{\pm}^2 + \frac{C_3}{2} \psi_{\pm} - \epsilon^2 = 0. \quad (10.12)$$

Each solution $\psi(y)$, periodic in y , will be determined by location of zeros and the slope at a representative zero. The former feature is uninteresting if we can shift the coordinates, appropriate at very high Y values. Thus, the essential parameter for each solution $\psi(y)$ is an initial slope, $\psi'(0)$, assuming $\psi(0) = 0$.

An illuminating discussion of bubbles and droplets expected in a system described by (4) can be found in Ref (28).

To illustrate the qualitative features expected, as in Ref (29), we scale variables in the equation

$$\ddot{\psi} = \epsilon^2 \psi - \epsilon^2 \lambda^2 \psi^3 \quad (10.13)$$

where $\epsilon\lambda \sim (.005)^{1/2}$ if the C_4 coefficient has the order of magnitude suggested by eqn (10.11). We can simplify if we scale the magnitude of ψ with λ , and scale the rapidity by ϵ ; thus writing

$$u = \lambda\psi \quad (10.14)$$

and

$$x = \epsilon y, \quad (10.15)$$

we obtain

$$\ddot{u} - u + u^3 = 0 \quad (10.16)$$

(where $\dot{u} = \partial u / \partial x$, etc). The solutions of (10.16) are given by³⁰

$$u(x) = \text{sn}(x, k) \quad , \quad (10.17)$$

where k , the modulus of the elliptic function sn , is related to the derivative of u with respect to x at $x = 0$; if

$$q = \dot{u}(0) \quad , \quad (10.18)$$

then³⁰ :

$$k^2 = \frac{1 - (1 - q^2)^{1/2}}{1 + (1 - q^2)^{1/2}} \quad (10.19)$$

and there are solutions only with $q^2 < 1$.

The probability of observing rapidity densities near the configuration ψ is determined by the weight-function $\exp \left\{ - \int_{-Y/2}^{+Y/2} [\psi(y)] dy \right\}$, which is now a function of q .

For the following, we have borrowed from Langer's discussion²¹ of magnet theory, as well as the Krumhansl-Schrieffer²⁹ theory of structural phase transitions.

If q^2 is not close to unity, the function ψ_q will have many oscillations, and the integral of $(\psi)^2$ will be large, sharply reducing the contributions from such functions in the functional integral. Thus, only for q^2 near unity will the saddle-point functions have a appreciable weight.

For $q^2 \approx 1$, we find $k^2 \approx 1$; the period $K(k)$ of the elliptic function is very long compared to unity (a short-range correlation length), and the transitions between $u = +1$ and $u = -1$ regions become very sharp. Thus, we find the phase-space in y is divided evenly into domains, between which the density changes suddenly between high and low values.

A sketch of a typical-event density, $\rho(y) \sim \exp[\psi(y)]$, with $\psi = \lambda^{-1}u$, is given in Fig 3a. We see dense regions, which could be called "clusters", alternating with sparse regions, which could be called "gaps". We have chosen $\epsilon^2 = 0.05$, for illustration, since this is a typical value³¹ of $(\alpha_{\text{eff}} - 1)$ seen in total cross section analyses; this gives $\lambda \approx 0.3$, with our assumptions on the order of magnitude for C_4 .

The observable densities are obtained from functional averages over all solutions ψ . We may picture the true typical-event density as a superposition of solutions of (10.18), with various q^2 values near unity, when the fluctuations around the saddle-point functions are taken into account. The resulting density will be represented as an aperiodic set of domains,²⁹ whose wall-locations are distributed in a relatively random fashion. A sketch of such a typical-event density is given in Fig 3b.

As pointed out by K.S.,⁽²⁹⁾ and in a similar theory with N internal degrees of freedom discussed Dashen, Ma, and Rajaraman,³² the statistical distribution following from (10.4) of domain sizes and numbers can be estimated if one takes the domain walls or "kinks" to be a non-interacting gas of quasi-particles. (This is reasonable with finite-range correlations in one dimension, but not when long-range terms are included). The distribution of kink-number n_k [for $Y \rightarrow \infty$] follows a Poisson probability distribution:

$$P(n_k) = \frac{(\bar{n}_k)^{n_k}}{n_k!} e^{-\bar{n}_k} \quad (10.20)$$

where the average kink-density in our notation is given by ⁽²⁹⁾:

$$\frac{\bar{n}_k}{Y} \approx \exp\left(-\frac{\epsilon^2}{2\lambda^2}\right) \quad (10.21)$$

If there are N internal degrees of freedom, with $O(N)$ symmetry, it can be shown [as in Ref (32)] that:

$$\bar{n}_k/Y \approx \epsilon \exp(-N\epsilon^2/2\lambda^2) \quad (10.22)$$

As noticed by D.M.R.³², this means that in the $N \rightarrow \infty$ limit, no kinks will be found in an asymptotic expansion in powers of N^{-1} .

