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APERIODICITY IN ONE-DIMENSIONAL CELLULAR AUTOMATA

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ABSTRACT:

Cellular automata are a class of mathematical systems characterized by discreteness (in space, time, and state values), determinism, and local interaction. A certain class of one-dimensional, binary site-valued, nearest-neighbor automata is shown to generate infinitely many aperiodic temporal sequences from arbitrary finite initial conditions on an infinite lattice. The class of automaton rules that generate aperiodic temporal sequences are characterized by a particular form of injectivity in their interaction rules. Included are the nontrivial "linear" automaton rules (that is, rules for which the superposition principle holds); certain nonlinear automata that retain injectivity properties similar to those of linear automata; and a wider subset of nonlinear automata whose interaction rules satisfy a weaker form of injectivity together with certain symmetry conditions. A technique is outlined here that maps this last set of automata onto a linear automaton, and thereby establishes the aperiodicity of their temporal sequences.

APERIODICITY IN ONE-DIMENSIONAL CELLULAR AUTOMATA

§1 Introduction

Cellular automata are a class of mathematical systems characterized by discreteness (in space, time, and state values), determinism, and local interaction. A cellular automaton consists of a lattice of sites whose values are restricted to a finite (typically small) set of integers $Z_k = \{0, 1, \cdots, k-1\}$. The value of each site at any time step is then determined as a function of the values of the neighboring sites at the previous time step. The general form of a one-dimensional cellular automaton, for example, is given by

$$x_i^{t+1} = f(x_{i-r}^t, \cdots, x_i^t, \cdots, x_{i+r}^t), \qquad f : Z_k^{2r+1} \rightarrow Z_k \tag{1.1}$$

where x_i^t denotes the value of site i at time t, f represents the "rule" defining the automaton, and r is a non-negative integer specifying the radius of the rule. The simplest cellular automata are those with $r = 1$ and $k = 2$; designated by Wolfram [1] as "elementary," these automata are defined on a one-dimensional spatial lattice, and consist of binary-valued sites evolving in time according to a nearest-neighbor interaction rule.

Cellular automata's distinctive set of features has attracted, in recent years, substantial attention as simple models for complex physical and biological phenomena ([2-5]). In particular, cellular automata can be viewed as prototypical models for systems consisting of a large number of simple, identical, and locally connected components. Examples of phenomena that have been modeled using cellular automata include turbulent flow resulting from the collisions of fluid molecules, dendritic growth of crystals resulting from aggregation of atoms, and patterns of electrical activity in simple neural networks resulting from neuronal interaction. Such problems are conventionally studied using continuous models based on partial differential equations. Models based on cellular automata differ from, but are qualitatively and to some extent quantitatively consistent with, those obtained from a continuous approach. Simulations based on cellular automata may provide an increase in computational efficiency, as well as insight into the relation between continuous and discrete modeling of the phenomena being studied.

In addition to providing modeling tools, cellular automata represent intriguing and little-understood mathematical systems. At present, few tools exist for the analysis of their behavior. Problems in cellular automata research pose special difficulties since they often fall outside the purvey of traditional continuous mathematics; cellular automata problems typically reflect, in both their formulation and their solution, the features of discreteness and local interaction that make these systems distinctive. Although some cellular automata are clearly equivalent to other standard mathematical constructs, including shift-commuting maps and finite-difference schemes for solving partial differential equations, others are not. Many automata, such as that defined by

$$x_i^{t+1} = x_{i+1}^t \dot{+} \max(x_i^t, x_{i-1}^t), \tag{1.2}$$

where "$\dot{+}$" denotes addition modulo 2, cannot easily be identified as discretizations of a continuous system.

In fact, the evolution of a typical cellular automaton is governed typically not by a function expressed in closed-form, but by a "rule table" consisting of a list of the discrete states that occur in an automaton together with the values to which these states are to be mapped in one iteration of the rule. (The rule table can be converted to a function involving Boolean expressions but such re-formulation is not usually profitable.) Not much of calculus applies to such systems.

Thus, while it is natural to view cellular automata as dynamical systems, problems arise in that many concepts standard to that field (for example, "stability," "attractors," "sensitive dependence," "chaotic behavior") do not have unambiguous analogues in this new context. In particular, the lack of a natural metric in either cellular automaton state space or cellular automaton rule space makes it difficult to translate these concepts for cellular automata and other spatially extended systems defined with a finite set of state values. In recent years, a number of simulation studies [1,2] have been undertaken to identify appropriate statistical quantities for cellular automata – analogous to, say, dimensions, entropies, and Lyapunov exponents – and to use these quantities to characterize the dynamical behavior of these systems.

This paper focuses on one aspect of cellular automata for which the analogy with continuous systems is relatively clear; namely, the periodicity of the behavior they generate. The question to be considered is, more precisely, the *aperiodicity* of the sequence of values $\{x_i^t, t = 0, 1, \cdots\}$ assumed by a particular site x_i under successive iterations of the automaton rule. The purpose of the paper is not to provide a systematic or exhaustive study of aperiodicity, but to present a collection of analytical results representative of the phenomenon for these systems. A major theme throughout is the distinction between *linearity* and *nonlinearity*, and the development of techniques that permit results on aperiodic behavior in linear automata to be extended to certain nonlinear cases.

