Higgs-Type Lagrangians and Vortex-Line and Soliton Solutions*

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ABSTRACT

Approximate vortex-line and soliton solutions of the Higgs-type Lagrangian are obtained analytically, and the magnetic field, magnetic energy and its root mean square radius, Higgs energy, and interaction energy between the vector field A_μ and the Higgs scalar ϕ are calculated. The soliton solution for another case is commented on.

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I. INTRODUCTION

Nielsen and Olesen¹ obtained solutions of the Higgs-type Lagrangian and showed how Nambu strings in dual resonance theories can be related to a localized classical solution of Landau Ginzberg Lagrangian which has vortex line solutions in a type II-superconductor. There is a large number of articles on such localized classical solutions or solitary waves (kinks) in physics. They have become interesting because they may represent new states that have some features of extended hadrons.

The solutions of the Higgs-type Lagrangian have been given so far asymptotically¹ for small and large r . We obtain approximate solution valid over the whole range of r , and calculate physical quantities such as magnetic field, magnetic energy, Higgs energy and interaction energy between the vector field A_μ and scalar Higgs field ϕ . We study the situations when the Higgs-type Lagrangian has solitary wave solutions. The plan of the paper is as follows. Solutions are obtained to the equations of motion of the A_μ and $|\phi|$ in Sec.II, and physical quantities are calculated in Sec.III. The soliton solutions are studied in Sec.IV, and remarks are given in Sec.V.

II. SOLUTION OF HIGGS-TYPE LAGRANGIAN

We start from the Higgs-type Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}|(\partial_\mu - ieA_\mu)\phi|^2 + c_2|\phi|^2 - c_4|\phi|^4, \quad (1)$$

where $F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$, $c_2 = -\mu^2$ (ϕ mass) > 0 , so that we have the Higgs mechanism, and $c_4 > 0$ for stability. We are interested in a static cylindrically sym-

metric solution along the z direction. So the equations of motion for $A(r)$ and $|\phi(r)|$ with $A_0 = A_z = A_r = 0$, $\underline{A}(r, z, \omega) = A(r)\hat{z} \times \hat{r} = A(r)\hat{\omega}$, are

$$\frac{1}{r} \frac{d}{dr} r \frac{d|\phi|}{dr} - \left(\frac{1}{r} - eA\right)^2 |\phi| + 2C_2 |\phi| - 4C_4 |\phi|^3 = 0, \quad (2)$$

$$\frac{d}{dr} \frac{1}{r} \frac{d}{dr} r A(r) + e \left(\frac{1}{r} - eA\right) |\phi|^2 = 0. \quad (3)$$

When the absolute bracket of $|\phi|$ is suppressed and $eA = \frac{1}{r} + K$ is substituted in (2) and (3), one gets

$$\phi'' + \frac{1}{r} \phi' - (K^2 + 4C_4 \phi^2 - 2C_2) \phi = 0, \quad (4)$$

$$K'' + \frac{1}{r} K' - \frac{1}{r^2} K - e^2 \phi^2 K = 0. \quad (5)$$

The solution of Eq.(4) is found to be²

$$K = - \left(\frac{4C_2}{y \sinh y} \right)^{\frac{1}{2}}, \quad (6)$$

$$\phi = \left(\frac{C_2}{2C_4} \right)^{\frac{1}{2}} \tanh \frac{y}{2}, \quad (7)$$

and the solution of Eq.(5) is K in (6) and

$$\tilde{\phi} = \left(\frac{c_2}{e^2}\right)^{\frac{1}{2}} \left(3\left(\frac{1}{\tanh y}\right)^2 - \frac{3}{y^2} - 2\right)^{\frac{1}{2}}, \quad (8)$$

where $y = 2c_2^{\frac{1}{2}}r$. It is shown below that $(\phi - \tilde{\phi})/2\phi < 0.055$ over the whole range of r when $2|c_4| = e^2$ so that we regard (6) and (7) as an approximate solution with the condition $2|c_4| = e^2$, which we impose. This restricts the model somewhat. On the other hand, this condition guarantees that both of the functions ϕ and $\tilde{\phi}$ give the same vacuum expectation value asymptotically and it is also consistent with the physical requirements we find later. The asymptotic solutions of (4) and (5) are from (6) and (7)

$$K = -\frac{1}{r} + \frac{c_2}{3}r, \quad \phi = \left(\frac{c_2}{2c_4}\right)^{\frac{1}{2}} \left(r - \frac{c_2}{3}r^3\right) \quad r \rightarrow 0, \quad (9)$$

$$K = -2c_2^{\frac{1}{4}} \frac{1}{\sqrt{r}} e^{-\sqrt{c_2}r}, \quad \phi = \left(\frac{c_2}{2c_4}\right)^{\frac{1}{2}} \quad r \rightarrow \infty. \quad (10)$$

