DYNAMICS OF TWO COAXIAL CYLINDRICAL SHELLS
CONTAINING VISCOUS FLUID

by

Tsyh-tyan Yeh and Shoei-sheng Chen

COMPONENTS TECHNOLOGY DIVISION

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ABSTRACT

This study was motivated by the need to design the thermal shield in reactor internals and other system components to avoid detrimental flow-induced vibrations. The system component is modeled as two coaxial shells separated by a viscous fluid. In the analysis, Flügge's shell equations of motion and linearized Navier-Stokes equation for viscous fluid are employed. First, a traveling-wave type solution is taken for shells and fluid. Then, from the interface conditions between the shells and fluid, the solution for the fluid medium is expressed in terms of shell displacements. Finally, using the shell equations of motion gives the frequency equation, from which the natural frequency, mode shape, and modal damping ratio of coupled modes can be calculated. The analytical results show a fairly good qualitative agreement with the published experimental data. Some important conclusions are as follows: (1) In computing the natural frequencies and mode shapes of uncoupled modes and coupled modes, the fluid may be considered inviscid and incompressible. (2) There exists out-of-phase and in-phase modes. The lowest natural frequency is always associated with the out-of-phase mode. (3) The lowest natural frequency of coupled modes is lower than the uncoupled modes. (4) The fluid viscosity contributes significantly to damping, in particular, the modal damping of the out-of-phase modes is relatively large for small gaps. (5) If the fluid gap is small, or the fluid viscosity is relatively high, the simulation of the vibration Reynolds number should be included to ensure that modal damping of the model is properly accounted for. With the presented analysis and results, the frequency and damping characteristics can be analyzed and design parameters can be related to frequency and damping.

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NOMENCLATURE

A_1, A_2, A_3	Constants
a_{pq}	Matrix defined in Eq. (17)
α_{pq}	Stress coefficient matrix defined by Eq. (21)
B_1, B_2, B_3	Constants
b_{pq}	Matrix defined by Eq. (23)
C_o	Speed of sound in fluid
C_{pq}	Matrix associated with shell property defined by Eq. (24)
c_f, c_s	Fluid and structure damping coefficients
E_i	Young's modulus
F_n	n-th order Bessel functions
f, f_R, f_I	Frequency, $f = f_R + if_I$
f_a, f_w	Frequencies in air and in water
f_N	Natural frequency
G_n	n-th order Bessel function
$H_n^{(1)}, H_n^{(2)}$	Hankel function
h_i	Shell wall thickness
h_{pq}	Matrix defined in Eq. (19)
$\text{Im}(\cdot)$	Imaginary part of $(\cdot)$
i	$\sqrt{-1}$
J_n	First kind Bessel function
k	Axial wave number
k_s	Structure stiffness
k_1	$-k^2 + \frac{\omega^2/C_o^2}{1 + i \frac{\omega}{\omega_o}}$
k_2	$-k^2 - i\omega/v_o$
M_{oi}	$\frac{\omega r_i^2}{C_o}$

NOMENCLATURE (Contd.)

m_a, m_s	Added and shell mass per unit shell surface area
n	Circumferential wave number
P, P_o	Total pressure and reference pressure
$P_{\theta i}, P_{z i}, P_{r i}$	Shell surface loading stresses due to fluid
R_i	Mean radius of shell
$\text{Re}(\cdot)$	Real part of $(\cdot)$
Re_i	$\frac{\omega r_i^2}{\nu_o}$, Reynolds number
r	Radial coordinate
r_i	Radius of interface
t	Time
$\tau_{rr}, \tau_{r\theta}, \tau_{rz}$	Fluid stresses
$u_i, \bar{u}_i$	Axial displacement of shell
$\vec{V}$	Fluid velocity, (V_r, V_θ, V_z)
$v_i, \bar{v}_i$	Circumferential displacement of shell
$w_i, \bar{w}_i$	Radial displacement of shell
ω	Circular frequency, $2\pi f$
ω_o	$C_o^2 / (\frac{4}{3} \nu_o + \nu_o')$
Y_n	Second kind Bessel function
z	Axial coordinate
α_i	$k r_i$
δ_i	h_i / R_i
θ	Angular coordinate
μ_i	$\frac{\rho_o r_i}{\rho_i h_i}$
ν_i	Poisson's ratio
ν_o, ν_o'	Kinematic and secondary viscosity of the fluid
ρ_i	Shell density
ρ_o, ρ	Fluid density

NOMENCLATURE (Contd.)

Ω_i	$R_i \omega \left[\frac{\rho_i (1 - \nu_i^2)}{E_i} \right]^{1/2}$
$\vec{\psi}$	$(\psi_z, \psi_\theta, \psi_r)$, vector potential of the velocity field
$\phi, \bar{\phi}$	Scalar potential of the velocity field
ζ	Modal damping ratio associated with fluid viscosity
ζ_s	Structure damping ratio in vacuo
ζ_t	$\zeta_s + \zeta$, total damping ratio
δ_{pq}	$\delta_{pq} = 1$ if $p = q$, otherwise $\delta_{pq} = 0$

I. INTRODUCTION

In this report, the dynamics of a system of two concentric cylindrical shells coupled by a viscous fluid is studied analytically. The objective is to investigate the effect of fluid viscosity on fluid structure interactions.

This study was motivated by the need to design reactor system components to avoid detrimental flow-induced vibrations. Several reactor system components consist of nominally circular cylindrical shells coupled to other shells through a fluid. Examples include shrouds and thermal liners. Those components are subject to various excitation sources including fluid flow and structural borne disturbance. To design a component such that over the design life its performance will not be affected by vibrations, one must understand the system characteristics.

Vibrations of two coaxial shells separated by a fluid have been studied extensively [1-4]. However, all those investigations omit the effect of fluid viscosity. Although for many practical applications, the viscosity is small and the fluid may be considered inviscid as a first approximation, near the interface of the structure and fluid there exists a thin layer of rotational flow. This flow regime in which the viscous effect is significant is of great concern to the dynamic response of coupled shell systems. In particular, when the annular gap is small, the fluid viscous effect becomes more pronounced.

In this study, Flügge's shell equations of motion and the linearized Navier-Stokes equation for fluid are employed. First, a traveling-wave type solution is taken for shells and fluid. Then, from the interface conditions between the shells and fluid, the solution for fluid medium

is expressed in terms of shell displacements. Finally, using the shell equations of motion gives the frequency equation from which the natural frequency, mode shape, and the modal damping ratio can be calculated.

The analytical results are also compared with an experimental investigation for two concentric shells coupled by water.

II. GOVERNING EQUATIONS OF MOTION

Consider two concentric circular cylindrical shells separated by a viscous fluid annulus as shown in Fig. 1. The motion of the shells is described by the following Flügge's shell equation [5]:

$$\begin{aligned}
 & \left[\frac{\partial^2}{\partial z^2} + \left(\frac{1 - \nu_i}{2R_i^2} \right) \left(1 + \frac{h_i^2}{12 R_i^2} \right) \frac{\partial^2}{\partial \theta^2} \right] u_i + \frac{1 + \nu_i}{2 R_i} \frac{\partial^2 v_i}{\partial z \partial \theta} \\
 & + \left[\frac{\nu_i}{R_i} \frac{\partial}{\partial z} - \frac{h_i^2}{12 R_i} \frac{\partial^3}{\partial z^3} + \frac{(1 - \nu_i) h_i^2}{24 R_i^3} \frac{\partial^3}{\partial z \partial \theta^2} \right] w_i \\
 & = \frac{\rho_i (1 - \nu_i^2)}{E_i} \frac{\partial^2 u_i}{\partial t^2} - \frac{1 - \nu_i}{E_i h_i} p_{zi} ; \\
 & \frac{1 + \nu_i}{2 R_i} \frac{\partial^2 u_i}{\partial z \partial \theta} + \left[\frac{1}{R_i^2} \frac{\partial^2}{\partial \theta^2} + \frac{1 - \nu_i}{2} \left(1 + \frac{h_i^2}{4 R_i^2} \right) \frac{\partial^2}{\partial z^2} \right] v_i \\
 & + \left[\frac{1}{R_i^2} \frac{\partial}{\partial \theta} - \frac{(3 - \nu_i) h_i^2}{24 R_i^2} \frac{\partial^3}{\partial \theta \partial z^2} \right] w_i = \frac{\rho_i (1 - \nu_i)^2}{E_i} \frac{\partial^2 v_i}{\partial t^2} \\
 & - \frac{1 - \nu_i^2}{E_i h_i} p_{\theta i} ; \\
 & \left[- \frac{h_i^2}{12 R_i} \frac{\partial^3}{\partial z^3} + \frac{\nu_i}{R_i} \frac{\partial}{\partial z} + \frac{(1 - \nu_i) h_i^2}{24 R_i^3} \frac{\partial^3}{\partial z \partial \theta^2} \right] u_i \\
 & + \left[- \frac{(3 - \nu_i) h_i^2}{24 R_i^2} \frac{\partial^3}{\partial \theta \partial z^2} + \frac{1}{R_i^2} \frac{\partial}{\partial \theta} \right] v_i
 \end{aligned} \tag{1}$$

$$+ \left[\frac{1}{R_i^2} + \frac{h_i^2}{12 R_i^4} + \frac{h_i^2}{12} \frac{\partial^4}{\partial z^4} + \frac{h_i^2}{6 R_i^2} \frac{\partial^4}{\partial z^2 \partial \theta^2} + \frac{h_i^2}{12 R_i^4} \frac{\partial^4}{\partial \theta^4} + \frac{h_i^2}{6 R_i^4} \frac{\partial^2}{\partial \theta^2} \right] w_i \quad (1)$$

$$= - \frac{\rho_i (1 - \nu_i^2)}{E_i} \frac{\partial^2 w_i}{\partial t^2} + \frac{1 - \nu_i^2}{E_i h_i} P_{ri} ; \quad (\text{Contd.})$$

where the index i denotes the variables associated with the inner shell ($i = 1$) and outer shell ($i = 2$); u_i , v_i , w_i are the displacement components of the shell middle surfaces; z , θ , and r are cylindrical coordinates; P_{zi} , $P_{\theta i}$, and P_{ri} are the surface loading components per unit area, and t is the time. The physical characteristics of shells are defined by the mean radius R_i , wall thickness h_i , mass density ρ_i , Young's modulus E_i and Poisson's ratio ν_i .

For a non-steady, small-amplitude oscillatory motion, the equations of motion for the contained viscous fluid can be expressed as follows [6]:

$$\left. \begin{aligned} \frac{\partial p}{\partial t} + \rho_0 \nabla \cdot \vec{V} &= 0, \\ \frac{\partial \vec{V}}{\partial t} &= -\frac{1}{\rho_0} \nabla p + \left(\frac{4}{3} v_0 + v_0'\right) \nabla (\nabla \cdot \vec{V}) - v_0 \nabla \times \nabla \times \vec{V}, \\ \frac{\partial p}{\partial \rho} &= c_0^2, \end{aligned} \right\} \quad (2)$$

where ρ_0 and ρ are the mean and instantaneous fluid mass densities, ν_0 and ν' are the kinematic and second viscosities of the fluid, C_0 is the speed of sound in fluid, p is fluid pressure and $\vec{V}$ is fluid velocity vector.

Fig. 1. Schematic of two concentric cylindrical shells containing a viscous fluid.

