ON NONLINEAR FORMULATION OF FINITE ELEMENT BY COROTATIONAL COORDINATES

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BASE TECHNOLOGY

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NOMENCLATURE

t	time
X_i	Langrangian coordinates
x_i	Eulerian or spatial coordinates
G_{ij}	deformation gradient tensor
u_i	displacement
R_{ij}	orthogonal rigid body rotation tensor
U_{ij}	stretch tensor
A_{ij}^{-1}	the inverse tensor of A_{ij}
E_{ij}	Langrangian strain tensor for finite deformation
α_{ij}	engineering strain tensor
u_i^{rig}	displacement corresponding to rigid body rotation referred to the spatial coordinates
u_i^{def}	displacement corresponding to deformation referred to the spatial coordinates
$\hat{\epsilon}_{ij}$	corotational strain tensor referred to the corotational coordinates
$\hat{u}_i^{def}$	displacement corresponding to deformation referred to the corotational coordinates
u_j^{tra}	displacement corresponding to rigid body translation referred to the spatial coordinates
δ	variational symbol
S°	undeformed surface of the continuum or the element
V°	undeformed volume of the continuum or the element
t_i°	surface traction per undeformed surface referred to the spatial coordinates

b_i°	body force per undeformed volume referred to the spatial coordinates
w^{ext}	external work
w^{int}	internal work
S_{ij}	second Piola-Kirchhoff stress tensor
σ_{ij}	Cauchy stress tensor referred to the spatial coordinates
$\hat{\sigma}_{ij}$	corotational stress tensor referred to the corotational coordinates
$\hat{w}^{\text{int}}$	corotational internal work
J	Jacobian of the deformation gradient tensor; or $J = \det(G_{ij})$
ρ°	density referred to the undeformed volume
$(\cdot); (\cdot)$	time derivative of ()
σ_{ij}^∇	Jaumann stress rate tensor referred to the spatial coordinates
V_{ij}	rate of deformation tensor referred to the spatial coordinates
C_{ijkl}	elastic coefficient tensor
λ, G	Lame's constants for an elastic material
$Y; f$	yield function for an elastoplastic material
κ	work hardening parameter for an elastoplastic material
w_p	plastic work
$\hat{V}_{ij}$	rate of deformation tensor referred to the corotational coordinates
$\hat{V}_{ij}^e$	reversible elastic part of the rate of deformation tensor referred to the corotational coordinates
$\hat{V}_{ij}^p$	irreversible plastic part of the rate of deformation tensor referred to the corotational coordinates
g_{ij}	gradient tensor of the yield function f
$\hat{s}_{ij}$	corotational stress deviation tensor referred to the corotational coordinates
m	constitutive parameter for an elastoplastic material

$\hat{d}_i^{\text{def}}$	nodal deformation displacement vector of an element referred to the corotational coordinates
$\hat{\phi}_{ij}$	corotational deformation shape functions
d_i	nodal displacement vector of an element referred to the spatial coordinates
$\hat{\psi}_{ij}$	functions of the relations between $\hat{d}_i^{\text{def}}$ and d_i
ϕ_{ij}	shape functions
ϕ'_{ij}	rigid body motion part of ϕ_{ij}
f_i^{ext}	generalized external nodal force vector of an element, that is referred to the spatial coordinates
$\hat{f}_i^{\text{int}}$	generalized internal nodal force vector of an element, that is referred to its corotational coordinates
f_i^{int}	generalized internal nodal force vector of an element, that is referred to the spatial coordinates
T_{ij}	transformation tensor that relates f_i^{int} and $\hat{f}_i^{\text{int}}$
m_{ij}	element mass tensor
D_j	generalized nodal displacements vector of the discretized continuum
$I^{(L_{ij})}$	Boolean connectivity operator for element I
M_{ij}	mass tensor of the discretized continuum
F_i^{ext}	generalized external nodal force vector of the discretized continuum
F_i^{int}	generalized internal nodal force vector of the discretized continuum

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B. J. Hsieh

Abstract

This study is concerned with certain aspects of the finite element formulation in corotational coordinates. This is a coordinate system that rotates and translates with the element to which it is allocated. It is shown that the displacement field of each element can be separated into rigid-body-motion displacements and deformation displacements by using Gauss's polar decomposition theorem. The strains, stresses, and internal forces are computed in the corotational coordinates. This approach greatly simplifies the computations of strains and stresses when the displacements and strains are not small.