To estimate the difference of the solutions ϕ and $\tilde{\phi}$, we form the combination

$$\Delta(y) = \left(\frac{\phi - \tilde{\phi}}{2\phi}\right) = \frac{1}{2} \left[1 - \left(\frac{2c_4}{e^2}\right)^{\frac{1}{2}} \left(3\left(\frac{1}{\tanh y}\right)^2 - \frac{3}{y^2} - 2\right)^{\frac{1}{2}} (\tanh \frac{y}{2})^{-1} \right] \quad (11)$$

where

$$\Delta(y \rightarrow 0) = \frac{1}{2} \left(1 - \left(\frac{8c_4}{5e^2}\right)^{\frac{1}{2}}\right), \quad (12)$$

$$\Delta(y \rightarrow \infty) = \frac{1}{2} \left(1 - \left(\frac{2c_4}{e^2}\right)^{\frac{1}{2}}\right). \quad (13)$$

We see from (12) and (13) that if we have $e^2/2c_4 = 1$, then $\Delta(y)$ vanishes as $y \rightarrow \infty$, and if we have $e^2/2c_4 = 4/5$, then $\Delta(y)$ vanishes as $y \rightarrow 0$. $\Delta(y)$ is plotted in Fig.1 in the case $e^2/2c_4 = 1$. In other words, if we substitute (6) and (7) into Eq.(5) and determine $e^2(y)$, where $y = 2(c_2)^{\frac{1}{2}}r$, we find

$$e^2(y)/2c_4 = \left(\frac{3}{(\tanh y)^2} - \frac{3}{y^2} - 2 \right) \left(\coth \frac{y}{2} \right)^2 = (e^2/2c_4) (\tilde{\phi}/\phi)^2$$

$$\rightarrow \frac{4}{5} \quad r \rightarrow 0,$$

$$\rightarrow 1 \quad r \rightarrow \infty,$$

i.e., if $e^2(y)$ has the above r -dependence, then the pair (6) and (7) is an exact solution of (4) and (5). The present choice $e^2 = 2c_4$ gives $\phi = \tilde{\phi}$ as $y \rightarrow \infty$.

It is possible to obtain more accurate solutions by adding more terms to K and ϕ or getting numerical solutions. The present analytical solution of K and ϕ appears to be sufficiently accurate so that it is used to obtain information on the magnetic field H and the total energy E that are computed in Sec.III.

III. MAGNETIC FIELD AND ENERGY OF SYSTEM

The magnetic field is given with the aid of (6)

$$\begin{aligned} |\underline{H}| &= |\text{curl}_z A(r)| = \frac{1}{r} \frac{d}{dr} r |\underline{A}| = \frac{1}{e r} \frac{d}{dr} r K \\ &= \frac{1}{e} \left(K' + \frac{K}{r} \right) = \frac{2c_2}{e} \frac{1}{(y \sinh y)^{\frac{1}{2}}} \left(\frac{1}{\tanh y} - \frac{1}{y} \right). \end{aligned} \quad (14)$$

The value of $|\underline{H}|$ near $r=0$ is $|\underline{H}| = (2c_2/3e) - (2c_2^2/5e)r^2$, and near $r=\infty$,
 $|\underline{H}| = (2c_2^{\frac{3}{4}}/e) 1/\sqrt{r} e^{-c_2 r}$.