At the interfaces between the shells and fluid, the following conditions must be satisfied:

$$V_z \Big|_{r=r_i} = \frac{\partial u_i}{\partial t}; \quad V_\theta \Big|_{r=r_i} = \frac{\partial v_i}{\partial t}; \quad \text{and} \quad V_r \Big|_{r=r_i} = \frac{\partial w_i}{\partial t} \quad \text{with } i = 1 \text{ and } 2. \quad (3)$$

The surface loading acting on the shells is given by

$$P_{\ell 1} = \tau_{r\ell} \Big|_{r=r_1} \quad \text{and} \quad P_{\ell 2} = -\tau_{r\ell} \Big|_{r=r_2} \quad \text{with } \ell = z, \theta, \text{ and } r, \quad (4)$$

where $r_1 = R_1 + h_1/2$ and $r_2 = R_2 - h_2/2$ are the interface radii, and

τ_{rr} , $\tau_{r\theta}$, and τ_{rz} are the fluid stresses:

$$\begin{aligned} \tau_{rr} &= -p + 2\mu \frac{\partial V_r}{\partial r}, \\ \tau_{r\theta} &= \mu \left[r \frac{\partial}{\partial r} \left(\frac{V_\theta}{r} \right) + \frac{1}{r} \frac{\partial V_r}{\partial \theta} \right], \\ \tau_{rz} &= \mu \left(\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r} \right). \end{aligned} \quad (5)$$

Here μ is fluid viscosity.

Equations (1) through (5) are the complete mathematical statement of the coupled viscous fluid/shell system.

III. FLUID STRESS COEFFICIENT MATRIX

A. Analysis

Letting

$$\vec{V} = \nabla \times \vec{\psi} + \nabla \phi \quad (6)$$

and inserting it into equations (2) yields

$$\left. \begin{aligned} \frac{\partial \nabla \phi}{\partial t} &= \frac{1}{\rho_o} \nabla p + \left(\frac{4 v_o}{3} + v_o' \right) \nabla \nabla^2 \phi, \\ \frac{\partial \nabla \times \vec{\psi}}{\partial t} &= -v_o \nabla \times \nabla \times \nabla \times \vec{\psi} \end{aligned} \right\} \quad (7)$$

Integration of the first equation of Eq. (7) gives

$$p = p_o - \rho_o \frac{\partial \phi}{\partial t} + \rho_o \left(\frac{4 v_o}{3} + v_o' \right) \nabla^2 \phi. \quad (8)$$

From equations (2) and (8), we have

$$\left[\left(1 + \frac{1}{\omega_o} \frac{\partial}{\partial t} \right) \nabla^2 - \frac{1}{c_o^2} \frac{\partial^2}{\partial t^2} \right] \phi = 0, \quad (9)$$

where

$$\omega_o = c_o^2 / \left(\frac{4}{3} v_o + v_o' \right).$$

Equation (8) shows that the fluid pressure is not affected by the waves produced from the vector potential, $\vec{\psi}$, which is associated with the fluid viscosity. Since the trivial solution ($\nabla \times \vec{\psi} = 0$) is to be excluded, equation (7) can further be reduced to

$$\left(\frac{\partial}{\partial t} - v_o \nabla^2 \right) \vec{\psi} = 0, \quad (10)$$

or in cylindrical coordinates,

$$\left(\frac{1}{v_0} \frac{\partial}{\partial t} - \nabla^2\right) \psi_z = 0 ,$$

$$\left(\frac{1}{v_0} \frac{\partial}{\partial t} - \nabla^2\right) \psi_\theta + \frac{\psi_\theta}{r^2} - \frac{2}{r^2} \frac{\partial \psi_r}{\partial \theta} = 0, \quad (11)$$

and

$$\left(\frac{1}{v_0} \frac{\partial}{\partial t} - \nabla^2\right) \psi_r + \frac{\psi_r}{r^2} + \frac{2}{r^2} \frac{\partial \psi_\theta}{\partial \theta} = 0 .$$

Solutions of the following form are assumed for the shells:

$$\begin{aligned} u_i &= i \bar{u}_i \cos(n \theta) \exp[i(\omega t - kz)] , \\ v_i &= \bar{v}_i \sin(n \theta) \exp[i(\omega t - kz)] , \\ w_i &= \bar{w}_i \cos(n \theta) \exp[i(\omega t - kz)] , \end{aligned} \quad (12)$$

where $i = \sqrt{-1}$, n is the circumferential wave number, ω is the circular frequency, k is the axial wave number, and $\bar{u}_i$, $\bar{v}_i$ and $\bar{w}_i$ are arbitrary constants to be determined.

Similarly, the fluid velocity potential may be assumed as follows:

$$\begin{aligned} \phi &= \bar{\phi}(r) \cos n \theta \exp[i(\omega t - kz)] , \\ \psi_z &= \bar{\psi}_z(r) \sin n \theta \exp[i(\omega t - kz)] , \\ \psi_\theta &= i \bar{\psi}_\theta(r) \cos n \theta \exp[i(\omega t - kz)] , \\ \psi_r &= i \bar{\psi}_r(r) \sin n \theta \exp[i(\omega t - kz)] . \end{aligned} \quad (13)$$

Substituting Eqs. (13) into (9) and (11) gives the following forms of Bessel's equations:

$$\left. \begin{aligned}
& \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + (k_1^2 - \frac{n^2}{r^2}) \right] \bar{\phi} = 0, \\
& \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + (k_2^2 - \frac{n^2}{r^2}) \right] \bar{\psi}_z = 0, \\
& \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \bar{\psi}_\theta \right) + (k_2^2 - \frac{n^2 + 1}{r^2}) \bar{\psi}_\theta + \frac{2n}{r^2} \bar{\psi}_r = 0, \\
& \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \bar{\psi}_r \right) + (k_2^2 - \frac{n^2 + 1}{r^2}) \bar{\psi}_r + \frac{2n}{r^2} \bar{\psi}_\theta = 0,
\end{aligned} \right\} \quad (14)$$

where

$$k_1 = \left(-k^2 + \frac{\omega^2/c_o^2}{1 + i \frac{\omega}{\omega_o}} \right)^{1/2},$$

$$k_2 = \left(-k^2 - i \frac{\omega}{v_o} \right)^{1/2}.$$

The general solutions of equations (14) are

$$\left. \begin{aligned}
\bar{\phi}(r) &= i \omega [A_1 F_n(k_1 r) + B_1 G_n(k_1 r)] \\
\bar{\psi}_z(r) &= i \omega [A_2 F_n(k_2 r) + B_2 G_n(k_2 r)] \\
\bar{\psi}_1(r) &= \frac{1}{2}(\bar{\psi}_r - \bar{\psi}_\theta) = i \omega [A_3 F_{n+1}(k_2 r) + B_3 G_{n+1}(k_2 r)], \\
\bar{\psi}_2(r) &= \frac{1}{2}(\bar{\psi}_r + \bar{\psi}_\theta) = i \omega [A_4 F_{n-1}(k_2 r) + B_4 G_{n-1}(k_2 r)],
\end{aligned} \right\} \quad (15)$$

where F_n and G_n are the n -th order Bessel functions. They can be either the first and second kind Bessel functions, J_n and Y_n , or the Hankel functions $H_n^{(1)}$ and $H_n^{(2)}$. The selection of the functions mainly depends on the computational consideration.

It may be shown that any one of the three potentials, $\bar{\psi}_z$, $\bar{\psi}_1$, and $\bar{\psi}_2$, can be set equal to zero without any loss of generality of the solution.

Thus, let $\bar{\psi}_2 = 0$, and the general solution becomes

$$\begin{aligned}\bar{\phi}(r) &= i \omega [A_1 F_n(k_1 r) + B_1 G_n(k_1 r)] , \\ \bar{\psi}_z(r) &= i \omega [A_2 F_n(k_2 r) + B_2 G_n(k_2 r)] ,\end{aligned}\quad (16)$$

$$\bar{\psi}_r(r) = -\bar{\psi}_\theta(r) = i \omega [A_3 F_{n+1}(k_2 r) + B_3 G_{n+1}(k_2 r)] .$$

Substituting equations (12), (13), and (16) into the interface conditions, Eq. (3), gives six linear, algebraic equations:

$$\{a_{pq}\} \begin{Bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ A_3 \\ B_3 \end{Bmatrix} = \begin{Bmatrix} r_1 \bar{u}_1 \\ r_1 \bar{v}_1 \\ r_1 \bar{w}_1 \\ r_2 \bar{u}_2 \\ r_2 \bar{v}_2 \\ r_2 \bar{w}_2 \end{Bmatrix}, \quad p, q = 1 \text{ to } 6, \quad (17)$$

where

$$\begin{aligned}a_{11} &= -\alpha_1 F_n(\gamma_1), & a_{12} &= -\alpha_1 G_n(\gamma_1), & a_{13} &= a_{14} = 0, \\ a_{15} &= -\beta_1 F_n(\beta_1), & a_{16} &= -\beta_1 G_n(\beta_1), & a_{21} &= -n F_n(\gamma_1), \\ a_{22} &= -n G_n(\gamma_1), & a_{23} &= \beta_1 F_{n+1}(\beta_1) - n F_n(\beta_1), \\ a_{24} &= \beta_1 G_{n+1}(\beta_1) - n G_n(\beta_1), & a_{25} &= a_{35} = \alpha_1 F_{n+1}(\beta_1), \\ a_{26} &= a_{36} = \alpha_1 G_{n+1}(\beta_1), & a_{31} &= n F_n(\gamma_1) - \gamma_1 F_{n+1}(\gamma_1), \\ a_{32} &= n G_n(\gamma_1) - \gamma_1 G_{n+1}(\gamma_1), & a_{33} &= n F_n(\beta_1), \\ a_{34} &= n G_n(\beta_1) \quad \text{and} \quad \alpha_i = k_i r_i, \quad \beta_i = k_2 r_i, \quad \gamma_i = k_1 r_i.\end{aligned}$$

The expression of a_{pq} for $p = 4, 5, 6$ is similar to those for $p = 1, 2, 3$

and can be obtained by replacing α_1 , β_1 , and γ_1 by α_2 , β_2 , and γ_2 .