INTRODUCTION

Most of the nonlinear dynamic analysis of solid continua use the Lagrangian formulation. The merits and disadvantages of these formulations were reported in [1] and [2]. The corotational coordinates formulation which reduces the efforts in computations for problems where the nonlinearities arise both from material nonlinearity and large displacement is presented here. The corotational coordinate is a coordinate that only rotates and translates with the element, but does not deform with it. This formulation linearizes the strain-displacement relations in terms of the deformation-displacements in the element's corotational coordinates. Furthermore, since the effects of rigid body rotations are eliminated in the corotational coordinates, the Jaumann stress rate is simply the time derivative of the corotational stress.

The first use of local coordinates to solve large deflection static problems was by Argyris, et al [3]. In their paper, the pure deformation modes rather than the usual nodal displacements were used as the unknown parameters. This makes it possible to use linearized strain expressions when the large displacement effects are due to rigid body translations and rotations.

The corotational coordinates formulation presented here uses the global nodal displacements as unknown parameters. The displacements are separated into additive rigid-body-motion displacements and deformation displacements by utilizing the Gauss's polar decomposition theorem from nonlinear continuum mechanics. The corotational strains and stresses are defined and related to each other. This formulation shows the linearity in the corotational strain expressions and the simplicity in

the corotational constitutive equations. The governing equations of motions are obtained by applying the principle of virtual work in the corotational coordinates.

Cartesian tensor index notations are used in this study. Any repeated indices indicate summation over the range of the indices, unless otherwise stated.

KINEMATICS

Let the undeformed configuration of a solid continuum at time $t = 0$ be described by the coordinates X_i ($i = 1, 3$). Later the deformed configuration is described by the spatial coordinates x_i ($i = 1, 3$). These two sets of coordinates are related by the deformation gradient tensor G_{ij} as

$$dx_i = G_{ij} dX_j \quad ; \quad i, j = 1, \text{ number of dimensions} \quad (1)$$

where dx_i and dX_i are the differences in the values of x_i and X_i between two neighboring points, and the deformation gradients tensor is given by

$$G_{ij} = \frac{\partial x_i}{\partial X_j} \quad ; \quad i, j = 1, \text{ number of dimensions} \quad (2)$$

The displacements u_i are defined as the difference between the values of these two coordinates, that is

$$u_i = x_i - X_i \quad ; \quad i = 1, \text{ number of dimensions} \quad (3)$$

It is well known that the deformation gradient can be separated into two components which are additive if the displacements are small.

When the displacements are large, however, the decomposition of the deformation gradient into two distinct (deformation and rigid body motion) additive components is impossible. But a decomposition of the deformation gradient into two multiplying components, one of which represents a rigid body rotation, while the other represents the stretch of the continuum is always possible [4,5].

Mathematically, this can be written as

$$G_{ij} = R_{ik} U_{kj} \quad ; \quad i, j, k = 1, \text{ number of dimensions} \quad (4)$$

where R_{ij} is an orthogonal tensor (i.e., $R_{ij}^{-1} = R_{ji}$) which is the rigid body rotation part of the deformation process. The symmetric tensor U_{ij} (i.e., $U_{ij} = U_{ji}$) is the stretch part of the deformation process. Let the stretch tensor be written as

$$U_{ij} = \delta_{ij} + \alpha_{ij} \quad ; \quad (5)$$

where δ_{ij} is the Kronecker symbol and α_{ij} can be identified as the engineering strain tensor.

A substitution of eqs. (4) and (5) into eq. (1) yields

$$dx_i = R_{ik} (\delta_{kj} + \alpha_{kj}) dX_j \quad (6)$$

To identify the physical meaning of eq. (6), a corotational coordinate system, $\hat{x}_i$, is defined by rotating the Lagrangian or the spatial coordinates through the rigid body rotation R_{ij} . That is

$$d\hat{x}_i = R_{ji} dx_j \quad ; \quad dx_i = R_{ij} d\hat{x}_j \quad (7)$$

Introducing eq. (8) into eq. (7), we have

$$d\hat{x}_i = (\delta_{ij} + \alpha_{ij}) dX_j \quad (8)$$

Hence the difference in motion between two neighboring points can be considered as first a stretch $(\delta_{ij} + \alpha_{ij})$ followed by a rigid body rotation R_{ij} . The same state of deformation can be obtained by first rotating the continuum through the rigid body rotation then stretching it to the same intrinsic amount. The derivation for this case is very similar to the case we have studied and hence not repeated here.