In the cylindrical system the energy density is obtained from Eq. (1)

$$E(r) = \frac{1}{2} \left(\frac{1}{r} \frac{d}{dr} rA \right)^2 + \frac{1}{2} \left(\frac{1}{r} - eA \right)^2 |\phi|^2 + \frac{1}{2} \left(\frac{d|\phi|}{dr} \right)^2 - c_2 |\phi|^2 + c_4 |\phi|^4 + \frac{c_2^2}{4c_4}. \quad (15)$$

The last constant term is added so that the scalar field energy density is normalized to zero as $r \rightarrow \infty$. The energy per unit length along the z axis is computed using the solution (6) and (7). The magnetic energy is

$$\begin{aligned} E_H &= \int \frac{1}{2} \left(\frac{1}{r} \frac{d}{dr} rA \right)^2 2\pi r dr \\ &= \frac{c_2 \pi}{2c_4} \int_0^\infty \frac{dy}{\sinh y} \left(\frac{1}{\tanh y} - \frac{1}{y} \right)^2 = \frac{c_2}{2c_4} (0.785). \end{aligned} \quad (16)$$

where we used $2|c_4| = e^2$. The scalar field energy is

$$\begin{aligned} E_\phi &= \int \left\{ \frac{1}{2} \left(\frac{d|\phi|}{dr} \right)^2 - c_2 |\phi|^2 + c_4 |\phi|^4 + \frac{c_2^2}{4c_4} \right\} 2\pi r dr \\ &= \frac{c_2 \pi}{2c_4} \int_0^\infty \frac{y dy}{2 \left(\cosh \frac{y}{2} \right)^4} = \frac{c_2 \pi}{2c_4} \frac{1}{3} (4 \ln 2 - 1) = \frac{c_2}{2c_4} (1.86). \end{aligned} \quad (17)$$

The interaction energy E_I which includes the mass term $\frac{1}{2} e^2 |\phi|^2 A_\mu^2$ of the vector field is

$$\begin{aligned} E_I &= \int \frac{1}{2} \left(\frac{1}{r} - eA \right)^2 |\phi|^2 2\pi r dr \\ &= \frac{c_2 \pi}{2c_4} \int_0^\infty \frac{(\tanh \frac{y}{2})^2}{\sinh y} dy = \frac{c_2}{2c_4} \frac{\pi}{2} = \frac{c_2}{2c_4} (1.57). \end{aligned} \quad (18)$$

We have also calculated³ E_ϕ and E_I using (6) and (8) and found that the difference in energy is at most about 20%.

The partial energies E_H , E_ϕ and E_I are respectively 20, 45 and 35% of the total energy. The energy density at $r = 0$ is $E(0) = 3lc_2^2/36c_4$. The root mean square radius of the magnetic energy distribution is

$$\langle r^2 \rangle = \frac{1}{E_H} \int E_H(r) r^2 2\pi r dr = \frac{\pi}{8c_4 E_H} \int_0^\infty \frac{y^2}{\sinh y} \left(\frac{1}{\tanh y} - \frac{1}{y} \right)^2 dy$$

or

$$\begin{aligned} (\langle r^2 \rangle)^{\frac{1}{2}} &= \left(\frac{\pi}{8c_4 E_H} \left(\frac{7}{4} \zeta(3) - \frac{1}{2} \right) \right)^{\frac{1}{2}} \\ &= 1.1 (c_2)^{-\frac{1}{2}} \end{aligned} \quad (19)$$

where $\zeta(3) = \sum_{n=1}^{\infty} n^{-3}$.

The magnetic field $|H(r)|$, scalar field $\phi(r)$, and vector field $A(r)$ are plotted as an illustration with $c_2 = 2c_4 = e^2 = 9$ in Fig.2. One observes that $A(r)$ and $\phi(r)$ can be regarded as a soliton solution.⁴

IV. SOLITON SOLUTIONS

In Sec.II, it was shown that the equations of motion for A and ϕ of the Higgs-type Lagrangian possess soliton solutions when $c_2 > 0$ and $c_4 > 0$. The potential in this case for $|\phi|$ is

$$V(r) = -c_2 |\phi|^2 + c_4 |\phi|^4$$

which has a minimum at $|\phi| = \left(\frac{c_2}{2c_4} \right)^{\frac{1}{2}}$. This is the case when the Higgs mechanism is effective. When $c_2 < 0$, $c_4 > 0$ we have the ordinary case where the mass of ϕ is real and the potential is minimum at $|\phi| = 0$. The situation is similar

when $c_2 > 0$ and $c_4 < 0$. We now examine the intriguing case of $c_2 < 0$, $c_4 < 0$, because there is a minimum at $|\phi| = 0$ surrounded by a potential barrier at $|\phi| = \left(\frac{c_2}{2c_4}\right)^{\frac{1}{2}}$.