Now we are in the position to calculate the loading stresses on the shell surfaces. Equation (8) is used to calculate the fluid pressure

$$p = p_o - \frac{i \rho_o \omega \phi}{1 + i \frac{\omega}{\omega_o}} .$$

Here only the dynamic quantity is of interest and thus the reference pressure p_o will not be considered. The equation (5) is used to obtain the fluid stresses. Define the new variables $\bar{P}_{zi}$, $\bar{P}_{\theta i}$, and $\bar{P}_{ri}$ as follows:

$$\begin{aligned} P_{zi} &= i \rho_o \omega^2 \bar{P}_{zi} \cos(n \theta) \exp[i(\omega t - kz)], \\ P_{\theta i} &= \rho_o \omega^2 \bar{P}_{\theta i} \sin(n \theta) \exp[i(\omega t - kz)], \end{aligned} \quad (18)$$

and

$$P_{ri} = \rho_o \omega^2 \bar{P}_{ri} \cos(n \theta) \exp[i(\omega t - kz)] .$$

Substituting equation (18) into (4) and using (13) and (16) yield another six linear, algebraic equations:

$$\left\{ h_{pq} \right\} \begin{pmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ A_3 \\ B_3 \end{pmatrix} = \begin{pmatrix} \bar{P}_{z1} \\ \bar{P}_{\theta 1} \\ \bar{P}_{r1} \\ \bar{P}_{z2} \\ \bar{P}_{\theta 2} \\ \bar{P}_{r2} \end{pmatrix} ; \quad p, q = 1 \text{ to } 6 \quad (19)$$

where

$$\begin{aligned} h_{11} &= 2 S_1 \sigma_1 [F_{n+1}(\gamma_1) - \frac{n}{\gamma_1} F_n(\gamma_1)] , \\ h_{12} &= 2 S_1 \sigma_1 [G_{n+1}(\gamma_1) - \frac{n}{\gamma_1} G_n(\gamma_1)] , \end{aligned}$$

$$h_{13} = -\frac{S_2 \sigma_2^n}{\beta_1} F_n(\beta_1) , \quad h_{14} = -\frac{S_2 \sigma_2^n}{\beta_1} G_n(\beta_1) ,$$

$$h_{15} = -S_2 \{ (\sigma_2^2 - 1) F_{n+1}(\beta_1) + \frac{n}{\beta_1} F_n(\beta_1) \} ,$$

$$h_{16} = -S_2 \{ (\sigma_2^2 - 1) G_{n+1}(\beta_1) + \frac{n}{\beta_1} G_n(\beta_1) \} ,$$

$$h_{21} = \frac{2S_1^n}{\gamma_1} [F_{n+1}(\gamma_1) - \frac{n-1}{\gamma_1} F_n(r_1)] ,$$

$$h_{22} = \frac{2S_1^n}{\gamma_2} [G_{n+1}(\gamma_1) - \frac{n-1}{\gamma_1} G_n(r_1)] ,$$

$$h_{23} = -S_2 \{ [\frac{2n(n-1)}{\beta_1^2} - 1] F_n(\beta_1) + \frac{2}{\beta_1} F_{n+1}(\beta_1) \} ,$$

$$h_{24} = -S_2 \{ [\frac{2n(n-1)}{\beta_1^2} - 1] G_n(\beta_1) + \frac{2}{\beta_1} G_{n+1}(\beta_1) \} ,$$

$$h_{25} = S_2 \sigma_2 [F_n(\beta_1) - \frac{2(n+1)}{\beta_1} F_{n+1}(\beta_1)] ,$$

$$h_{26} = S_2 \sigma_2 [G_n(\beta_1) - \frac{2(n+1)}{\beta_1} G_{n+1}(\beta_1)] ,$$

$$h_{31} = [\frac{2 S_1 n(n+1)}{\gamma_1^2} - (2 S_1 + \frac{1}{1 + i \frac{\omega}{\omega_o}})] F_n(\gamma_1) + \frac{2S_1}{\gamma_1} F_{n+1}(\gamma_1) ,$$

$$h_{32} = [\frac{2 S_1 n(n+1)}{\gamma_1^2} - (2 S_1 + \frac{1}{1 + i \frac{\omega}{\omega_o}})] G_n(\gamma_1) + \frac{2S_1}{\gamma_1} G_{n+1}(\gamma_1) ,$$

$$h_{33} = \frac{2 S_2^n}{\beta_1} [\frac{n-1}{\beta_1} F_n(\beta_1) - F_{n+1}(\beta_1)] ,$$

$$h_{34} = \frac{2 S_2^n}{\beta_1} [\frac{n-1}{\beta_1} G_n(\beta_1) - G_{n+1}(\beta_1)] ,$$

$$h_{35} = 2 S_2 \sigma_2 [F_n(\beta_1) - \frac{n+1}{\beta_1} F_{n+1}(\beta_1)] ,$$

$$h_{36} = 2 S_2 \sigma_2 [G_n(\beta_1) - \frac{n+1}{\beta_1} G_{n+1}(\beta_1)] ,$$

$$\text{and } S_1 = i \frac{\gamma_1^2}{\text{Re}_1} , \quad S_2 = i \frac{\beta_1^2}{\text{Re}_1} , \quad \frac{\omega}{\omega_o} = \frac{\text{Mo}_1^2}{\text{Re}_1} (\frac{4}{3} + \frac{v_o}{v_o}) ,$$

$$\sigma_1 = \alpha_1 / \gamma_1 , \quad \sigma_2 = \alpha_1 / \gamma_2 , \quad \text{Re}_1 = \frac{\omega r_1^2}{v_o} , \quad \text{Mo}_1 = \frac{\omega r_1}{C_o} .$$

Again, the expression of h_{pq} for $p = 4, 5$, and 6 is also omitted here since it is similar to those of $p = 1, 2$, and 3 ; they can be obtained by replacing β_1 and γ_1 by β_2 and γ_2 , and multiplied by -1 [e.g., $h_{43} = \frac{S_2 \sigma_2^n}{\beta_2} F_n(\beta_2)$].

Using equations (17) and (19) gives the surface loading expressed in terms of the interface radii and the shell displacement as

$$\begin{pmatrix} \bar{P}_{z1} \\ \bar{P}_{\theta 1} \\ \bar{P}_{r1} \\ \bar{P}_{z2} \\ \bar{P}_{\theta 2} \\ \bar{P}_{r2} \end{pmatrix} = \{\alpha_{pq}\} \begin{pmatrix} r_1 \bar{u}_1 \\ r_1 \bar{v}_1 \\ r_1 \bar{w}_1 \\ r_2 \bar{u}_2 \\ r_2 \bar{v}_2 \\ r_2 \bar{w}_2 \end{pmatrix} \quad (20)$$

where

$$\{\alpha_{pq}\} = \{h_{pq}\} \{a_{pq}\}^{-1} \quad (21)$$

The coefficient α_{pp} is proportional to the dynamic fluid stress acting on a shell surface due to its own movement; while the others, α_{pq} for $p \neq q$, are proportional to the dynamic fluid stresses acting on a shell surface in one direction due to the movement in another direction.

The fluid stresses acting on the shells are linear functions of shell motions. In general, the coefficients α_{pq} are complex. The fluid stress can be separated into two components: one proportional to $\text{Re}(\alpha_{pq})$ is in phase with the shell accelerations and is related to the added mass effect, while the other, proportional to $\text{Im}(\alpha_{pq})$, is opposing to the movement of the shells and is related to damping mechanism. If the fluid is inviscid, the second component of the stress opposing shell motion will be zero.

As shown in equations (17), (19), and (21), the dynamic fluid stress coefficient matrix α_{pq} is a function of the radius ratio, r_2/r_1 , the circumferential wave number, n , the axial wave number, α_1 , the Mach number $Mo_1 = \frac{\omega r_1}{C_0}$, the Reynolds number, $Re_1 = \frac{\omega r_1^2}{\nu_0}$, and the ratio of the fluid viscosities ν'_0/ν_0 .

It should be noted that the circular frequency ω and so Mach number Mo_1 and Reynolds number Re_1 are in general complex numbers. The Mach number Mo_1 is considered to include the compressibility effects of the fluid. However, the analysis is valid for a small compressible effect or a small Mach number Mo_1 . The Reynolds number Re_1 and viscosity ratio ν'_0/ν_0 are the additional function parameters associated with fluid viscosity. The viscous effect is discussed in detail in the study.

For the case of potential flow where $\nu_0 = 0$, and $\nu'_0 = 0$, the Reynolds number Re_1 and viscous ratio ν'_0/ν_0 are no longer defined and the coefficient is a function of r_2/r_1 , n , α_1 and Mo_1 only. Furthermore, all the elements of the coefficient matrix are zero except the real parts of the four elements: $Re\{\alpha_{33}\}$, $Re\{\alpha_{36}\}$, $Re\{\alpha_{63}\}$, and $Re\{\alpha_{66}\}$, and no damping is introduced to the system by the ideal fluid [1].

B. Numerical Results

The preceding analytical method to evaluate the stress coefficients is programmed on a digital computer which is listed in the appendix (HDSC) of this report. This program can be used to calculate all elements of the stress coefficient matrix, α_{pq} , for two concentric shells separated by viscous fluid for any given set of input parameters n , r_2/r_1 , α_1 , Mo_1 , Re_1 , and ν'_0/ν_0 .

As mentioned in Section III.A, the n -th order Bessel functions $F_n(Z)$ and $G_n(Z)$, ($Z = \beta_i$ or γ_i), in principle, can be either the first and second kind of Bessel functions, J_n and Y_n , or the Hankel functions $H_n^{(1)}$ and $H_n^{(2)}$. However, in practical applications a proper selection of the functions is required because of the limitations imposed by a digital computer and required accuracy. In general the function J_n and Y_n can be used for a small argument (e.g., $|Z| \leq 15$), while $H_n^{(1)}$ and $H_n^{(2)}$ are recommended for a large argument. Furthermore, for a very large argument (e.g., $|Z| > 50$) as in most practical cases, a normalized Bessel function instead of Bessel function itself is used not only to avoid the overflow interruption but also to maintain the accurate result for the calculated stress coefficient matrix.

Some numerical examples of the stress coefficients were obtained and shown in Table 1 and Figs. 2 to 5. All examples given are for incompressible fluid ($Mo_1 = 0$), the length of both shells equal to radius r_1 ($\alpha_1 = \pi$), and $v_o' = 0$. Also the circular frequency ω is limited to a purely real number. That is, $\text{Im}(\omega) = \text{Im}(\text{Re}_1) = 0$.

Table 1 shows the stress coefficient matrix for $\alpha_1 = \pi$, $r_2/r_1 = 1.2$ and $Mo_1 = 0$. Each element is specified by a real part (upper number) and an imaginary part (lower number).

For $n = 3$ (Table 1a) all the values of α_{pq} are very small except $\text{Re}\{\alpha_{33}\} = 0.324$, $\text{Re}\{\alpha_{36}\} = \text{Re}\{\alpha_{63}\} = -0.234$, and $\text{Re}\{\alpha_{66}\} = 0.318$. In this case, the Reynolds number is large ($\text{Re}_1 = 10^{10}$); as expected the imaginary part of α_{pq} is not important. The results are consistent with the solutions of the potential flow [1], which shows $\alpha_{33} = 0.324$, $\alpha_{36} = \alpha_{63} = -0.234$, $\alpha_{66} = 0.318$ and all others are zero. However, for the

TABLE 1. Dynamic Stress Coefficient Matrix α_{pq} for $\alpha_1 = \pi$,
 $r_2/r_1 = 1.2$ and $M_{01} = 0.0$.