The deformation displacement $\hat{u}_i^{\text{def}}$ between two neighboring points in the corotational coordinates is obtained from eqs. (3) and (5) to be

$$d\hat{u}_i^{\text{def}} = \alpha_{ij} dX_j \quad (9)$$

The relationship between the corotational deformation displacements $\hat{u}_i^{\text{def}}$ and the global spatial displacement u_i between two neighboring points is obtained by combining eqs. (3), (7), and (10) as

$$d\hat{u}_i^{\text{def}} = R_{ji} du_j + (R_{ji} - \delta_{ji}) dX_j \quad (10)$$

By combining eqs. (1), (3), (4), and (5), we also obtain the following relations:

$$du_i = du_i^{\text{rig}} + du_i^{\text{def}} \quad (11)$$

where

$$du_i^{\text{rig}} = (R_{ij} - \delta_{ij}) dX_j \quad (12)$$

$$du_i^{\text{def}} = R_{ij} d\hat{u}_j^{\text{def}} \quad (13)$$

and the superscripts rig and def represent rigid body and deformation motions, respectively.

The symmetric corotational strain in the corotational coordinates is defined by the following linear equation

$$\hat{\epsilon}_{ij} = \hat{\epsilon}_{ji} = \frac{1}{2} \left(\frac{\partial \hat{u}_i^{\text{def}}}{\partial X_j} + \frac{\partial \hat{u}_j^{\text{def}}}{\partial X_i} \right) \quad (14)$$

By substituting eq. (9) into eq. (14), we obtain the corotational strain as

$$\hat{\epsilon}_{ij} = \alpha_{ij} \quad (15)$$

The relationship between the corotational strain and the Langrangian strain can be obtained by first noting that the relationship between the Langrangian strain and the stretch [1] is

$$U_{ij} = (\delta_{ij} + 2 E_{ij})^{1/2} \quad (16)$$

where the Langrangian strain E_{ij} is given by

$$E_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i} \frac{\partial u_k}{\partial X_j} \right) \quad (17)$$

A combination of eqs. (5), (15), and (16) yields the relationship between the Langrangian strain and the engineering strain to be

$$E_{ij} = \alpha_{ij} + \frac{1}{2} \alpha_{ik} \alpha_{kj} \quad (18)$$

A comparison between eqs. (15) and (18) shows that $\hat{\epsilon}_{ij}$ and E_{ij} are the same in the first order term in α_{ij} , regardless of the magnitude of

the rigid body motion. In the particular case when α_{ij} is small compared to unity, the two strains are the same numerically. This is a great advantage of the corotational coordinate formulation since the linear strain expression is valid no matter how large the displacement may be.

The above analysis is applicable between any two neighboring points in the continuum. If the rotation is assumed constant between these two neighboring points then we can integrate eq. (10) explicitly, and it follows

$$\hat{u}_i^{def} = R_{ji}(u_j - u_j^{tra}) + (R_{ji} - \delta_{ji})X_j \quad (19)$$

where u_j^{tra} are the integration constants, which can be identified to be the rigid body translations of these two points.

Equation (19) shows how to separate the corotational deformation displacements from the global spatial displacements once the rigid body rotations and translations are known. This is the only relation which was derived under the constrain of constant rotation. Since no assumptions of smallness were made, all the equations in this section are valid for large displacements and strains.

KINETICS

The exact equations of motion of a continuum are generally very difficult to solve. Hence approximate equations obtained from the principle of virtual work and the d'Alembert's principle are employed to solve practical problems. In order to clarify the principle of virtual work in the corotational coordinates, we first present this principle in fixed Cartesian coordinates.