The organization of the paper is as follows. Section 2 provides definitions and other introductory material on elementary cellular automata. Section 3 considers the behavior of linear rules and a certain subset of nonlinear but "injective" rules applied to finite initial conditions (that is, initial conditions with a compact support) on an infinite lattice, and in particular establishes that these rules generate infinitely many aperiodic temporal sequences. Section 4 outlines an exact linearization technique that leads to the extension of this result to a wider class of nonlinear automata. Concluding remarks are made in Section 5.

§2 Preliminaries

This section contains definitions and introductory discussion as groundwork for the sections which follow.

An "elementary" [1] (that is, nearest-neighbor, binary site-valued) cellular automaton is defined by

$$x_i^{t+1} = f(x_{i-1}^t, x_i^t, x_{i+1}^t), \quad f : \{0,1\}^3 \to \{0,1\},$$

where x_i^t denotes the value of site i at time t, and f represents the "rule" defining the automaton.

Since the domain of f is the set of 2^3 possible 3-tuples, the rule function f is completely defined by specifying the "rule table" of values $a_i \in \{0,1\}$ with $i = 0, 1, \cdots, 7$ such that

$$000 \rightarrow a_0,\ 001 \rightarrow a_1,\ \cdots, 111 \rightarrow a_7,$$

where $xyz \rightarrow a_i$ indicates that $f(xyz) = a_i$. There is a total of 256 distinct elementary rules.

The conventional labelling scheme [1] assigns the integer

$$R = \sum_{i=0}^{i=7} a_i 2^i$$

to the elementary rule defined by f. The rule number thus assumes an integer value between 0 and 255.

The major focus of the sections which follow will be on the aperiodic behavior of temporal sequences generated by automaton on infinite lattices. Note that in the case of automaton rules operating on finite spatial lattices, the discreteness of state values and determinism of the interaction rule imply that all initial conditions will be attracted into "limit cycle" behavior; that is, aperiodic behavior cannot occur. Here, a limit cycle of period p on a cylinder size n is defined to be a set of spatial sequences $\{x_i^t; i = 0, 1, \cdots n-1; t = T, T+1, \cdots, T+p-1\}$ such that $x_i^t = x_i^{t+p}$ for all $t \geq T$; that is, a set of spatial sequences on the cylinder that repeat themselves periodically in time.

For clarity, notational distinctions will be made between an automaton's *spatial sequences* S^t with components $\{S_i^t \equiv x_i^t; -\infty < i < \infty\}$ at time t, and the *temporal sequences* W_i with components $\{W_i^t \equiv x_i^t, t = 0, 1, \cdots\}$ representing the values assumed by the site x_i with successive iterations of the automaton rule.

Definition: The temporal sequence W_i is periodic of period $0 < p < \infty$ with transience $0 < T < \infty$ if, $x_i^{t+p} = x_i^t$ for $t > T$.

It will be assumed that the automaton rules are operating on infinite lattices with arbitrary finite initial conditions; that is, arbitrary spatial sequences of finite length with compact support.

Definition: A finite initial condition on an infinite lattice is an initial condition $\{x_i^0, -\infty < i < \infty\}$ such that for some finite M, N, $x_i^0 = 0$ for $i < M$ and $i > N$ with $x_M^0 = x_N^0 = 1$.

Finally, a distinction will be made throughout between "linear" and "nonlinear" automaton rules.

Definition: A rule R is defined to be linear if it satisfies the additivity condition: that is, for any 3-tuples y and z, the function f defining the rule R satisfies

$$f(y) \dot{+} f(z) = f(y \dot{+} z),$$

where "$\dot{+}$" denotes binary addition.

Linear elementary automata include Rules 0 (zero rule), 15 (right-shift with toggle rule), 51 (toggle rule), 60, 90, 105 (sum-rule with toggle), 150 (sum-rule), 170 (left-shift rule), and 240 (identity rule); together with their equivalents under symmetry transformations.

From the point of view of dynamical systems, most linear rules generate extremely simple behavior. Extensive work [6,7] has been done, on the other hand, on the behavior of the "nontrivial" linear rules (that is, Rules 60, 90, and 150) operating on finite lattices with periodic or fixed boundary conditions. Techniques have been developed for enumeration of their limit cycles, and computation of maximal limit cycle periods and maximal transience length. It has further been shown [7] that a given spatial sequence appears in limit cycles for linear rules iff its values satisfy a linear recurrence relation defined by the automaton rule, and thus the considerable mathematical machinery of recurring sequences over finite fields can be used for for the analysis of the detailed structure of limit cycle sequences.

Nonlinear cellular automata are not susceptible to the approaches of [6,7] since many of those results derive from algebraic properties of the linear operator representing the automaton rule, and further, the inability to invoke the superposition principle cripples the analysis of the behavior of nonlinear rules for arbitrary initial conditions. In the following sections, a simple result establishing aperiodic behavior for a linear rule on infinite lattices will be shown to hold for certain nonlinear rules as well.