		p \ q	1	2	3	4	5	6
(a)	n = 3	1	7.07E-6	0	-7.19E-6	0	0	5.20E-6
			-7.07E-6	0	7.19E-6	0	0	-5.20E-6
		2	0	7.07E-6	-6.86E-6	0	0	4.97E-6
			0	-7.07E-6	6.86E-6	0	0	-4.97E-6
		3	-7.19E-6	6.86E-6	3.24E-1	-5.20E-6	-4.14E-6	-2.34E-1
			7.19E-6	6.86E-6	-2.15E-5	5.20E-6	4.14E-6	2.03E-5
Re ₁ = 10 ¹⁰		4	0	0	-5.20E-6	5.89E-6	0	7.05E-6
			0	0	5.20E-6	-5.89E-6	0	-7.05E-6
		5	0	0	-4.14E-6	0	5.89E-6	5.61E-6
			0	0	4.14E-6	0	-5.89E-6	-5.61E-6
		6	5.20E-6	4.97E-6	-2.34E-1	7.05E-6	5.61E-6	3.18E-1
			-5.20E-6	-4.97E-6	2.03E-5	-7.05E-6	-5.61E-6	-2.11E-5
(b)	n = 3	1	4.14E-2	-1.67E-2	-2.13E-2	7.46E-3	-1.59E-2	1.25E-2
			-1.49E 0	-7.32E-1	2.32E 0	-3.33E-1	-6.21E-1	-2.60E 0
		2	-1.67E-2	4.64E-2	-2.22E-2	-1.61E-2	1.20E-2	1.25E-2
			-7.32E-1	-1.42E 0	2.03E 0	-6.64E-1	-1.52E-1	-2.34E 0
		3	-2.13E-2	-2.22E-2	3.85E-1	-1.28E-2	-1.02E-2	-2.91E-1
			2.32E 0	2.03E 0	-9.38E 0	2.47E 0	2.06E 0	8.93E 0
Re ₁ = 10		4	7.46E-3	-1.61E-2	-1.28E-2	3.48E-2	-1.41E-2	2.24E-2
			-3.33E-1	-6.64E-1	2.47E 0	-1.20E 0	-6.11E-1	-2.20E 0
		5	-1.59E-2	1.20E-2	-1.02E-2	-1.41E-2	3.86E-2	1.66E-2
			-6.21E-1	-1.52E-1	2.06E 0	-6.11E-1	-9.20E-1	1.89E 0
		6	1.25E-2	1.25E-2	-2.91E-1	2.24E-2	1.66E-2	3.78E-1
			-2.60E 0	-2.34E 0	8.93E 0	-2.20E 0	-1.89E 0	-8.87E 0
(c)	n = 1	1	2.97E-2	-9.55E-3	-3.20E-2	-3.64E-3	-8.94E-3	2.24E-2
			-1.95E 0	-3.96E-1	3.89E 0	-7.97E-1	-3.42E-1	-4.19E 0
		2	-9.55E-3	6.01E-2	-1.11E-2	-9.01E-3	2.48E-2	7.53E-3
			-3.96E-1	-8.44E-1	1.14E 0	3.65E-1	3.20E-1	-1.25E 0
		3	-3.20E-2	-1.11E-2	5.82E-1	-2.38E-2	-6.23E-3	-4.86E-1
			3.89E 0	1.14E 0	-1.41E 1	3.95E-1	1.10E 0	1.38E 1
Re ₁ = 10		4	-3.64E-3	-9.01E-3	-2.38E-2	2.48E-2	-8.03E-3	3.28E-2
			-7.97E-1	-3.65E-1	3.95E 0	-1.62E 0	-3.31E-1	-3.69E 0
		5	-8.94E-3	2.48E-2	-6.23E-3	-8.03E-3	5.01E-2	8.18E-3
			-3.42E-1	3.20E-1	1.10E 0	-3.31E-1	-4.67E-1	-1.05E 0
		6	2.24E-2	7.53E-3	-4.86E-1	3.28E-2	8.18E-3	5.73E-1
			-4.19E 0	-1.25E 0	1.38E 1	-3.69E 0	-1.05E 0	-1.38E 1

Fig. 2. Stress coefficient α_{33} as a function of Reynolds number Re_1 for $\alpha_1 = \pi$, $r_2/r_1 = 1.1$, and $Mo_1 = 0.0$.

case of small Reynolds number, the viscous effect becomes important; the imaginary part of α_{pq} becomes more important and, in some cases, the imaginary part may be larger than the real part. This can be seen from Table 1b and 1c; for $Re_1 = 10$, the imaginary parts of α_{pq} are, indeed, much larger than their real parts.

Figure 2 shows the stress coefficient α_{33} as a function of Reynolds number $Re_1 = \frac{\omega r_1^2}{v_0}$ as well as n for $\alpha_1 = \pi$, $r_2/r_1 = 1.1$ and $Mo_1 = 0.0$. α_{33} decreases as the Reynolds number Re_1 increases; this behavior is the same as that of a vibrating rod in confined viscous fluids [7]. For fixed values of r_1 , r_2 , and v_0 , $Re\{\alpha_{33}\}$ decreases with an increasing ω ; this behavior is consistent with the experimental observations [8]. It is interesting to note that the sensitivity of all $Re\{\alpha_{pq}\}$ with respect to Re_1 is much smaller than $Im\{\alpha_{pq}\}$, although both $Re\{\alpha_{pq}\}$ and $Im\{\alpha_{pq}\}$ increase with decreasing Re_1 . Figure 2 also shows that both $Re\{\alpha_{33}\}$ and $Im\{-\alpha_{33}\}$ are the decreasing function of n .

Figures 3 and 4 show the stress coefficients α_{33} and α_{66} as functions of r_2/r_1 , n , and Re_1 for $\alpha_1 = \pi$ and $Mo_1 = 0$. In all cases, as the values of r_2/r_1 increase, α_{33} approaches a constant value while α_{66} monotonically decreases. The different behaviors between α_{33} and α_{66} is attributed to the choice of inner radius r_1 (rather than r_2) as a reference length scale. As the value of r_2/r_1 becomes large, the movement of either shell is less dependent on the existence of the other and the stress coefficients are expected to approach a constant value if all other parameters for the shell are kept unchanged. However, the present example is set to keep α_1 , Re_1 , and Mo_1 unchanged, and so α_2 , Re_2 and Mo_2 which are the important parameters for the outer shell are increased as r_2/r_1 increases. This means there will be a larger

Fig. 3. Real parts of stress coefficients as functions of radius ratio r_2/r_1 for $\alpha_1 = \pi$ and $\text{Mo}_1 = 0.0$. —: $\text{Re}\{\alpha_{66}\}$; ---: $\text{Re}\{\alpha_{33}\}$.

Fig. 4. Imaginary parts of stress coefficients as functions of r_2/r_1 for $\alpha_1 = \pi$ and $M_0 = 0.0$. —: $\text{Im}\{-\alpha_{66}\}$; ---: $\text{Im}\{-\alpha_{33}\}$.

three-dimensional effect, a larger compressibility effect and a smaller viscous effect for the outer shell as r_2/r_1 increases, although these effects are relatively constant for the inner shell.

For $r_2/r_1 > 2.0$ the outer shell has practically no effects on α_{33} and the effect of the inner shell on the outer shell seems no longer important and the decrease of α_{66} is primarily due to the increase of α_2 or the decrease of length/radius for the outer shell. With respect to the dependence of n , both α_{33} and α_{66} are larger for $n = 1$ than those for $n = 4$. Also the decreasing rate for $\text{Re}\{\alpha_{33}\}$ and $\text{Re}\{\alpha_{66}\}$ is smaller as Re_1 increases, while this behavior for $\text{Im}\{-\alpha_{33}\}$ and $\text{Im}\{-\alpha_{66}\}$ is much pronounced.

Figure 5 shows the stress coefficient $\alpha_{36}(=\alpha_{63})$ as a function of r_2/r_1 , n , and Re_1 for $\alpha_1 = \pi$ and $\text{Mo}_1 = 0$. Both $\text{Re}\{-\alpha_{36}\}$ and $\text{Im}\{\alpha_{36}\}$ are monotonically decreasing as r_2/r_1 increases. It is attributed not only to the increasing separation between two shells, but also to the increasing three-dimensional effect on the outer shell. Note that the example given here is for $\text{Mo}_1 = 0$. For the case of $\text{Mo}_1 > 0$, the value of α_{pq} is expected to be smaller. It can be concluded that the stress coefficient is a decreasing function of the parameters α_1 , Mo_1 , Re_1 , and r_2/r_1 and in most cases it is a decreasing function of n . In some cases α_{pq} for $p(\text{or } q) = 2$ or 4 could be slightly increased with the increase of n .

Fig. 5. Stress coefficient α_{36} as a function of r_2/r_1 for $\alpha_1 = \pi$ and $Mo_1 = 0.0$.
 —: $n = 1$; ---: $n = 4$.

IV. FREQUENCY EQUATION

A. Analysis

Substituting Eqs. (12), (18), and (20) into the shell Eq. (1) gives six linear algebraic homogeneous equations

$$\{b_{pq}\} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{w}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{w}_2 \end{Bmatrix} = 0, \quad (22)$$

where

$$\begin{aligned} b_{pq} &= C_{pq} - \Omega_1^2 (\delta_{pq} + \mu_1 \alpha_{pq}), \quad \text{for } p, q = 1, 2 \text{ and } 3, \\ &= -\Omega_1^2 \left(\delta_{pq} + \mu_1 \frac{r_2}{r_1} \alpha_{pq} \right), \quad \text{for } p = 1, 2, \text{ and } 3, \\ &\quad \text{and } q = 4, 5, \text{ and } 6, \\ &= -\Omega_2^2 \left(\delta_{pq} + \mu_2 \frac{r_1}{r_2} \alpha_{pq} \right), \quad \text{for } p = 4, 5, \text{ and } 6, \\ &\quad \text{and } q = 1, 2, \text{ and } 3, \\ &= C_{pq} - \Omega_2^2 (\delta_{pq} + \mu_2 \alpha_{pq}), \quad \text{for } p, q = 4, 5, \text{ and } 6, \end{aligned} \quad (23)$$

$$\Omega_i = R_i \omega \left[\frac{\rho_i (1 - \nu_i^2)}{E_i} \right]^{1/2}, \quad \mu_i = \frac{\rho_o r_i}{\rho_i h_i},$$

$$\delta_{pq} = 1 \quad \text{for } p = q, \text{ otherwise } \delta_{pq} = 0,$$

$$\begin{aligned} C_{11} &= \alpha_1^2 + \frac{1 - \nu_1}{2} n^2 \left(1 + \frac{\delta_1^2}{12} \right), \\ C_{12} &= C_{21} = \frac{1 + \nu_1}{2} \alpha_1 n, \\ C_{22} &= n^2 + \frac{1}{2} (1 - \nu_1) \left(1 + \frac{\delta_1^2}{4} \right) \alpha_1^2, \end{aligned} \quad (24)$$

$$\begin{aligned}
C_{13} &= C_{31} = \alpha_1 \nu_1 + \frac{\alpha_1^3}{12} \delta_1^2 - \frac{\alpha_1}{24} (1 - \nu_1) \delta_1^2 n^2, \\
C_{23} &= C_{32} = n + \frac{\alpha_1^2}{24} (3 - \nu_1) \delta_1^2 n, \\
C_{33} &= 1 + \frac{\delta_1^2}{12} [1 - 2n^2 + (\alpha_1^2 + n^2)^2], \\
\delta_i &= h_i / R_i.
\end{aligned} \tag{24}$$

(Contd.)

The expression of C_{pq} for the outer shell can be obtained easily by replacing α_1 and δ_1 by α_2 and δ_2 and changing the subscript for C_{pq} from 1, 2, and 3 to 4, 5, and 6 respectively.

The frequency equation of the coupled fluid/shell system is obtained by setting the determinant of the coefficient matrix b_{pq} in equation (22) equal to zero; it can be written as $|b_{pq}| = 0$ or in the function form as

$$F(\alpha_1, \gamma_1, r_2/r_1, Mo_1, Re_1, \nu'_o/\nu_o, n, \Omega_i, \delta_i, \mu_i) = 0. \tag{25}$$

As pointed out before, the stress coefficient for a given physical condition is not a constant but functions of the frequency parameter, ω , which in general is a complex number. Therefore, in order to determine the natural frequency and the damping ratio of the coupled system, an iteration procedure is in general required.