Let the variations of internal and external work of a continuum due to small variations of the displacements from its equilibrium configuration be defined as

$$\delta W^{\text{ext}} = \int_{S^0} (t_i^0 \delta u_i) ds + \int_{V^0} (b_i^0 \delta u_i) dv; \quad i = 1, \text{ number of dimensions} \quad (20)$$

$$\delta W^{\text{int}} = \int_{V^0} (S_{ij} \delta E_{ij}) dv; \quad i, j = 1, \text{ number of dimensions} \quad (21)$$

where S^0 and V^0 are the undeformed boundary surface and volume of the continuum; t_i^0 and b_i^0 are the surface traction and body force per undeformed unit surface and volume, respectively; and S_{ij} is the symmetric Piola-Kirchhoff stress tensor.

The principle of virtual work [6] states that if

$$\delta W^{\text{ext}} = \delta W^{\text{int}} \quad (22)$$

for any arbitrary infinitesimal admissible virtual displacements, δu_i , then the equations of equilibrium and the boundary conditions of the continuum are satisfied.

To study the principle of virtual work in the corotational variables, we first define the work done internally in the continuum as

$$\delta \hat{W}^{int} = \int_{V_0} (\hat{\sigma}_{ij} \delta \hat{\epsilon}_{ij}) dv \quad (23)$$

where $\hat{\epsilon}_{ij}$ is the corotational strain defined previously. The corotational stress $\hat{\sigma}_{ij}$ remains to be defined. We require σ_{ij} to be symmetric since any antisymmetric part would not contribute to the internal work because of the symmetry of the corotational strains.

The principle of virtual work in the corotational variables can now be stated as follows: The equilibrium and boundary conditions of a continuum are satisfied, if

$$\delta W^{ext} = \delta \hat{W}^{int} \quad (24)$$

are satisfied for any arbitrary admissible virtual displacements. A necessary condition for the above statement to be true is obtained by combining eqs. (22) and (24) to be

$$\delta \hat{W}^{int} = \delta W^{int} \quad (25)$$

The relationship between the corotational stress tensor and the Piola-Kirchhoff stress tensor in terms of α_{ij} is obtained by substituting eqs. (15) and (18) into eq. (25) and evoking the arbitrariness of α_{ij} as

$$\hat{\sigma}_{ij} = \frac{1}{2} [S_{ik} (\delta_{kj} + \alpha_{kj}) + S_{jk} (\delta_{ki} + \alpha_{ki})] \quad (26)$$

To obtain the relation between the corotational stress tensor and the Cauchy's stress tensor we first note the relationship between the Piola-Kirchhoff stress and Eulerian stress is [6]

$$S_{ij} = J[(\delta_{ik} + \alpha_{ik})^{-1} \beta_{kl} (\delta_{jl} + \alpha_{jl})^{-1}] \quad (27)$$

where the tensor β_{ij} is the Cauchy's stress tensor referred to the corotational coordinates, and J is the Jacobian of the deformation gradient G_{ij} . They are given by

$$\beta_{ij} = \beta_{ji} = R_{ki} \sigma_{kl} R_{lj} \quad (28)$$

$$J = 1 + \alpha_{ii} + \frac{1}{2}(\alpha_{ii}\alpha_{jj} - \alpha_{ij}\alpha_{ji}) + \frac{1}{6}e_{ijk}e_{rst}\alpha_{ir}\alpha_{js}\alpha_{kt} \quad (29)$$

where e_{ijk} is the permutation tensor. Substituting eq. (27) into eq. (26), we obtain the relationship between the corotational stress tensor and the Cauchy's stress tensor

$$\hat{\sigma}_{ij} = \frac{J}{2}[(\delta_{ik} + \alpha_{ik})^{-1} \beta_{kj} + \beta_{ik} (\delta_{kj} + \alpha_{kj})^{-1}] \quad (30)$$

Equations (26) and (30) can be physically interpreted as follows: The corotational stress tensor component $\hat{\sigma}_{ij}$ has the same numerical value as the Piola-Kirchhoff stress tensor component which has been "stretched" by the stretch tensor U_{ij} ; or $\hat{\sigma}_{ij}$ has the same value as the component of the Cauchy's stress tensor that is "unstretched" by U_{ij}^{-1} and is referred to the original surface in the corotational coordinates. In the particular case when the stretch is small, the corotational stress tensor components have the same values as the components of the Piola-Kirchhoff stress tensor or the same values as the components of the Cauchy's stress tensor that are referred to a coordinate system that rotates with the continuum.