In extending results from linear to nonlinear rules, a central concept will be that of a certain type of "injectivity" as a feature of certain nonlinear as well as linear rules. An automaton rule will be said to be *injective in a particular component* if, for every 3-tuple, the component uniquely determines the value assigned by the rule tables to that tuple (assuming fixed values for the other components). The precise definition is as follows:

Definition: A rule R is injective in the $(i+k)$-th component $(k=-1,0,1)$ if for every tuple $(x_{i-1}x_i x_{i+1})$, the rule table for R represents a one-to-one mapping between x_{i+k} and $f(x_{i-1}x_i x_{i+1})$ when the two other components $x_{i+j}, j \neq k$ are fixed.

Thus, for example, Rule 15 defined by

$$\{100,101,110,111\} \to 0, \qquad \{000,001,010,011\} \to 1,$$

representing the action of a right-shift with toggle, is injective in the $(i-1)$-th component; and Rule 150 defined by

$$\{000,011,101,110\} \to 0, \qquad \{001,010,100,111\} \to 1,$$

is injective in all three components. these two automata.

The following proposition asserts that all linear automaton rules (with the exception of Rule 0) are injective in at least one component. The result is stated for elementary automata, but applies to rules of arbitrary neighborhood size with sites assuming values in $F_q = \{0,1,\cdots,q-1\}$, where q is a prime power. .

Proposition 1. Let R be a linear automaton rule. Then either R is injective in at least one component, or R is the rule that maps all tuples to 0.

Proof: Suppose R is not injective in any component. Then for instance there must exist $x, y \in \{0,1\}$ such that

$$f(xy0) = f(xy1),$$

where f is the interaction rule defining R. The additivity principle of linear rules implies that

$$\begin{aligned} f(001) &= f(xy0 \dot{+} xy1), \\ &= f(xy0) \dot{+} f(xy1), \\ &= 0. \end{aligned}$$

Similarly, $f(010) = f(100) = 0$, from which it follows using superposition that all tuples are mapped to 0.

The converse is not true. Rule 30, for example, defined by

$$\{000, 101, 110, 111\} \rightarrow 0, \qquad \{001, 010, 011, 100\} \rightarrow 1,$$

is injective in the $(i-1)$-th component, but does not satisfy the definition of linearity.

§3 Aperiodic Behavior of Injective Rules on Infinite Lattices

In this section, it will be shown that, starting from arbitrary finite initial conditions on infinite lattices, a certain class of elementary automaton rules generates infinitely many aperiodic temporal sequences; that is, sequences that are not periodic of any period. This class of rules is characterizable as exhibiting injectivity together with a feature that ensures infinite-time "propagation" of the automata.

The aperiodicity of temporal sequences generated of a particular nontrivial linear rule will be singled out for discussion here. The rule is defined by

$$\{000, 010, 101, 111\} \rightarrow 0, \qquad \{001, 011, 100, 110\} \rightarrow 1, \tag{3.1a}$$

alternatively specified by the functional form

$$x_i^{t+1} = x_{i-1}^t \dot{+} x_{i+1}^t, \tag{3.1b}$$

to be referred to hereafter as Rule 90. Figure 1 provides an example of the rule's evolution on a finite initial. Note that Rule 90 is injective in both its $(i-1)$ and $(i+1)$ components. A simple argument (easily extended to the case of the other nontrivial linear automata, including Rules 60 and 150) establishes the aperiodicity of all temporal sequences generated by Rule 90 (with the exception of a trivial case to be described below). The proof rests on showing that there exists a K such that every temporal sequence contains a subsequence of 2^k 0's for all $k > K$.

Proposition 2: With the exception of the trivial case, every temporal sequence generated by Rule 90 with arbitrary finite initial conditions on an infinite lattice is aperiodic. The trivial case is the temporal sequence of all 0's generated by Rule 90 from an initial condition that is spatially symmetric, of odd length, with the central component being 0.

Proof: Let the initial condition be specified by $\{x_i^0, -\infty < i < \infty\}$ with $x_M^0 = x_N^0 = 1$ being the "leftmost" and "rightmost" 1's. For convenience, assume $M < 0 < N$. From

(3.1), it can be seen that the left and right "borders" of 1's propagate with unit speed. Also from (3.1), it follows that

$$x_i^t = x_{i-t}^0 \dot{+} x_{i+t}^0 \tag{3.2}$$

whenever t is a power of 2. Consider an arbitrary temporal sequence W_j and assume, without loss of generality, $j > 0$. Define K to be a positive integer such that

$$j - 2^K < M \qquad \text{and} \qquad j + 2^K > N.$$

Then (3.2) implies that for any k such that $k \geq K$,

$$x_j^t = 0, \qquad \text{for } 2^k \leq t \leq 2^k + \min[2^k - j - M, 2^k + j - N],$$

since the values of the temporal sequence W_j in this range are given as sums of the values of the temporal sequences W_{j-2^k} and W_{j+2^k}, each of which consists of 0's until terminated by the propagation of the left and right borders. (Termination on the left occurs at $t = 2^k - j - M$ and on the right at $t = 2^k + j - N$.) Moreover the temporal sequence W_j cannot have all values equal to 0 since the the spatial sequence generated at time step 2^k consists of two "copies" of the initial condition with a block of 0's in the center, and the two 1's bordering the block propagate toward the center with unit speed, thereby generating a 1 at each site (except possibly the center). Since the result holds for k arbitrary large, it follows that the temporal sequence W_j is aperiodic except in the trivial case where it is the "center" sequence of all 0's generated by an odd-length spatially symmetric initial condition.