It should be noted that, for the case of a viscous fluid, the dynamic fluid stress is not only a function of n , r_2/r_1 , α_1 and Mo_1 , but also a function of the vibrational Reynolds number Re_1 . This additional parameter Re_1 makes the simulation of a scaled model test for a coupled viscous fluid/shell system to be very difficult when fluid viscosity effect is important. In a reduced-scale model test, the geometrical simulation commonly employed tends to overestimate the fluid damping and thus the test result will not be conservative.

B. Numerical Example

For a specific numerical example, consider the following dimensional values: $R_1 = 86.52$ cm (34.0625 in.), $R_2 = 88.74$ cm (34.9375 in.), $h_1 = 0.635$ cm (0.25 in.), $h_2 = 1.5875$ cm (0.625 in.), $E_1 = E_2 = 1.896 \times 10^{11}$ Pa (2.75×10^7 psi), $\nu_1 = \nu_2 = 0.27$, $\rho_1 = \rho_2 = 7.986 \times 10^3$ kg/m³ (0.2885 lbm/in.³), $\rho_o = 9.217 \times 10^2$ kg/m³ (0.0333 lbm/in.³), $\nu_o = 7.432 \times 10^{-7}$ m²/s (8.0×10^{-6} ft²/sec), $\nu'_o = 0$ and $C_o = \infty$. This is the same example as that given in Ref. 1, with the exception that, in the present case, there is no fluid inside the inner shell and the fluid is viscous in the annulus region. The fluid is considered to be incompressible, $Mo_1 = 0$, and the shell is assumed to be simply supported at both ends.

The frequencies of the system depend on the axial wave number α_i and circumferential wave number n . The lowest frequency is associated with the lowest axial wave number, i.e., the wavelength, is equal to twice the shell length which is assumed to be 104.14 cm (41 in.).

Natural frequencies for the coupled shell/fluid system can be calculated from Eq. (25). The real and imaginary parts of frequencies, f_R and f_I , are obtained first and then the system natural frequencies f_N and the related modal damping ratio ζ are given by $f_N \approx f_R$ and:

$$\zeta = \frac{f_I}{f_R} \left[1 + \left(\frac{f_I}{f_R} \right)^2 \right]^{-1/2}$$

Tables 2 and 3 and Figs. 6, 7 and 8 summarize the results. The lower natural frequency is associated with the mode in which two shells move out-of-phase, while the higher frequency is associated with in-phase mode.

Table 2 gives the lowest natural frequencies and the related modal damping ratios of a shell system for several cases: I) inner shell in vacuo; II) outer shell in vacuo; III) an uncoupled shell system with rigid outer shell and $\nu_o = \nu'_o = 0$; IV) same as Case III except $\nu_o = 7.432 \times 10^{-7}$ m²/s (8.0×10^{-6} ft²/sec); V) an uncoupled shell system

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Case I	-	-	-	369 0	263 0	197 0	162 0	152 0	161 0	184 0	215 0	254 0	297 0	346 0
Case II	-	-	-	395 0	281 0	238 0	241 0	278 0	338 0	412 0	498 0	-	-	-
Case III	66.5 0	62.0 0	52.5 0	43.3 0	36.3 0	31.7 0	29.9 0	31.7 0	37.5 0	47.1 0	60.4 0	77.1 0	98.3 0	128.4 0
Case IV	66.0 .274	61.8 .283	52.3 .308	43.2 .338	31.1 .369	31.5 .393	29.8 .401	31.6 .385	37.3 .350	47.0 .308	60.3 .268	76.9 .233	98.1 .197	128.2 .145
Case V	102.0 0	95.2 0	81.6 0	68.6 0	60.0 0	59.0 0	68.2 0	88.3 0	119.0 0	159 0	208 0	267 0	340 0	445 0
Case VI	101 .218	94.9 .224	81.4 .240	68.4 .259	59.9 .273	58.8 .270	68.0 .246	88.1 .211	119.0 .177	158 .149	208 .126	267 .107	340 .089	445 .062
Case VII	55.7 0	52.1 0	44.3 0	36.8 0	31.2 0	28.1 0	27.6 0	30.0 0	36.0 0	45.6 0	58.6 0	74.9 0	95.6 0	125 0
	-	-	-	366 0	269 0	221 0	216 0	243 0	291 0	354 0	428 0	-	-	-
Case VIII	55.6 .299	51.9 .309	44.2 .335	36.7 .367	31.1 .398	28.0 .418	27.5 .420	29.9 .398	35.9 .360	45.4 .315	58.4 .274	74.7 .239	95.4 .202	125 .149
	-	-	-	366 0	269 0	221 0	216 0	243 0	291 0	354 0	428 0	-	-	-

TABLE 2. Natural Frequencies (Hz) and Related Modal Damping Ratio (%) of Shell Systems

with rigid inner shell and $v_o = v'_o = 0$; VI) same as Case V except $v_o = 7.432 \times 10^{-7} \text{ m}^2/\text{s}$; VII) a coupled shell system with $v_o = v'_o = 0$; VIII) same as Case VII except $v_o = 7.432 \times 10^{-7} \text{ m}^2/\text{s}$. The natural frequencies, f_N , and related modal damping ratio, ζ , (in %) are given by the upper and lower numbers in each group.

Table 2 shows that natural frequencies of the system decrease due to the existence of the fluid. Note that these natural frequencies are practically independent of the variation of fluid viscosity, although the related modal damping ratio is noticeably increased in some cases when the fluid viscosity is included. These results are expected since the fluid viscosity (related to Re_1) has a smaller effect on $\text{Re}\{\alpha_{pq}\}$ and has a larger effect on $\text{Im}\{\alpha_{pq}\}$. The decrease in natural frequencies is due to fluid inertia effect, which is proportional to $\text{Re}\{\alpha_{pq}\}$, while the increase in damping is mainly attributed to fluid drag, which is proportional to $\text{Im}\{\alpha_{pq}\}$. For $n \leq 2$ or $n \geq 11$ some of the data are not presented because of high natural frequencies.

The natural frequencies for the cases of viscous fluid are also shown on Fig. 6. It can be also seen from Table 2 and Fig. 6 that for each circumferential wavenumber n the natural frequencies for the first coupled modes (out-of-phase modes) are lower than either of the uncoupled natural frequencies, while the frequencies for the second coupled modes (in-phase modes) are higher than either of the uncoupled natural frequencies. For $n \leq 2$ or $n \geq 11$ the in-phase mode data were not presented because of the high natural frequencies.

Fig. 7 shows the modal damping ratio ζ of the shell systems. The effects of fluid viscosity on coupled and uncoupled modes are clearly shown. For the coupled shell systems the effects are mostly pronounced for the out-of-phase modes, but these effects are much smaller and in fact are

Fig. 6. Natural frequencies of shell systems.

Fig. 7. Modal damping ratio ζ of a fluid/shell system.

considered to be negligible for the in-phase modes. However, for the uncoupled vibrations, the effects of fluid viscosity remain comparable for both shells. The reason for large damping ratios on out-of-phase modes and small damping ratios on in-phase modes can be seen from the differences of the vibrational mode-shapes which are shown on Table 3 and Fig. 8. For in-phase modes, both shells as well as the fluid in the annular region are moved almost in the same way (i.e., amplitude ratios ≈ 1.0) and so the existence of the fluid is hardly noticed by the shells; the frequencies are close to those of outer shell in vacuo. However, for the out-of-phase modes, two shells are moved in opposite directions and the existence of the fluid is mostly detected by the shells. For $n \geq 6$ the movement of inner shell is much larger than that of outer shell and the result is close to that of uncoupled shell system with rigid outer shell.

Additional calculations also have been made to understand the effect of fluid compressibility. It is found that the effect of fluid compressibility, in general, is very small on natural frequency and damping, and in practical applications for structural vibrations, the compressibility of the fluid may be neglected. On the other hand, if the propagation of waves in the system is of interest, fluid compressibility has to be included.

TABLE 3. Modal Shapes of Out-of-Phase and In-Phase Modes of a Coupled Shell System, Case VIII. $\bar{w}_1 = 1.0$

n	f(Hz)	ζ (%)	$\bar{u}_1$	$\bar{v}_1$	$\bar{u}_2$	$\bar{v}_2$	$\bar{w}_2$	$\bar{u}_2/\bar{u}_1$	$\bar{v}_2/\bar{v}_1$
0	55.6	.299	-.104	0	.042	0	-.419	-.409	-
1	51.9	.309	-.036	-.27	.016	.107	-.414	-.433	-.397
2	44.2	.335	.048	-.333	-.018	.131	-.402	-.369	-.392
3	36.7	.367	.075	-.294	-.028	.112	-.385	-.373	-.381
	366	0	.079	-.299	.076	-.293	.988	.959	.978
4	31.1	.398	.071	-.242	-.025	.084	-.350	-.346	-.348
	269	0	.072	-.243	.071	-.240	.990	.979	.985
5	28.0	.418	.060	-.200	-.016	.054	-.272	-.272	-.272
	221	0	.060	-.200	.060	-.199	.995	.995	.994
6	27.5	.420	.049	-.169	-.009	.030	-.178	-.179	-.177
	216	0	.049	-.169	.050	-.170	1.01	1.01	1.01
7	29.9	.398	.040	-.145	-.005	.017	-.116	-.117	-.116
	243	0	.040	-.145	.041	-.148	1.02	1.03	1.02
8	35.9	.360	.032	-.127	-.003	.011	-.086	-.087	-.086
	291	.001	.033	-.127	.034	-.131	1.03	1.05	1.03
9	45.4	.315	.027	-.113	-.002	.008	-.074	-.075	-.074
	354	.001	.027	-.113	.029	-.118	1.05	1.06	1.05
10	58.4	.274	.022	-.101	-.002	.007	-.069	-.070	-.069
	428	.002	.023	-.102	.025	-.108	1.06	1.08	1.06
11	74.7	.239	.019	-.092	-.001	.006	-.067	-.068	-.067
12	95.4	.202	.016	-.084	-.001	.005	-.064	-.066	-.064
13	125	.149	.014	-.078	-.001	.005	-.061	-.062	-.061

Fig. 8. Modal shape of a coupled fluid/shell system.

V. COMPARISON OF ANALYTICAL AND EXPERIMENTAL RESULTS

An experimental study on a related problem was reported recently by Chung et al. [9]. In the experiment, a steel shell free at the top edge and soldered to a disc at the bottom was tested. The fluid gap was provided by using a thick concrete shell as the outer cylinder and the fluid is water. A series of tests was made for four different fluid gap sizes: 2.616 cm (1.03 in.), 1.367 cm (0.538 in.), 0.643 cm (0.253 in.), and 0.384 cm (0.151 in.).

To compare the experimental and analytical results, an analysis was made for the steel shell. The physical properties of the shell and water are as follows: $r_1 = 21.69$ cm (8.54 in.), $r_2 = r_1 + G$, $h_1 = 0.1473$ cm (0.058 in.), $E_1 = 1.93 \times 10^{11}$ Pa (2.8×10^7 psi), $\nu_1 = 0.3$, $\rho_1 = 7.47 \times 10^3$ kg/m³ (0.27 lbm/in.³), $\rho_o = 1.00 \times 10^3$ kg/m³ (0.0362 lbm/in.³), $\nu_o = 1.003 \times 10^{-6}$ m²/s (1.08×10^{-5} ft²/sec) and $L = 51.11$ cm (20.125 in.). In the analysis, the fluid is considered to be incompressible (i.e., $Mo_1 = 0$, or $C_o = \infty$) and $\nu'_o = 0$. The numerical result is given only for the lowest axial wavenumber for which the wavelength is assumed to be equal to four times the shell length. Under the stated conditions, one notes at least two important differences between the experiment and the analyses: 1. In the experiment, the boundary condition of the shell is close to fixed-free end condition, while a simply-supported end condition is assumed in the analysis; 2. In the experiment, there exists a relatively large value of structure damping, while in the analysis, the structure damping is assumed to be zero.