In summary, the equilibrium and boundary conditions are satisfied provided

$$\int_{S^0} (t_1^0 \delta u_1) ds + \int_{V^0} (b_1^0 \delta u_1) dv = \int_{V^0} (\hat{\sigma}_{ij} \delta \hat{\epsilon}_{ij}) dv \quad (31)$$

is satisfied for arbitrary admissible virtual displacements δu_1 . The dynamic counterpart of eq. (31) is obtained by introducing the inertial forces as body forces as

$$\int_{S^0} (t_1^0 \delta u_1) ds + \int_{V^0} [(b_1^0 - \rho^0 \ddot{u}_1) \delta u_1] dv = \int_{V^0} (\hat{\sigma}_{ij} \delta \epsilon_{ij}) dv \quad (32)$$

where ρ^0 is the density at the undeformed state and the acceleration $\ddot{u}_1$ is given by

$$\ddot{u}_1 = \frac{\partial^2 u_1}{\partial t^2} \quad (33)$$

CONSTITUTIVE EQUATIONS

The constitutive equations in the corotational system are the expressions that display the corotational stress $\hat{\sigma}_{ij}$ as a function of corotational strain $\hat{\epsilon}_{ij}$. Most experimental one dimensional test results show either the nominal stress S_{11} or the true stress σ_{11} as a function of the engineering strain α_{11} . Both the stress and strain are measured referring to a coordinate system whose 1-axis lies along the specimen. This choice of coordinate system is the same as the corotational coordinates we have defined. Hence, the one dimensional corotational constitutive equation can be obtained either directly from test data or after some modification by using eqs. (15), (18), (26), and (30). For isotropic materials, the three dimensional constitutive equations are obtained by generalizing the one dimensional test results and imposing the isotropic characteristics on the constitutive laws. We limit our attention only to elastic and elastoplastic materials in this section.

To define the elasticity and the elastoplasticity in terms of the corotational variables, we first show the relationship between the Jaumann stress rate σ_{ij}^V and the corotational stress rate $\hat{\sigma}_{ij}^*$, the rate of deformation V_{ij} and the corotational strain rate $\hat{\epsilon}_{ij}^*$. The rate of deformation V_{ij} is given by [6]

$$V_{ij} = \frac{1}{2} \left(\frac{\partial u_i^*}{\partial x_j} + \frac{\partial u_j^*}{\partial x_i} \right) ; \quad i, j = 1, \text{ number of dimensions} \quad (34)$$

where u_i^* is the velocity. The Jaumann stress rate σ_{ij}^V is defined by [6]

$$\sigma_{ij}^{\nabla} = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} [\sigma'_{ij}(t + \Delta t) - \sigma'_{ij}(t)]; i, j = 1, \text{ number of dimensions} \quad (35)$$

where the superscript prime on σ'_{ij} indicates the quantity is referred to a coordinate system that rotates with the continuum.

This rotated coordinates are exactly the corotational coordinates we defined. Since the corotational stress rate is referred to the corotational coordinates while the Jaumann stress rate is referred to the spatial coordinates, we, by using eqs. (7) and (28), obtain

$$\hat{\sigma}_{ij}^{\bullet} = \frac{\partial \hat{\sigma}_{ij}}{\partial t} = R_{pi} \sigma_{pq}^{\nabla} R_{qj} \quad (36)$$

The relationship between the rate of deformation and the corotational strain rate is obtained by combining eqs. (34), (11), (14), (15), and (5) to be

$$\hat{v}_{ij} = \frac{1}{2} [\hat{\epsilon}_{ik}^{\bullet} (\delta_{kj} + \hat{\epsilon}_{kj})^{-1} + \hat{\epsilon}_{jk}^{\bullet} (\delta_{ki} + \hat{\epsilon}_{ki})^{-1}] \quad (37)$$

where $\hat{v}_{ij}$ are the components of the rate of deformation tensor referred to the corotational coordinates and $\epsilon_{ij}^{\bullet}$ is the corotational strain rate. They are given by

$$\hat{v}_{ij} = R_{pi} v_{pq} R_{qj} \quad (38)$$

and

$$\hat{\epsilon}_{ij}^{\bullet} = \frac{\partial \hat{\epsilon}_{ij}}{\partial t} \quad (39)$$