The above result depends on the use of the (essentially) linear representation of Rule 90. In what follows, it will be shown that by discarding linearity but retaining injectivity, certain nonlinear rules exhibit aperiodic behavior as well.

Consider now the class of automata injective in either their $(i-1)$-th or their $(i+1)$-the component. Each such class contains a total of 2^4 elementary rules, with linear Rules 90 and 150 the only automata belonging to both classes.

In order to establish aperiodicity of temporal sequences generated by these injective rules, a lemma [8] is needed. The lemma asserts that the existence of two periodic temporal sequences anywhere in an automaton implies the periodicity of every temporal sequence in between. Note that the lemma makes no assumption on the automaton rule (in other words, it is not restricted to injective rules).

Lemma: Let W_i and W_j, $i < j$, be two temporal sequences periodic of periods p_i, p_j, respectively. Then for $i < k < j$, the temporal sequence W_k is periodic of period $p \mid \mathrm{lcm}(p_i, p_j)$ (where $x|y$ indicates that x divides y evenly), and of transience bounded by $T < \infty$, where T is a constant independent of k.

On the basis of the above lemma, the following proposition [8] asserts that (propagating) automaton rules belonging to Class A and B generate at most one aperiodic temporal sequence.

Proposition 3: Let R be a rule injective in its $(i+1)$-th component with $\{100\} \to 1$ (or injective in its $(i-1)$-th component with $\{001\} \to 1$). Then with arbitrary finite initial conditions, there can exist at most one periodic temporal condition.

Proof: The argument will be given for rules injective in their $(i+1)$-th components. The fact that $\{100\} \to 1$ implies that the "rightmost" 1 in the initial condition propagates with unit speed to the right. Suppose that within the "borders" of the automaton there exists a periodic temporal sequence W_i with transience T_i, and a periodic sequence W_j with transience T_j, with $i < j$. Then the lemma establishes that every sequence W_k is also periodic with some transience $T < \infty$ for $i < k < j$. In particular, W_{j-1} – that is, the temporal sequence immediately to the "left" of W_j is periodic with transience T. The fact that R is injective in its $(i+1)$-th component (the "mirror" argument holds for injectivity on the left) implies that the values of the two adjacent temporal sequences W_{j-1} and W_j determine the values (and hence the periodicity properties) of the temporal sequence W_{j+1}. Hence that temporal sequence is periodic with transience T, as is in fact *every* temporal sequence to the right of W_j. A contradiction results however since there must exist beyond the righthand "border" a temporal sequence that is not periodic with transience T.

The conditions of the above theorem hold for elementary automata Rules 30, 86, 90, 150, 154, 210. The nature of the aperiodicity of temporal sequences generated by these automaton rules differs significantly from case to case. In the case of Rule 90 (the purely linear rule) evolving on, say, an initial condition consisting of a single non-zero site, the aperiodicity of the temporal sequences takes the form of an orderly sequence of 1's separated by sequences of 0's with lengths that are powers of 2. By contrast, the temporal sequences generated by Rule 30 (depicted in Figure 2) are aperiodic in a much stronger sense, and have been proposed by Wolfram [9] as highly efficient pseudo-random number generators.

§4 Aperiodicity in Nonlinear Rules That Mimic Rule 90

As has been noted elsewhere [10,11], several nonlinear rules – in particular, Rules 18, 22, 122, 146, 182 – have been observed to "simulate" Rule 90 in that their behaviors coincide when restricted to certain spatial subsequences. In addition, as indicated by the numerical simulations in [10], the evolution of these rules on an infinite lattice can be viewed as consisting of multiple "domains" within which the system emulates Rule 90. In the infinite case, the positions of the domain walls vary in an ostensibly random fashion, with collision of walls leading to annihilation and merging of domains. This annihilation process leads gradually to the emergence of a more and more "ordered" configuration, much as in the case of spontaneous symmetry breaking [10].

A technique is outlined here (see [12] for details) for the exact linearization of nonlinear one-dimensional automata rules that "mimic" Rule 90 in a sense to be defined below. The technique will be discussed in detail for Rule 18 whose rule table is given by

$$\{000, 010, 011, 101, 110, 111\} \to 0, \qquad \{001, 100\} \to 1. \tag{4.1}$$

Like the automaton rules of the previous section, Rule 18 exhibits injective behavior on the subsets

$$\{00*\}, \{10*\}$$

and

$$\{*00\}, \{*01\},$$

where "$*$" denotes the two values $\{0,1\}$, and as before, "injective" implies that the tuples in each subset are mapped to pairwise differing values. Unlike those rules, however, Rule 18 maps

$$\{01*\}, \{11*\}$$

and

$$\{*10\}, \{*11\}$$

to pairwise equal values; it is in the values to which it maps these subsets – in particular, the tuples $\{011, 110\}$ – that Rule 18 differs from Rule 90. (The mapping of the tuple 111 can be ignored here since it is easy to show that the rule never generates more than two adjacent 1's.)