The equation of motion for $c_2 = -|c_2|$ and $c_4 = -|c_4|$ is from (4) and (5)

$$\phi'' + \frac{1}{r} \phi' - (K^2 - 4|c_4| \phi^2 + 2|c_2|) \phi = 0, \quad (20)$$

$$K'' + \frac{1}{r} K' - \frac{1}{r^2} K - e^2 \phi^2 K = 0. \quad (21)$$

The solution of ϕ is $I_1(\sqrt{2|c_2|}r)$ near the origin and $K_1(\sqrt{2|c_2|}r)$ at large distances. The asymptotic solution is for $r \rightarrow 0$

$$\phi = \left(\frac{|c_2|}{2}\right)^{\frac{1}{2}} \left(r + \frac{|c_2|}{4} r^3 \right), \quad (22)$$

$$K = -\frac{1}{r}, \quad (23)$$

and for $r \rightarrow \infty$

$$\phi = \left(\pi / 2 \sqrt{2|c_2|} r \right)^{\frac{1}{2}} e^{-\sqrt{2|c_2|} r}, \quad (24)$$

$$K = -\frac{1}{2r}. \quad (25)$$

The case of $c_2 < 0$, $c_4 < 0$ [Eq.(20)] possesses a soliton solution for ϕ . Equation (20) is similar to the equation of the modulus of the azimuthal electric field in a medium with a nonlinear susceptibility which has a soliton solution.⁵

V. REMARKS

The Higgs-type Lagrangian (1) provides an interesting model to study vortex-line and soliton solutions of the vector boson A_μ and the Higgs scalar ϕ . For small r , the $|\frac{1}{r} - eA|^2 |\phi|^2$ term would dominate the $c_2 |\phi|^2$ term so that the ground state is $|\phi| = 0$ and there is no Higgs mechanism. For large r , $|\phi|$ approaches its asymptotic value $(c_2/2c_4)^{\frac{1}{2}}$. It is interesting that the vortex-line solution of A is associated with a soliton solution of $|\phi|$.

The masses associated with the fields A_μ and ϕ are from Eqs. (6) and (7) $m_v = e|\phi| = c_2^{\frac{1}{2}}$ and $m_s = (2c_2)^{\frac{1}{2}}$, respectively, so that the radius over which the magnetic field is appreciably different from zero (penetration depth) is $c_2^{-\frac{1}{2}}$ and the radius for $|\phi|$ to reach its asymptotic value is $(2c_2)^{-\frac{1}{2}}$. We note that in the string picture magnetic energy/unit length = $E_H = (2\pi\alpha')^{-1}$ or with Eq. (16)

$$\alpha' = 2c_4/5c_2 = e^2/5c_2,$$

where α' is the uniform slope of the Regge trajectory. In order for the string picture to be valid, one obtains

$$\sqrt{\alpha'}/m_v^{-1} \sim e \gg 1,$$

$$\sqrt{\alpha'}/m_s^{-1} \sim e \gg 1,$$

so that the model is far from being classical.¹ This is the case in which the masses m_v and m_s are much larger than a typical hadron mass. If there are monopoles located at the ends of the string the energy depends linearly on the distance of separation.⁶ We note from Eq. (19) that $(\langle r^2 \rangle)^{\frac{1}{2}} \sim m_v^{-1}$ so that the string contains most of the magnetic energy.

The solutions of the Higgs-type Lagrangian for the case $c_2 > 0$, $c_4 > 0$ is related to the vortex-line solution in type II-superconductors and for the case $c_2 < 0$, $c_4 < 0$ is related to the soliton solution of the azimuthal electric field in a medium with a nonlinear susceptibility.

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FIGURE CAPTIONS

Fig.1. Plot of $\Delta(y) = (\phi - \tilde{\phi})/2\phi$ versus $y = 2(c_2)^{\frac{1}{2}}r$ with the condition $2|c_4| = e^2$.

Fig.2. Plot of magnetic field $|H(r)|$, scalar field $|\phi(r)|$ and vector field $A(r)$ versus r with $c_2 = 2c_4 = e^2 = 9$.

Fig.1

Fig.2