Equation (36) shows that the corotational stress rate has the same value as that of the Jaumann stress rate which is referred to the corotational coordinates regardless of how large the displacements or strains may be. Equation (37) indicates that the components of the rate of

deformation in the corotational coordinates have the same values as the components of the corotational strain rate which have been "unstretched" by the amount U_{ij}^{-1} regardless of the motion of the continuum. In the particular case when the displacements may be large, but the intrinsic deformation is small, that is α_{ij} or $\hat{\epsilon}_{ij}$ is much smaller than unity, the corotational strain rate tensor has the same value as that of the rate of deformation which is referred to the corotational coordinates.

We now define the constitutive equations in the corotational coordinate formulation. The materials considered are separated into two parts: (A) elastic and (B) elastoplastic.

(A) Elasticity: A material is defined as elastic if its constitutive equation is given by

$$\hat{\sigma}_{ij}^{\cdot} = C_{ijkl} \hat{v}_{kl}; \quad i, j, k, l = 1, \text{ number of dimensions} \quad (40)$$

where C_{ijkl} may depend on the stress. Substituting eqs. (37), (38), and (36) into eq. (40), we see that this definition of elasticity is exactly the same as that of the hypoelasticity in the nonlinear continuum mechanics.

For isotropic and linearly elastic materials, the constitutive equation can be simplified to

$$\hat{\sigma}_{ij}^{\cdot} = \lambda \delta_{ij} \hat{v}_{kk} + 2G \hat{v}_{ij} \quad (41)$$

where λ and G are constants and are related to the Poisson's ratio and Young's modulus, which can be obtained by performing one dimensional tests.

(B) Elastoplasticity: The elastoplastic constitutive equations in terms of the corotational variables are developed by following very closely the classic elastoplastic theory used for small displacements problems.

The corotational rate of deformation is assumed to be composed of two additive parts: elastic reversible corotational rate of deformation $\hat{V}_{ij}^e$ and plastic irreversible corotational rate of deformation $\hat{V}_{ij}^p$.

A yield function $Y(\hat{\sigma}_{ij}, \hat{V}_{ij}^p, \kappa)$ is defined so that the plastic deformation can occur only when $Y = 0$ and $\dot{Y} \geq 0$, where the super-script dot denotes time derivative and κ is the work hardening parameter. The yield function may also be written as

$$f(\hat{\sigma}_{ij}) = \kappa = F(W_p) \quad (42)$$

where W_p is the plastic work and its time derivative is defined by

$$\dot{W}_p = \hat{\sigma}_{ij} \hat{V}_{ij}^p \quad (43)$$

Assuming that the plastic corotational rate of deformation is incompressible and that the plastic corotational rate of deformation is always normal to the yield surface, we can show [1] that

$$\hat{\sigma}_{ij}^* = C_{ijkl} [\delta_{pk} \delta_{ql} - \frac{g_{kl} g_{mn} C_{pqmn}}{g_{rs} C_{rsuv} g_{uv} + m \hat{s}_{rs} g_{rs}}] \hat{V}_{pq} \quad (44)$$

when plastic flow occurs, where the gradient of the yield function, g_{ij} , the corotational stress deviation tensor $\hat{s}_{ij}$ and the parameter m are given by

$$g_{ij} = \frac{\partial f}{\partial \hat{\sigma}_{ij}} \quad ; \quad i, j = 1, \text{ number of dimensions} \quad (45)$$

$$\hat{s}_{ij} = \hat{\sigma}_{ij} - \frac{1}{3} \hat{\sigma}_{kk} \delta_{ij} ; \quad i, j = 1, \text{ number of dimensions} \quad (46)$$

$$m = \frac{\partial F}{\partial W_p} \quad (47)$$

For a linearly elastic-plastic material that follows the Hooke's law and Von Mises' yield criterion, the corotational constitutive equations can be summarized as follows:

$$\hat{\sigma}_{ij} = \lambda \delta_{ij} (\hat{v}_{kk}^e) + 2G (\hat{v}_{ij}^e) \quad (48)$$