Hence for any sequence consisting only of isolated 1's (separated by 0's), Rule 18 mimics exactly the behavior of Rule 90. Figure 3 depicts the evolutions of Rules 18 and 90 on various finite initial conditions. It is easy to see that the a pair of adjacent 1's is generated in the evolution of Rule 18 whenever, and only whenever, a block of 0's of even length has been generated at some previous time step. In the discussion that follows, such a block will be termed a "deviant" block.

Suppose now an arbitrary spatial sequence is to be iterated for one time step under Rule 18. Assume for the moment that the deviant blocks in the sequence are sufficiently separated (in space) so as not to affect one another. The action of Rule 18 and Rule 90 is identical on all but the deviant blocks. Moreover, since the definition of Rule 18 specifies that

$$\{000, 010, 011, 101, 110\} \rightarrow 0,$$

it follows that the insertion of an extra 0 in each deviant block, and subsequent application of Rule 18, produces a spatial sequence that is identical to the result of applying Rule 90 to the transformed sequence, and yet faithfully reproduces – except with the retention of the extra 0 – the effect of applying Rule 18 to the original sequence. For example, suppose that the sequence contains the block 0001101, which under iteration by Rule 18 generates 01000. The transformed sequence is given by 00010101, which under iteration by either Rule 18 or Rule 90, generates $010\tilde{0}00$, where $\tilde{0}$ denotes the inserted 0.

Thus, Rule 18 possesses the property that a subset of its rule table is injective (and in fact identical to the linear Rule 90), and the remainder – the part that deviates from linearity – collapses onto 0. This combination of features permits linearization by the procedure consisting of the following steps:

(i) identification of the subsequences on which the action of the rule deviates from linearity;

(ii) transformation of the spatial lattice (specifically, insertion of new site values) so as to exclude the occurrence of these subsequences while leaving the evolution of the rest of the system undisturbed;

(iii) analysis of the resultant linear system.

To summarize, the linearization procedure relies upon transformation of the nonlinear system's spatial lattice so as to induce the rule to mimic the evolution of a linear rule on the transformed lattice while preserving its own evolution on the unperturbed lattice. The transformation has the effect of suppressing the occurrence of spatial subsequences upon which the action of the rule deviates from linearity ("deviant blocks"), and can be defined whenever these subsequences possess certain symmetry properties related to the injectivity properties of the nonlinear rule table. The technique may be applied to the analysis of nonlinear rules evolving on either finite or infinite lattices. In the finite case, it permits the analysis of the limit cycle structure (including features such as transience length and maximal limit cycle period) [12]; in the infinite case, as will be discussed here, it leads to an extension of the results on aperiodicity of temporal sequences.

As suggested by the discussion above, the linearization of Rule 18 depends critically on the definition of a quantity to be denoted c_t^* that represents the number of deviant blocks in a spatial sequence S^t. Precisely, define

$$c_t^* = \text{ number of even-length blocks of 0's in sequence } S^t,$$

noting that 11 is encompassed in this definition.

Further, define a transformation $G(S)$ that acts upon a spatial sequence S^t so as to insert an extra zero into each of the deviant blocks, thereby converting it into an odd-length block. The transformation G applied to the sequence, for example,

$$S \equiv \cdots 0100011010010000010 \cdots$$

produces the sequence

$$G(S) = \cdots 010001010100010000010 \cdots .$$

Note that the transformation is many-to-one.

Finally, for any spatial sequence S, let $R_\alpha^\beta[S]$ denote the spatial sequence generated by iteration of the automaton with rule number α for β time steps on the sequence S. For example, $R_{18}^T[S]$ denotes the result of T iterations of Rule 18 applied to the sequence S.

In [12], the following proposition is established.

Proposition 4. For Rule 18 evolving on (either finite or infinite) lattices with arbitrary initial conditions,

(i) the quantity c_t^* is monotonically non-increasing as a function of t;

(ii) for any range of iterations $[T_0, T_1]$ for which the quantity c_t^* is conserved, the sequences S^t evolving under Rule 18 "mimic" the sequences $G(S^t)$ evolving under Rule 90; that is,

$$G(R_{18}^t[S^{T_0}]) = R_{90}^t[G(S^{T_0})],$$

for $0 \leq t \leq T_1 - T_0$.