The estimate (10.21) gives a fairly low average density of kinks; with our choice of parameters,

$$\bar{n}_k = (.22) \exp(-.25) = \frac{1}{6} . \quad (10.23)$$

Note that fluctuations with several times this density of kink boundaries will not be very rare, according to (10.20). We conclude that for $Y < 6$, only one kink is expected on the average (although realistic boundary conditions probably require an even number); but for $Y \gg 6$, we expect frequent occurrence of several kinks.

Since present accelerator experiments are limited to $Y < 6$ [where Y is the effective plateau length], it may not be reasonable to directly test these predictions by comparison with available data. There may be some evidence for dramatic density fluctuations in very high energy cosmic-ray experiments, but it does not seem possible to establish a systematic pattern with the limited number of events available.

At very high energies, if Eqns (10.1) and (10.2) remain a reasonable description, we expect the effective ϵ^2 to be smaller than 0.05, and the value of λ^2 correspondingly greater. Then the amplitude of the fluctuations should increase, while the average kink density decreases; "clusters" become more dense, but more widely separated on the average, and the "gaps" are populated more sparsely. However, the long-range rapidity correlations must play an increasingly important role, and we expect the V_y^σ terms omitted from (10.1) to influence the clustering structure in important ways.

XI SUMMARY

We have demonstrated mathematical methods for extracting quantitative results from the Multi-Eikonal hypothesis, which in turn may be motivated by a desire to incorporate a semi-classical limit for multiparticle production amplitudes in which Regge cuts play an important role.

Mechanisms then arise naturally for collective high-energy phenomena which are absent in pure Regge pole theories or MPM's. Some of these, including large- P_T cross-section enhancements and a rich spectrum of t-channel

Regge trajectories, are reminiscent of models of hadrons as composite structures of point-like entities (partons, quarks). We can interpret the Multi-Eikonal results as a complementary picture in which certain hadronic fluctuation structures appearing in the unitarity sums act as if they were heavy, spatially small, and as if the hadrons were composites of them.

This complementary picture is somewhat reminiscent of certain collective phenomena in plasmas, where the plasmon modes, strongly excited by short wavelength photons, can act as independent particles with a peculiar energy-momentum relationship. The plasma role is played by multihadron states; the Coulomb potential is replaced by Regge cuts; the plasmons correspond to quarks or partons; and the large energy-momentum losses in deep-inelastic lepton-hadron collisions are analogous to anomalous (collective) energy depositions by charged particle beams in a plasma, e.g. through two-stream instabilities. In hadron physics, the quasi-particles can be "seen" even at low energies through dispersion relations, which relate low energy amplitudes to high energy states, unlike any phenomenon familiar in classical plasma theory.

Such a picture, combined with currently fashionable ideas for unifying weak and strong interactions, leads to the hypothesis that leptons arise from the same mechanisms, but are decoupled from strong interactions at low energies because of high rapidity thresholds for certain collective modes. Exploiting this picture then may lead to a quantitative relation between weak and strong dynamics,⁽⁵⁾ as well as a successful theory of strong interaction production phenomena.

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FIGURE CAPTIONS

Figure 1. Combinatoric structure of A_4 , equation (2.1).

Wavy lines represent factors of K .

Dotted line indicates particle-producing t-channel exchange.

Figure 2. ~~Two~~ particle amplitude interpretation of A_0 , G , and K factors.

Figure 3. (a) Typical-event rapidity density pattern arising from one elliptic function, equation (10.17), with one value of q .

(b) Functional-averaged typical-event density with averaging over q determined by the parameters defined in equation (10.21).

(a) $A_0(Y, B) \xrightarrow{\quad}$

(b) $A_1 = A_0 G(r)$

(c) $A_2 = A_0 G(r_1) K(r_1, r_2) G(r_2)$

(d) Multi-Regge limit of $A_2 = A_0 G G$