where

$$\hat{v}_{ij}^e = \hat{v}_{ij} ; \text{ if } f(\hat{\sigma}_{ij}) < \kappa \quad (49)$$

$$\hat{v}_{ij}^e = [\delta_{ip} \delta_{jq} - \frac{\hat{s}_{ij} \hat{s}_{pq}}{\hat{s}_{nn} \hat{s}_{nn} (1 + m/2G)}] \hat{v}_{pq} ; \text{ if } f(\hat{\sigma}_{ij}) = 0 \text{ and } f^*(\hat{\sigma}_{ij}) \geq 0 \quad (50)$$

The value of m can be computed from the one dimensional stress-strain relation. For a bilinear material, it is found to be

$$m = \frac{2E E_p}{3(E + E_p)} \quad (51)$$

where E and E_p are the elastic and plastic moduli, respectively.

Though the equations in this section have similar forms as the constitutive equations used in the classic small strain, small displacement problems, they can be applied to problems where large displacements and large strains are involved because of the use of the corotational stress rate and rate of deformation.

FINITE ELEMENT APPLICATION

A continuum is divided into a finite number of subregions called elements appropriately connected at nodes [8]. A corotational coordinate system is allocated to each element, which rotates and translates with the element. The essence of this finite element formulation is the assumption of constant or nearly constant rotation in each element.

The corotational deformation displacements inside each element are assumed to be a known function of its nodal displacements. That is

$$\hat{u}_i^{\text{def}} = \hat{\phi}_{ij} \hat{d}_j^{\text{def}}; \quad i = 1, 3; \quad j = 1, \text{ number of degrees of freedom per element} \quad (52)$$

where $\hat{\phi}_{ij}$ is called the deformation shape function and $\hat{d}_j^{\text{def}}$ is the generalized nodal deformation displacements referred to the corotational coordinates.

The corotational nodal deformation displacements can be related to the global nodal displacements by using eq. (19) as

$$\hat{d}_i^{\text{def}} = \hat{\psi}_{ij} d_j; \quad i = 1, \quad j = 1, \text{ number of degrees of freedom per element} \quad (53)$$

where the elements of the tensor $\hat{\psi}_{ij}$ can be computed by inserting appropriate nodal Lagrangian coordinates into eq. (19). By using eqs. (7), (19), (52), and (53), the global displacements are obtained as

$$u_i = \phi_{ij} d_j; \quad i = 1, \quad j = 1, \text{ number of degrees of freedom per element} \quad (54)$$

where the global shape functions ϕ_{ij} are given by

$$\phi_{ij} = R_{ik} \hat{\phi}_{k\ell} \hat{\psi}_{\ell j} + \phi'_{ij} \quad (55)$$

$$\phi'_{ij} d_j = c_1 + (R_{ij} - \delta_{ij}) X_j \text{ when } i = j \quad (56)$$

$$\phi'_{ij} = 0 \text{ when } i \neq j \quad (57)$$

and X_1 is the Langrangian coordinate of any point in the element and c_1 is the rigid body translation of the element.

It is relevant to point out that the global displacement function defined in eq. (54) can represent any rigid body motion under appropriate nodal displacements regardless of the magnitude of displacements and strains provided the rotations in the element is constant and the deformation shape function can adequately represent the deformation of the element.

Substituting eq. (53) into eq. (20), we find

$$\delta W^{\text{ext}} = f_1^{\text{ext}} \delta d_1 \quad (58)$$

where the generalized external nodal forces for the element, f_1^{ext} , is given by

$$f_1^{\text{ext}} = \int_{S^0} \phi_{ij} t_j^0 ds + \int_{V^0} \phi_{ij} b_j^0 dv \quad (59)$$

Substituting eqs. (52) and (14) into eq. (23), we obtain

$$\delta W^{\text{int}} = \hat{f}_1^{\text{int}} \hat{\delta d}_1^{\text{def}} \quad (60)$$

where the generalized corotational internal nodal forces for the element referred to its corotational coordinates is given by

$$\hat{f}_i^{int} = \frac{1}{2} \int_{V^0} \hat{\sigma}_{jk} [\hat{\phi}_{ji,k} + \hat{\phi}_{ki,j}] dv \quad (61)$$