The above proposition asserts that in any range of iterations for which the number of deviant blocks is conserved, the evolution of Rule 18 may be mapped onto the evolution of Rule 90 by inserting an extra 0 into each of the deviant blocks. The analysis of Rule 18 is then achieved by characterizing the behavior of the appropriate linear automaton, and then "inverting" the transformation to re-capture the features of the original nonlinear system. The many-to-one nature of the transformation G, and its implications for the limit cycle structure for Rule 18, is the subject of much of the discussion in [12]. In the context of this paper, however, the inversion procedure is not necessary to characterize the periodicity properties of temporal sequences generated by Rule 18. The following proposition establishes the desired result.

Proposition 5: With the exception of the trivial case, every temporal sequence generated by Rule 18 with arbitrary finite initial conditions on an infinite lattice is aperiodic. The trivial case is the temporal sequence of all 0's generated by Rule 18 from an initial condition that is spatially symmetric, with all 0-blocks of odd length, and the central component being 0.

Proof: The proof closely resembles that used to prove the aperiodicity of temporal sequences generated by Rule 90, but requires a slight modification reflecting the lack of a linear operator representation for Rule 18.

Since for arbitrary initial conditions evolving under Rule 18, the quantity c_t^* representing the number of deviant blocks is a monotonically non-increasing function of t, it follows that it converges after some finite t^* iterations some integer $c^* \geq 0$. Take the infinite spatial sequence at time t^* to be the initial condition S^0 for the automaton. For all subsequent iterations, the evolution of Rule 18 mimics that of Rule 90 in the sense described by Proposition 4.

Suppose there exists a temporal sequence W_j which is periodic of period $p < \infty$ under Rule 18, and for convenience, suppose that $j > 0$ and that the sites in the initial condition S^0 are labelled so that x_0 lies in the "center" of the nonzero portion of S^0. (As before, x_M^0 and x_N^0 with $M < 0 < N$ are the left- and rightmost 1's in the initial condition.)

Consider now the evolution under Rule 90 of the transformed initial condition $G(S^0)$. From the relation (3.2) provided in the proof of Proposition 2,

$$R_{90}^t[G(S^0)]_i = G(S^0)_{i-t} \dot{+} G(S^0)_{i+t}, \tag{4.2}$$

for all i and any value of t that is a (sufficiently large) power of 2. The above implies that at each such time step t, the spatial sequence $R_{90}^t[G(S^0)]$ consists of two "copies" of the initial condition $G(S^0)$ separated by $t-1$ 0's (see, for example, Figure 1).

Let $K = 2^k$ be a power of 2 that

$$j - K < M, \qquad j + K > N, \qquad \text{and} \qquad K > p, \tag{4.3}$$

where p is the period of the temporal sequence W_j. Proposition 4 implies that the spatial sequence $R_{18}^K[S^0]$ mimics the sequence $R_{90}^K[G(S^0)]$. The discrepancies between the two sequences occur either in the "copies" of S^0 appearing in Rule 18, or in the length of the center block of 0's (the Rule 18 center block may be shorter by 1 unit than the Rule 90 block). Regardless of the discrepancies, the temporal subsequence W_j is entirely contained in the "cone" generated by the spatial subsequence of consisting of the block of 0's together with its two bordering 1's. From the definition of Rule 18, it follows that

$$x_j^t = 0 \qquad \text{for } K \le t < K + (\frac{2K - M - N - 1}{2} - j),$$

where the quantity in the parentheses represents the number of iterations within the cone for which the temporal sequence W_j is necessarily equal to 0. The values of W_j cannot be identically equal to 0, however, since the 1's on the border of the cone propagate with unit speed toward the center, and therefore

$$x_j^{K-j} = 1.$$

This relation holds for all powers of 2 greater than or equal to K, and hence there can be no periodic temporal sequence for Rule 18 with the possible exception of the center temporal sequence generated by a spatially symmetric initial condition with no deviant blocks and a 0 in its central site.

A similar result can be obtained for the other nonlinear rules (including Rules 22, 126, 146, 182) that mimic Rule 90.

§5 Concluding Remarks

The preceding sections have established the aperiodicity of the temporal sequences generated by a certain class of one-dimensional, binary site-valued, nearest-neighbor automata evolving from arbitrary finite initial conditions on an infinite lattice. Included in this class are those linear (here "linearity" implies that the superposition principle holds) automaton rules whose evolutions are nontrivial in that they represent something other than a simple shift, the identity, or the zero rule ; certain nonlinear rules exhibiting an "injective" property together with a feature that ensures infinite-time propagation; and a subset of nonlinear rules that exhibit linear behavior when restricted to certain subspaces, and thereby mimic the behavior of linear rules when applied together with a well-defined transformation to arbitrary spatial sequences.

It should not be concluded from the above that aperiodicity is in any sense generic to cellular automaton rules. As mentioned earlier, all automaton rules operating on finite lattices (with either fixed or periodic boundary conditions) generate limit cycle (that is, periodic) behavior. Even on infinite lattices, the systematic simulation studies performed by Wolfram [1] indicate that many automata evolve either to a homogeneous state or to limit cycle behavior (specifically, the generation of fixed domain walls within which the automaton undergoes periodic behavior) or to behavior that is extremely complicated but possibly not aperiodic. (Moreover, the periodicity properties of temporal sequences for these rules may well depend on the details of the initial conditions used. A simple example is that of a shift rule.) Loosely speaking, such automata may be viewed as possessing "dissipative" characteristics that force the contraction of arbitrary initial conditions onto limit cycle attractors. Again loosely speaking, the automaton rules that generate aperiodicity are distinctive in that the special "injective"-like features of their rule tables preclude such contractive behavior.