In order to make a more meaningful comparison, the normalized frequencies f_w/f_a , where f_w and f_a are frequencies in water and in air respectively, are given in Table 4 for various water gaps. The experimental data are enclosed by parentheses. Table 4 shows that the analytical and experimental results

TABLE 4. Comparison of Experimental and Analytical Frequency Ratio

n	Frequencies (f_w/f_a) for different water gaps (cm)					Frequencies in vacuo f_a (Hz)
	G = 0.1917	G = 0.3835	G = 0.6426	G = 1.366	G = 2.616	
3	0.066	0.095	0.123	0.177	0.235	176
4	0.085	0.122 (0.156)	0.157 (0.187)	0.224 (0.254)	0.294 (0.312)	151 (160)
5	0.105	0.150 (0.180)	0.192 (0.234)	0.271 (0.331)	0.348 (0.414)	195 (205)
6	0.125	0.177 (0.212)	0.227 (0.258)	0.315 (0.366)	0.397 (0.458)	273 (295)

have the similar trend as functions of the gap except the analytical results are smaller.

The frequency ratio f_w/f_a is a measure of fluid added mass and is equal to $\left(\frac{m_s}{m_s + m_f}\right)^{1/2}$, where m_s and m_f are shell and added mass per unit shell surface area respectively. Since the added mass depends on shell mode shape, the added mass will be different for cantilevered and simply-supported shells. The differences between the theoretical and experimental values are most likely attributed to different boundary conditions.

With respect to the damping, consider the shell vibrating at one of its natural frequencies in water. The equation of motion can be represented by

$$(m_s + m_f)\ddot{x} + (c_s + c_f)\dot{x} + k_s x = 0 \quad (25)$$

where m_s , c_s and k_s are mass, structure damping and stiffness corresponding to its particular mode, and m_f and c_f are the added mass and the added damping effects due to a viscous fluid. Both m_f and c_f are functions of the fluid gap.

Based on Eq. (25), the total damping ratio in water is

$$\zeta_t = \frac{c_s + c_f}{2\sqrt{(m_s + m_f)k_s}} = \sqrt{\frac{m_s}{m_s + m_f}} \zeta_s + \zeta \quad (26)$$

where $\zeta_s = \frac{c_s}{2\sqrt{m_s k_s}}$ is the structure damping ratio in vacuo and

$\zeta = \frac{c_f}{2\sqrt{(m_s + m_f)k_s}}$ is the damping ratio associated with fluid viscosity.

In the experiment, both ζ_t and ζ_s are measured and ζ_t is increased with the fluid gap. Therefore the viscous damping ratio due to fluid can be calculated from Eq. (26). The analytical and experimental results are given in Fig. 9. The general behavior of the theoretical fluid damping is similar to the experimental data. Quantitatively, the

Fig. 9. Comparison of experimental and analytical damping ratio.

agreement is good for circumferential wave number $n = 5$ and 6 and small gaps, and fair for large gaps. However, the experimental values for $n = 4$ are much larger than the analytical results. As pointed out earlier, one of the reasons for the discrepancy is attributed to the boundary conditions. Since the theoretical and experimental models have different boundary conditions, and the added mass m_f and viscous damping c_f depend on vibrational mode shape, the two models are not expected to give identical results.

In a similar study for tubes vibrating in a viscous fluid annulus, the analytical results based on the linear viscous theory are in good agreement with the experimental results for damping and added mass [7]. Since the same theory is used here for shells, as long as the motion is small, the analytical results based on the linear viscous theory are expected to be reliable.

Note that both m_f and c_f decrease with increasing fluid gap. Therefore the term $\sqrt{\frac{m_s}{m_s + m_f}} \zeta_s$ increases monotonically with fluid gap. On the other hand, the fluid damping, which is a function of c_f and m_f , usually decreases with fluid gap. Since the total damping ratio ζ_t consists of $\sqrt{\frac{m_s}{m_s + m_f}} \zeta_s$ and ζ , it is either an increasing function or a decreasing function of the gap depending on the values of structural damping and fluid damping. For the case of a small fluid viscosity and large structural damping, the total damping ratio ζ_t will decrease as the gap decreases. However, eventually as the gap size gets small enough such that fluid damping becomes important, the total damping will increase because of the large increase of the fluid damping. That is, the total damping ratio ζ_t in a dense fluid such as water can be smaller than that in air (or vacuo); the result is quite opposite to what we commonly have expected that damping ratio in water is always larger than that in air.

VI. CONCLUDING REMARKS

In this report, an analysis is presented for coupled vibrations of two concentric shells separated by a viscous fluid. The coupling effects are accounted for using fluid stress coefficient matrix of concentric shells. With this analysis, natural frequencies and modal damping of coupled concentric shells in viscous fluid can readily be obtained.

In the analysis the three-dimensional, linearized, Navier-Stokes equations governing the motion of viscous fluids are used. The displacements of the shells are assumed to be small such that the equations of state and motion can be linearized. The analytical results are in reasonable agreement with the published experimental data.

Numerical results are presented for a few selected problems for an incompressible fluid. It is shown that the fluid stress coefficient is always a decreasing function of α_1 , Mo_1 , Re_1 and r_2/r_1 and in most cases it will decrease as n increases. The sensitivity of Re_1 on $Re\{\alpha_{pq}\}$ is much smaller than that on $Im\{\alpha_{pq}\}$. For general cases, the magnitude of $Re\{\alpha_{pq}\}$ is larger than the magnitude of $Im\{\alpha_{pq}\}$; however when Re_1 is sufficiently small, the magnitude of $Im\{\alpha_{pq}\}$ could be much larger than that of $Re\{\alpha_{pq}\}$ and the damping ratio of the system can therefore be very large.

The lowest natural frequency of the coupled shell system with fluid is significantly lower than those of the individual shells. The frequencies for the first coupled modes (out-of-phase modes) are lower than either of the uncoupled natural frequencies. The effect of the fluid viscosity on the system natural frequencies is negligibly small in most practical systems. However, the modal damping ratio is noticeably increased for some cases when the fluid viscosity is included, especially for the lower

frequency cases. For a coupled shell the viscous effects are mostly pronounced for the out-of-phase modes, but these effects are considered to be negligible for the in-phase mode. In the preliminary design of practical system components, for $r_2/r_1 > 1.15$, the effect of fluid viscosity on damping can be estimated based on the corresponding structure vibrating in an infinite fluid. However, when $r_2/r_1 < 1.15$, it is recommended that the viscous damping should be calculated based on the proposed method.

Finally, it should be pointed out that results from scaled models, frequently used in practices for design evaluation, may not be conservative if the vibration Reynolds number is not simulated. If the gap is small, or the fluid viscosity is relatively high, the simulation of the vibration Reynolds number Re_1 should be included to ensure that modal damping of the model is properly accounted for.

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APPENDIX: COMPUTER PROGRAM (HDSC)

This appendix gives the listing of the computer program, HDSC, which was used to calculate the dynamic fluid stress coefficient matrix, α_{ij} , given in this report. This program can be used for a system of two concentric shells separated by a viscous fluid. The program consisting of a series of subroutines with the input parameters n , r_2/r_1 , α_1 , $\text{Re}\{\text{Re}_1\}$, $\text{Im}\{\text{Re}_1\}$, $\text{Re}\{\text{Mo}_1\}$ and v'_0/v_0 and the output parameter α_{ij} in a 6×6 complex matrix is written in Fortran IV for IBM/370 Computer system.

SUBROUTINE HDSC (NN,R2DR1,ALPR1,RER,REI,SMKR,RNEU,AA)

TO CALCULATE THE FLUID STRESS COEFF. OF TWO CONCENTRIC SHELLS
IN VISCOUS FLUID

***** INPUT PARAMETERS *****

NN= CIRCUMFERENTIAL WAVE NUMBER

R2DR1 = R2/R1, WHERE R1,R2 ARE THE INTERFACE RADIUS

ALPR1 =AXIAL WAVENUMBER PER R1

RER = R1**2*WR/NEUO, REAL PART OF REYNOLDS NUMBER

REI =R1**2*WI/NEUO, IMAGINARY PART OF REYNOLDS NUMBER

SMKR = R1*WR/CO, REAL PART OF MACH NO.

RNEU = VISCOSITY RATIO, NEUOP/NEUO

WHERE

WR,WI =REAL AND IMAGINARY PARTS OF CIRCULAR FREQUENCY

NEUO,NEUOP = FLUID VISCOSITY

CO = SPEED OF SOUND IN FLUID

***** OUTPUT PARAMETERS *****

COMPLEX*8 AA(6,6) FLUID STRESS COEFF. MATRIX

COMMON

C /AAA2/ N,AN,ANS,ANM1,ANP1,TN,TNP2,ANSP1

C /AAA3/ S1,S2,SIG1,SIG2,B1,B2,G1,G2

C /AAA7/ TS1NN1,TS1WP1,EPIB1,EMIB2,EMIDB,TNNM1,IARGZ

C /ABHD/ A(6,6),B(6,6),H(6,6),DETERM

C /FLUID/ SMK1PW,RN1PW,RNEUPO

C /WALP/ WR,WI,WSR,WSI,ALP1,ALP2,R21

COMPLEX*16 A,B,H,DETERM

COMPLEX*16 GJ(2),GY(2),BJ(2),RY(2)

C COMPLEX*8 S1SG1,S2SG2,TS1,TS2,SG2SM1,TNS1,BNS1,
C TS2SG2,TSGG,TS2B,GG,BB,TS1SG1,S2SG2N,DGG,DBB

C ,EMIG1,EPIG2,EPIDG

COMPLEX*8 S1,S2,SIG1,SIG2,B1,B2,G1,G2

COMPLEX*8 TS1NN1,TS1WP1,EPIB1,EMIB2,EMIDB

COMPLEX*8 TRC/(0.,1.0)/,AA(6,6)

SMK1PW=SMKR

RN1PW=RER

N=NN

ALP1=ALPR1

R21=R2DR1

RNEUPO=RNEU

WR=1.0

WI=REI/REF

CALL CONSTO

CALL CONST1

SG2SM1=SIG2*SIG2-1.