Recalling that the work done by the internal nodal forces under rigid body motion is zero and the frame indifference of work and using eq. (11), we can write

$$\delta W^{int} = f_i^{int} \delta d_i \quad (62)$$

where the generalized internal forces f_i^{int} are referred to the spatial coordinates and is given by

$$f_i^{int} = T_{ij} \hat{f}_j^{int}; \quad i, j = 1, \text{ number of degree of freedom element} \quad (63)$$

and T_{ij} is the transformation tensor that maps the internal nodal forces from the corotational coordinates to the spatial coordinates.

Substituting eqs. (58) and (62) into eq. (32), we obtain the following equation for each element

$$m_{ij} \ddot{d}_j \delta d_i = f_i^{ext} \delta d_i - f_i^{int} \delta d_i \quad (64)$$

where $\ddot{d}_i$ are the element nodal accelerations and the element mass tensor m_{ij} is given by

$$m_{ij} = \int_{V^0} (\rho^0 \phi_{ki} \phi_{kj}) dv \quad (65)$$

To obtain the equations of motion for the continuum, we first define a Boolean connectivity operator [7] that relates the element nodal displacements and the nodal displacements of the continuum by

$$d_i = I^{(L_{ij})} D_j; \quad i, j = 1, \text{ number of degrees of freedom of the element and continuum, respectively} \quad (67)$$

where the pre-subscript I on L_{ij} indicates the association with element

I. Substituting eq. (67) into eq. (64), we obtain

$$I^{(L_{ji})}(m_{jk}) I^{(L_{kl})} \ddot{D}_k \delta D_\ell = I^{(L_{lj})} [f_j^{\text{ext}} - f_j^{\text{int}}] \delta D_\ell ; \text{ no sum on } I \quad (68)$$

Equation (68) is the relationship of variations of internal work, external work and kinetic energy for each element. Assuming the variations of work and energy of the continuum is the sum of those of the elements and evoking the arbitrariness of δD_ℓ , we obtain the equations of motion for the continuum as

$$M_{ij} \ddot{D}_j = F_i^{\text{ext}} - F_i^{\text{int}} \quad (69)$$

where the global mass M_{ij} , the global generalized forces F_i^{ext} and F_i^{int} are given by

$$M_{ij} = \sum_I^N I^{(L_{ki})}(m_{kl}) I^{(L_{lj})} \quad (70)$$

$$F_i^{\text{ext}} = \sum_I^N I^{(L_{ji})} f_j^{\text{ext}} \quad (71)$$

$$F_i^{\text{int}} = \sum_I^N I^{(L_{ji})} f_j^{\text{int}} \quad (72)$$

It is important to note that the vital step which enables us to derive the final equations of motion in terms of global nodal displacements is the fact that no work is done by the internal forces under rigid body motion, which is used in deriving eq. (62) from (60) for each element. This statement is equivalent to saying that the internal forces for each element are always in equilibrium. Hence, it is significant that the self-equilibrium condition of internal forces for each element is satisfied in applying this method of corotational coordinates.

CONCLUSIONS

The advantages of the corotational finite element formulations are manifold. They are summarized as follows:

(1) The corotational shape functions can be assumed for the corotational deformation displacements in the corotational coordinates. Hence the same shape functions used for small displacement problems can be used to treat large displacement problems. In other words, the corotational formulation yields better results compared to conventional formulation when the same simple shape functions are used particularly when the corotational strains are small.

(2) The corotational strains are always linear in terms of the corotational deformation displacements. Hence, the nonlinearities in the conventional strain-displacement relation is completely eliminated regardless of the magnitudes of the displacements and strains.

(3) Due to the use of the corotational rate of deformation and the corotational stress rate rather than the conventional strain and stress as the independent and dependent variables in the corotational constitutive equations, the effects of rigid body motion on the stress and strain rates are completely absent. This makes the constitutive equations very simple for large displacement problems. An elastic and an elastoplastic constitutive equations that have the same functional forms as those of the classic elastic and elastoplastic constitutive equations for small displacement problems are extended to large displacement problems as examples. Other types of constitutive equations can also be derived by the same method.

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