The connection between the aperiodicity of temporal sequences in cellular automata and chaotic behavior in dynamical systems is as yet unclear. (Aperiodicity of course encompasses a much broader set of behavior.) Given a continuous dynamical system that exhibits chaotic behavior, it is probably correct to assume that aperiodicity should appear as a feature of any cellular automaton representing an "appropriate" discretization of that system. Since procedures do not yet exist (and may not in general be feasible) for construction of continuous systems corresponding to arbitrary cellular automata, the implications in reverse are not known.

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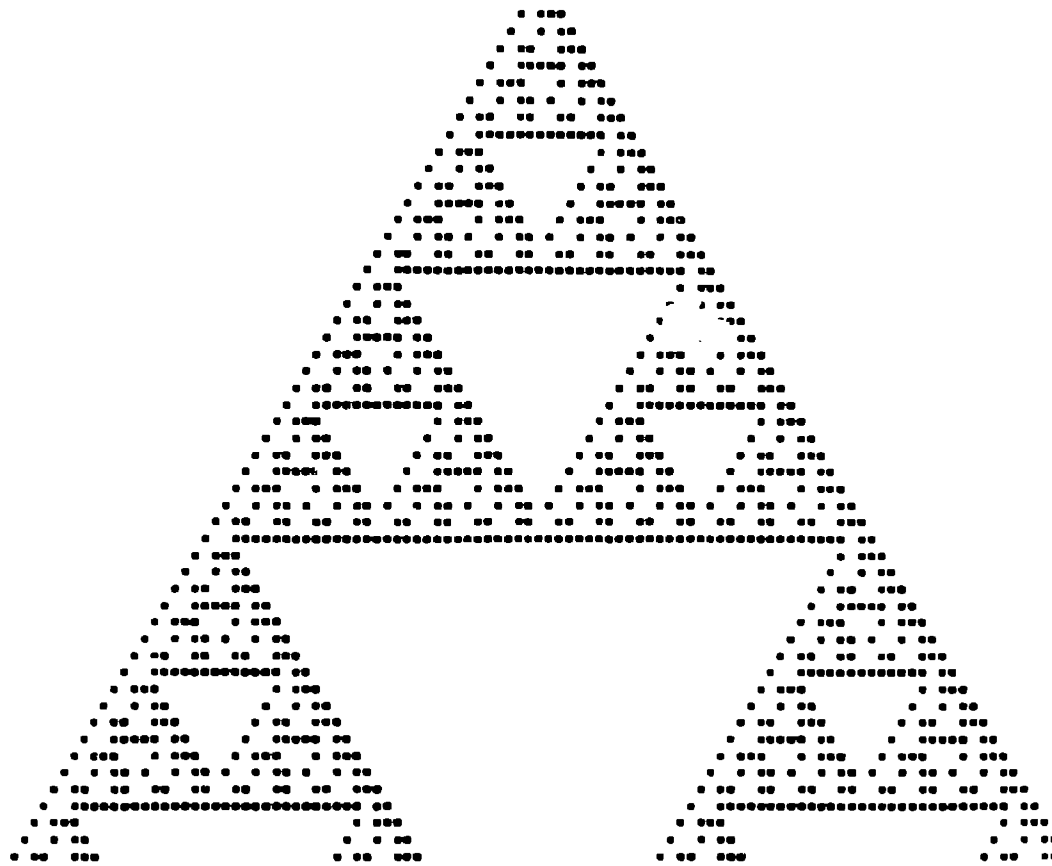

Figure 1. Evolution of Rule 90 defined by

$$\{000, 010, 101, 111\} \rightarrow 0, \qquad \{001, 011, 100, 110\} \rightarrow 1,$$

from a finite initial condition on an infinite lattice. In the figure, each row of dots represents the spatial sequence generated at time t, with the top row corresponding to $t = 0$. A dot represents a site value of 1, a blank represents a site value of 0. Proposition 2 establishes the aperiodicity of temporal sequences ("columns" in this figure) generated by Rule 90 from arbitrary finite initial conditions.