TNS1=TN*S1

S2SG2=S2*SIG2

TS1=2.*S1

TS2=2.*S2

TS1SG1=TS1*SIG1

TS2SG2=TS2*SIG2

S2SG2N=S2SG2*AN

IG2=3

IF(CABS(G2) .LT. 15.) GO TO 16

IF(ABS(AIMAG(G2)) .GT. 60.) GO TO 40

IG2=1

EPIG2=CEXP(TRC*G2)

```

      EMIG1=CEXP(-TRC*G1)
      GO TO 16
40  IG2=2
      EPIDG=CEXP(TRC*(G2-G1))
C   INNER INTERFACE
16  J=1
      JD=0
      ALP=ALP1
      BB=B1
      GG=G1
      2  JD1=JD+1
          JD2=JD+2
          JD3=JD+3
          GO TO (31,32,17),IG2
17  CALL BSJY(GG,GJ,GY,N,1)
      GO TO 19
31  CALL HANKEL(GG,GJ,GY,N,1)
      DO 25 L1=1,2
          GJ(L1)=GJ(L1)*EMIG1
25  GY(L1)=GY(L1)*EPIDG
      GO TO 19
32  CALL HANKEL(GG,GJ,GY,N,2)
      GO TO (33,34),J
33  GY(1)=GY(1)*EPIDG
      GY(2)=GY(2)*EPIDG
      GO TO 19
34  GJ(1)=GJ(1)*EPIDG
      GJ(2)=GJ(2)*EPIDG
19  IF(CABS(B2) .GT. 15.) GO TO 18
      CALL BSJY(BB,BJ,BY,N,1)
      GO TO 30
18  GO TO (20,21),IARGZ
20  CALL HANKEL(BB,BJ,BY,N,1)
      DO 22 L1=1,2
          BJ(L1)=BJ(L1)*EMIB2
22  BY(L1)=BY(L1)*EPIB1
      GO TO 30
21  CALL HANKEL(BB,BJ,BY,N,2)
      GO TO (23,24),J
23  BJ(1)=BJ(1)*EMIDB
      BJ(2)=BJ(2)*EMIDB
      GO TO 30
24  BY(1)=BY(1)*EMIDB
      BY(2)=BY(2)*EMIDB
C
C   TRANSPOSE OF MATRIX AIJ
30  A(1,JD1)=-ALP*GJ(1)
      A(2,JD1)=-ALP*GY(1)
      A(3,JD1)=0.
      A(4,JD1)=0.
      A(5,JD1)=-BB*BJ(1)
      A(6,JD1)=-BB*BY(1)
      A(1,JD2)=-AN*GJ(1)
      A(2,JD2)=-AN*GY(1)
      A(3,JD2)=BB*BJ(2)-AN*BJ(1)
      A(4,JD2)=BB*BY(2)-AN*BY(1)
      A(5,JD2)=ALP*BJ(2)
      A(6,JD2)=ALP*BY(2)
      A(1,JD3)=AN*GJ(1)-GG*GJ(2)

```



```

A (2,JD3)=AN*GY (1) -GG*GY (2)
A (3,JD3)=AN*BJ (1)
A (4,JD3)=AN*BY (1)
A (5,JD3)=A (5,JD2)
A (6,JD3)=A (6,JD2)
C BOUNDARY CONDITION      MATRIX HIJ
C TRANSPOSE OF HIJ
DGG=1./GG
DBB=1./BB
H (1,JD1)=TS1SG1*(GJ (2) -AN*GJ (1) *DGG)
H (2,JD1)=TS1SG1*(GY (2) -AN*GY (1) *DGG)
H (3,JD1)=-S2SG2N*BJ (1) *DBB
H (4,JD1)=-S2SG2N*BY (1) *DBB
H (5,JD1)=-S2*(SG2SM1*BJ (2) +AN*BJ (1) *DBB)
H (6,JD1)=-S2*(SG2SM1*BY (2) +AN*BY (1) *DBB)
BNS1=TNNM1*DBB*DBB-1.
H (1,JD2)=TNS1*(GJ (2) -ANM1*GJ (1) *DGG) *DGG
H (2,JD2)=TNS1*(GY (2) -ANM1*GY (1) *DGG) *DGG
H (3,JD2)=-S2*(BNS1*BJ (1) +2.*BJ (2) *DBB)
H (4,JD2)=-S2*(BNS1*BY (1) +2.*BY (2) *DBB)
H (5,JD2)=S2SG2*(BJ (1) -TNP2*BJ (2) *DBB)
H (6,JD2)=S2SG2*(BY (1) -TNP2*BY (2) *DBB)
TSGG=TS1NN1*DGG*DGG-TS1WP1
TS2B=AN*TS2*DBB
H (1,JD3)=TSGG*GJ (1) +TS1*GJ (2) *DGG
H (2,JD3)=TSGG*GY (1) +TS1*GY (2) *DGG
H (3,JD3)=TS2B*(ANM1*BJ (1) *DBB-BJ (2) )
H (4,JD3)=TS2B*(ANM1*BY (1) *DBB-BY (2) )
H (5,JD3)=TS2SG2*(BJ (1) -ANP1*BJ (2) *DBB)
H (6,JD3)=TS2SG2*(BY (1) -ANP1*BY (2) *DBB)
GO TO (3,4) ,J
C OUTER INTERFACE
3 J=2
JD=3
ALP=ALP2
BB=B2
GG=G2
GO TO 2
4 DO 5 L1=4,6
DO 5 L2=1,6
5 H(L2,L1)=-H(L2,L1)
C TO GET TRANSPOSE OF ADMASS MATRIX ALPIJ
CALL CROUTC(A,6,H,6,DETERM,6)
C ALPIJ=TRANSPOSE OF HIJ WHICH IS TRANSPOSE OF ALPIJ
DO 7 L1=1,6
DO 7 L2=1,6
7 AA(L1,L2)=H(L2,L1)
RETURN
END
C
C SUBROUTINE CONSTO
C TO SET UP THE CONSTS. FOR THE GIVEN ALP1
COMMON
C /AAA2/ N,AN,ANS,ANM1,ANP1,TN,TNP2,ANSP1
C /AAA3/ S1,S2,SIG1,SIG2,B1,B2,G1,G2
C /AAA7/ TS1NN1,TS1WP1,EPIB1,EMIB2,EMIDB,TNNM1,IARGZ
C /FLUID/ SMK1PW,RN1PW,RNEUPO
C /WALP/ W,WS,ALP1,ALP2,R21
COMPLEX*8 S1,S2,SIG1,SIG2,B1,B2,G1,G2

```

```

COMPLEX*8 TS1NN1,TS1WP1,EPIB1,EMIB2,EMIDB
COMPLEX*8 CC,G1S,B1S,RN1I,W,WS,R1SN,R2SN
COMPLEX*8 TRC/(0.,1.)/

```

```
AN=N
```

```
IF(SMK1PW .LT. 1.E-20 ) SMK1PW=0.
```

```
IF(RN1PW .GT. 1.E30 ) RN1PW=1.0E30
```

```
ALP2=ALP1*R21
```

```
TO1S=SMK1PW*SMK1PW
```

```
TO2S=TO1S*R21*R21
```

```
RWOI=(4./3.+RNEUPO)*TO1S/RN1PW*TRC
```

```
R1SN=TRC*RN1PW
```

```
R2SN=R1SN*R21*R21
```

```
ANM1=AN-1.
```

```
ANP1=AN+1.
```

```
ANS=AN*AN
```

```
ANSP1=ANS+1.
```

```
TN=2.*AN
```

```
TNNM1=TN*ANM1
```

```
TNP2=TN+2.
```

```
RETURN
```

```
ENTRY CONST1
```

C TO CALCULATE PARAMETERS FOR A GIVEN FREQUENCY, W= (WR,WI)

```
WS=W*W
```

```
A2S=ALP2*ALP2
```

```
A1S=ALP1*ALP1
```

```
CC=1./(1.+RWOI*W)
```

```
G1S=-A1S+TO1S*CC*WS
```

```
G1=CSQRT(G1S)
```

```
G2=CSQRT(-A2S+TO2S*CC*WS)
```

```
RN1I=R1SN*W
```

```
B1S=-A1S-RN1I
```

```
B1=CSQRT(B1S)
```

```
B2=CSQRT(-A2S-R2SN*W)
```

```
SIG1=ALP1/G1
```

```
SIG2=ALP1/B1
```

```
S1=-G1S/RN1I
```

```
S2=1.+A1S/RN1I
```

```
TS1WP1=2.*S1+CC
```

```
TS1NN1=TNNM1*S1
```

```
AIB2=AIMAG(B2)
```

```
IF(AIB2 .LT. -60.) GO TO 20
```

```
IARGZ=1
```

```
EPIB1=CEXP(TRC*B1)
```

```
EMIB2=CEXP(-TRC*B2)
```

```
EMIDB=EMIB2*EPIB1
```

```
RETURN
```

20 IARGZ=2

```
EPIB1=1.0E70
```

```
EMIB2=0.
```

```
EMIDB=0.
```

```
IF(AIMAG(B2-B1) .GT. -60. ) EMIDB=CEXP(TRC*(B1-B2))
```

```
RETURN
```

```
END
```

C

```
SUBROUTINE CROUTC(A,N,B,M,DETERM,NMAX)
```

C

C

```
SOLUTION OF LINEAR EQUATIONS BY THE CROUT METHOD(COMPLEX)
```

C

```
COMPLEX*16 A,B,DETERM,DOTC,V,AMAX,TEMP
```

```
REAL*8 T1,T2  
DIMENSION A(NMAX,N),B(NMAX,M)  
COMMON /F454/ V(100)  
REAL*8 DETMAX/Z6110000000000000/,DETMIN/Z2110000000000000/
```

INITIALIZE DETERMINANT

DETERM=1.0
IDET=0

PERFORM SUCCESSIVE PIVOT OPERATIONS (GRAND LOOP)

```
DO 1000 K=1, N
  KP1=K+1
  KM1=K-1
```

SEARCH FOR PIVOT ELEMENT AND EXTEND DETERMINANT PARTIAL PRODUCT

$$\dot{T}_1 = 0.0$$

DO 20 I=K, N

DO 2 L=1,K

$$2 \quad V(L) = A(I, L)$$
$$A(I, K) = A(I, K) - \text{DOTC}(V, A(1, K), \text{KM1})$$
$$T2 = CDABS(A(I,K))$$

```
IF (T2.LT.T1) GO TO 20
```

$$T_1 = T_2$$

IMAX=I

20 CONTINUE

$$A_{MAX} = A(I_{MAX}, K)$$
$$\text{DETERM}=\text{AMAX}*\text{DETERM}$$

```
IF (CDABS (DETERM) .LT. DETMAX) GO TO 30
```

$$\text{DETERM}=\text{DETERM}*\text{DETMIN}$$
$$IDET = IDET + 1$$

GO TO 40

```
30 IF (CDABS (DETERM) .GT. DETMIN) GO TO 40
```

DETERM=DETERM*DETMAX

IDET=IDET-1

40 CONTINUE

RETURN IF MATRIC IS SINGULAR (ZERO PIVOT)

```
IF (CDABS (DETERM) .EQ. 0.0) RETURN
```

INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL

```
IF (IMAX.EQ.K) GO TO 600
```

DETERM=-DETERM

DO 50 J=1,N

TEMP=A (K, J)

$$A(K, J) = A(IMAX, J)$$

```

A (IMAX,J) =TEMP

```

50 CONTINUE

IF (M.LE.0) GO TO 600

DO 400 J=1, M

TEMP=B (K, J)

$$B(K, J) = B(\text{IMAX}, J)$$

```
B (IMAX, J) = TEMP
```

400 CONTINUE

```
600 DO 666 L=1,K
```

```

666 V(L)=A(K,L)
C
C   DIVIDE PIVOT COLUMN BY PIVOT ELEMENT
C
   IF(K.EQ.N) GO TO 850
   DO 700 I=KP1,N
700 A(I,K)=A(I,K)/AMAX
C
C   REDUCE PIVOT ROW
C
   DO 800 J=KP1,N
800 A(K,J)=A(K,J)-DOTC(V,A(1,J),KM1)
850 IF (M.LE.0) GO TO 1000
   DO 900 J=1,M
900 B(K,J)=B(K,J)-DOTC(V,B(1,J),KM1)
1000 CONTINUE
C
C   BACK SUBSTITUTION INTO TRIANGULAR FORM
C
   IF(M.LE.0) GO TO 9000
   DO 8000 I=1,N
   K=N+1-I
   DO 6666 L=K,N
6666 V(L)=A(K,L)
   DO 7000 J=1,M
7000 B(K,J)=(B(K,J)-DOTC(V(K+1),B(K+1,J),I-1))/V(K)
8000 CONTINUE
9000 V(1)=IDET
   RETURN
   END
   FUNCTION DOTC(A,B,N)
   COMPLEX*16 A,B,DOTC
   DIMENSION A(1),B(1)
   DOTC=0.0
   IF(N.EQ.0) RETURN
   DO 100 I=1,N
100 DOTC=DOTC+A(I)*B(I)
   RETURN
   END
C
C   SUBROUTINE BSJY(Z,BJK,BYK,N,IARGZ)
C   BESSEL FUNCTIONS OF NTH AND N+1TH ORDER WITH A COMPLEX ARGUMENT Z
C
C   TWO ENTRY POINTS:
C   BSJY.....FOR FUNCTIONS J AND Y
C   HANKEL ...FOR HANKEL FUNCTIONS H1 AND H2
C
C   ***** INPUT PARAMETERS Z,N,IARGZ *****
C   Z=X+SQRT(-1)*Y .....ARGUMENT, COMPLEX*8
C   N .....ORDER OF THE BESSEL FUNCTIONS
C   IARGZ....INTEGER*4,CONTROL PARAMETER FOR OUTPUT FORMAT
C   =1 IS RECOMMENDED FOR A SMALL ARGUMENT FOR A NORMAL OUTPUT
C   =2 IS USED FOR A LARGE ARGUMENT Z. IN THIS CASE THE
C       EXPONENT FACTOR EXP(I*Z) OR EXP(-I*Z) IS OMITTED TO
C       AVOID OVERFLOW AND THE OUTPUT WILL BE
C       J(Z)*EXP(I*Z),Y(Z)*EXP(I*Z), IF Y>0.
C       J(Z)/EXP(I*Z),Y(Z)/EXP(I*Z), IF Y<0.
C       AND H1/EXP(I*Z),H2*EXP(I*Z)
C   ***** OUTPUT BJK(2),BYK(2) BOTH ARE COMPLEX*16 *****

```

C J(N), OR H1(N) STORED AT BJK(1)
 C J(N+1), OR H1(N+1) STORED AT BJK(2)
 C Y(N), OR H2(N) STORED AT BYK(1)
 C Y(N+1), OR H2(N+1) STORED AT BYK(2)

C REF. BESSEL FUNCTIONS WITH SOME PHYSICAL APPLICATIONS
 C BY C.J. TRANTER

C REAL*8 CNO,CN10,CNK,CN1K,GAMA,TDPI,QNO,QN10,ZZ1,ZZ2,PI,FNS
 C ,RPI,FNP1S,CHKN(10),CHKN1(10)
 C COMPLEX*16 JN,JN1,JNK,HZ,HZS,HZN,HZN1,HZNN,HZN1N1,TJCN,TJCN1
 C ,YNK,TYNK,YN1K,TYN1K,JN1K,CYO,E1,E2,E3,QNK,QN1K
 C ,QN,QN1,PN,PN1,EK21Z,CSE1,CSE2,SSE1,SSE2,EZZ1,EZZ2
 C COMPLEX*16 EIZ,HNKEL,TIZ,TIZK,BJK(2),BYK(2)
 C COMPLEX*8 Z,TRC/(0.,1.0)/
 C DATA PI,GAMA,RPI,TDPI/3.141592653589793,
 C 0.577215664901533,0.3183098862,0.6366197724/
 C ,INUM/-1/

LRT=1

10 IF(N.LT. 0) GO TO 900
 IF(N.EQ. INUM) GO TO 100

C TO INITIALIZE THE NEW CONSTANTS

IF(N.GT. 0) GO TO 20

CNO=0.

NM1F=0

NF=1

NP1F=1

CN10=1.

GO TO 25

20 NF=1

CNO=1.

IF(N.LT. 2) GO TO 24

DO 22 K=2,N

NF=NF*K

22 CNO=CNO+1./K

24 NM1F=NF/N

NP1F=NF*(N+1)

CN10=CNO+1./(N+1)

25 NM1=N-1

NP1=N+1

INUM=N

FNS=4.*N*N

FNP1S=4.*NP1*NP1

QNO=(FNS-1.)/8.

QN10=(FNP1S-1.)/8.

ZZ1=PI/4.*(2*N+1)

ZZ2=PI/4.*(2*N+3)

TRC=(0.,1.0)

EZZ1=CDEXP(-TRC*ZZ1)

EZZ2=CDEXP(-TRC*ZZ2)

CHKN(1)=2.*QNO

CHKN1(1)=2.*QN10

DO 101 K=2,10

FK=4.*K

TKM1S=(2*K-1)**2

CHKN(K)=CHKN(K-1)*(FNS-TKM1S)/FK

101 CHKN1(K)=CHKN1(K-1)*(FNP1S-TKM1S)/FK

C 100 HZ=Z*0.5

```

      HZS=HZ*HZ
      ABSVZ=CABS(Z)
      IF ( ABSVZ-10.0) 200,200,300
C
C      SMALL Z, USING SERIES EXPANSION
200 JN=1.
      JN1=1.
      JNK=1.
      CNK=CN10
      TJCN=CNO
      TJCN1=CN10
      KK=15+ABSVZ
      DO 1 K=1, KK
      JNK=-JNK*HZS/(K*(K+N))
      JN1K=JNK*NP1/(NP1+K)
      JN=JN+JNK
      JN1=JN1+JN1K
      CNK=CNK+1./K
      TJCN=TJCN+JNK*CNK
      CNK=CNK+1.0/(K+NP1)
      TJCN1=TJCN1+JN1K*CNK
      IF(CDABS(JN1K) .LE. 1.D-8) GO TO 2
1 CONTINUE
2 HZN=HZ**N
      HZNN=HZN/NF
      HZN1=HZN*HZ
      HZN1N1=HZN1/NP1F
      BJK(1)=HZNN*JN
      BJK(2)=HZN1N1*JN1
      CYO=GAMA+CDLOG(HZ)
      BYK(1)=TDPI*(CYO*BJK(1)-0.5*HZNN*TJCN)
      BYK(2)=TDPI*(CYO*BJK(2)-0.5*HZN1N1*TJCN1)
      YNK=NM1F
      YN1K=NF
      TYNK=YNK
      TYN1K=YN1K
      IF(NM1) 4,5,6
4 BYK(2)=BYK(2)-RPI/HZ
      GO TO 102
6 DO 3 K=1, NM1
      YNK=YNK*HZS/((N-K)*K)
      TYNK=TYNK+YNK
      YN1K=YNK*(N-K)
3 TYN1K=TYN1K+YN1K
5 TYN1K=TYN1K+YN1K*HZS/N
      BYK(1)=BYK(1)-RPI*TYNK/HZN
      BYK(2)=BYK(2)-RPI*TYN1K/HZN1
      GO TO (102,50), IARGZ
50 AIZ=AIMAG(Z)
      EIZ=CEXP(TRC*Z)
      IF(AIZ .LT. 0.) EIZ=1./EIZ
      DO 51 K=1,2
      BJK(K)=BJK(K)*EIZ
51 BYK(K)=BYK(K)*EIZ
102 IF(LRT .EQ. 1) RETURN
      DO 103 K=1,2
      HNKEL=BJK(K)+TRC*BYK(K)
      BYK(K)=BJK(K)-TRC*BYK(K)
103 BJK(K)=HNKEL

```

RETURN

C
C

LARGE Z, USING ASYMTOTIC EXPANSION

```

300 KK=12.-ALOG10 (ABSVZ)
    IF (KK .LE. 2) KK=2
    GO TO (104,105),LRT
104 QNK=QNO/Z
    QN1K=QN10/Z
    QN=QNK
    QN1=QN1K
    PN=1.
    PN1=1.
    DO 30 K=2, KK
    K4=K*4
    K2=K*2
    K3S=(K4-3)**2
    K5S=(K4-5)**2
    DKZ=1./(256*(K2-1)*(K2-2)*HZS)
    QNK=-QNK*(FNS-K3S)*(FNS-K5S)*DKZ
    QN1K=-QN1K*(FNP1S-K3S)*(FNP1S-K5S)*DKZ
    EK21Z=8.*(K2-1)*Z
    PN=PN+EK21Z*QNK
    PN1=PN1+EK21Z*QN1K
    QN=QN+QNK
30  QN1=QN1+QN1K
    GO TO (31,32), IARGZ
31  E1=Z-ZZ1
    E2=Z-ZZ2
    CSE1=CDCOS(E1)
    SSE1=CDSIN(E1)
    CSE2=CDCOS(E2)
    SSE2=CDSIN(E2)
    GO TO 33
32  AIZ=AIMAG(Z)
    CSE1=0.5*EZZ1
    SSE1=0.5*EZZ1/TRC
    CSE2=0.5*EZZ2
    SSE2=0.5*EZZ2/TRC
    IF(AIZ .LT. -10.) GO TO 33
    IF(AIZ .LT. 10.) GO TO 35
    SSE1=-SSE1
    SSE2=-SSE2
33  E3=1./CDSQRT(PI*HZ)
    BJK(1)=E3*(PN*CSE1-QN*SSE1)
    BYK(1)=E3*(PN*SSE1+QN*CSE1)
    BJK(2)=E3*(PN1*CSE2-QN1*SSE2)
    BYK(2)=E3*(PN1*SSE2+QN1*CSE2)
    RETURN
35  EIZ=CEXP(TRC*Z)
    IF(AIZ .LT. 0.) EIZ=1./EIZ
    E1=Z-ZZ1
    E2=Z-ZZ2
    CSE1=EIZ*CDCOS(E1)
    SSE1=EIZ*CDSIN(E1)
    CSE2=EIZ*CDCOS(E2)
    SSE2=EIZ*CDSIN(E2)
    GO TO 33
    ENTRY HANKEL(Z,BJK,BYK,N,IARGZ)
    LRT=2

```

```

      GO TO 10
105  TIZ=2.*TRC*Z
      PN=1.
      PN1=1.
      QN=1.
      QN1=1.
      TIZK=1.
      SIGN=1.
      DO 106 K=1, KK
      TIZK=TIZK*TIZ
      SIGN=-SIGN
      QNK=CHKN(K)/TIZK
      QN=QN+QNK
      PN=PN+SIGN*QNK
      QNK=CHKN1(K)/TIZK
      QN1=QN1+QNK
106  PN1=PN1+SIGN*QNK
      E3=1./CDSQRT(PI*HZ)
      GO TO (107,108), IARGZ
107  EIZ=CEXP(TRC*Z)
      BJK(1)=E3*EIZ*EZZ1*PN
      BJK(2)=E3*EIZ*EZZ2*PN1
      BYK(1)=E3/(EIZ*EZZ1)*QN
      BYK(2)=E3/(EIZ*EZZ2)*QN1
      RETURN
108  BJK(1)=E3*EZZ1*PN
      BJK(2)=E3*EZZ2*PN1
      BYK(1)=E3/EZZ1*QN
      BYK(2)=E3/EZZ2*QN1
      RETURN
900  PRINT 910, N
910  FORMAT(//10X, '***** STOP DUE TO ERROR IN BESSEL FUNCTION', /
C      10X, 'ORDER N=', I5)
      STOP 900
      RETURN
      END

```