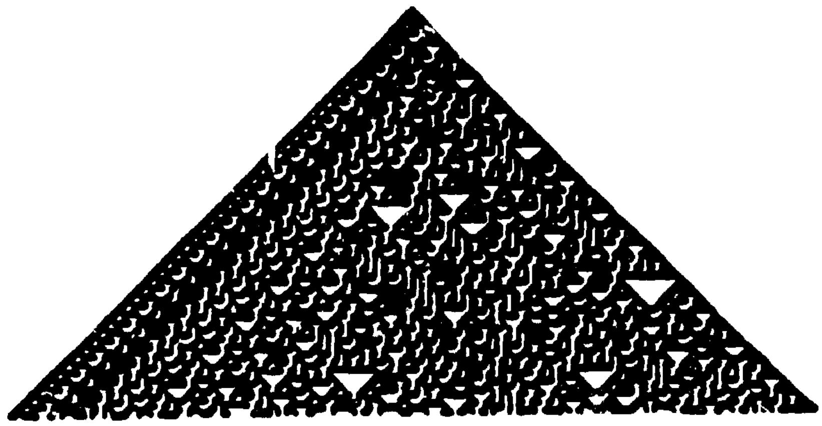

Figure 2. Evolution of Rule 30 defined by

$$\{000, 101, 110, 111\} \rightarrow 0, \qquad \{001, 010, 011, 100\} \rightarrow 1.$$

Proposition 3 establishes that, with arbitrary finite initial conditions, at most one temporal sequence can be periodic.

Figure 3. Evolutions of Rule 90 defined by

$$\{000, 010, 101, 111\} \rightarrow 0, \qquad \{001, 011, 100, 110\} \rightarrow 1,$$

and Rule 18 defined by

$$\{000, 010, 011, 101, 110, 111\} \rightarrow 0, \qquad \{001, 100\} \rightarrow 1.$$

Proposition 5 establishes the aperiodicity of temporal sequences generated by Rule 18 from arbitrary finite initial conditions.
a) Rule 18 "mimics" Rule 90 in the absence of "deviant blocks"; that is, blocks of 0's of even length.
b) The mimicking fails in the presence in the Rule 18 automaton of deviant blocks: the transformation G that inserts an extra 0 into each such block results in a mapping of the automaton into the automaton a).
c) For arbitrary initial conditions evolving under Rule 18, the number of deviant blocks is monotonically non-increasing and converges to a finite non-negative integer. In this case, there are two deviant blocks in the initial condition whose collision and annihilation at $t = 5$ precludes the occurrence of deviant blocks anywhere in the automaton's subsequent evolution.

```
                              1010001000001                                      t=0
                             100010101000101                                     t=1
                            10101000001010001                                    t=2
                           1000001000100010101                                   t=3
                          101000101010101000001                                  t=4
                         10001010000000001000101                                 t=5
                        1010100010000000101010001                                t=6
                       100000101010000010000010101                               t=7
                      10100010000010001010001000001                              t=8
                     1000101010001010100010101000101                             t=9
                    101010000010100000101000001010001                            t=10
                   10000010001000100010001000100010101                           t=11
                  1010001010101010101010101010101000001                          t=12
                 100010100000000000000000000000001000101                         t=13
                10101000100000000000000000000000101010001                        t=14
               1000001010100000000000000000000010000010101                       t=15
              101000100000100000000000000000001010001000001                      t=16
             10001010100010100000000000000000100010101000101                     t=17
            1010100000101000100000000000000010101000001010001                    t=18
           100000100010001010100000000000001000001000100010101                   t=19
          10100010101010100000100000000000101000101010101000001                  t=20
         1000101000000000100010100000000010001010000000001000101                 t=21
        101010001000000010101000100000001010100010000000101010001                t=22
       10000010101000001000001010100000100000101010000010000010101               t=23
      1010001000001000101000100000100010100010000010001010001000001              t=24
     100010101000101010001010100010101000101010001010100010101000101             t=25
```

Figure 3a.

```
101001000001                                                     t=0
10001101000101                                                   t=1
1010100001010001                                                 t=2
100000100100010101                                               t=3
10100010110101000001                                             t=4
1000101000000001000101                                           t=5
101010001000000101010001                                         t=6
10000010101000010000010101                                       t=7
1010001000001001010001000001                                     t=8
100010101000101100010101000101                                   t=9
10101000001010000101000001010001                                 t=10
1000001000100010010001000100010101                               t=11
101000101010101011010101010101000001                             t=12
10001010000000000000000000000001000101                           t=13
1010100010000000000000000000000101010001                         t=14
100000101010000000000000000000010000010101                       t=15
10100010000010000000000000000001010001000001                     t=16
1000101010001010000000000000000100010101000101                   t=17
101010000010100010000000000000010101000001010001                 t=18
10000010001000101010000000000001000001000100010101               t=19
1010001010101010000010000000000101000101010101000001             t=20
100010100000000010001010000000010001010000000001000101           t=21
10101000100000001010100010000001010100010000000101010001         t=22
1000001010100000100000101010000100000101010000010000010101       t=23
101000100000100010100010000010010100010000010001010001000001     t=24
10001010100010101000101010001011000101010001010100010101000101   t=25
```

RULE 90

```
101001000001                                                     t=0
10001101000101                                                   t=1
1010111001010001                                                 t=2
100001011100010101                                               t=3
10100100101101000001                                             t=4
1000110110011001000101                                           t=5
101011101111111101010001                                         t=6
10000101010000001000010101                                       t=7
1010010000010000101001000001                                     t=8
100011010001010010001101000101                                   t=9
10101110010100011010111001010001                                 t=10
1000010111000101110001011100010101                               t=11
101001001011010010110100101101000001                             t=12
10001101100110011001100110011001000101                           t=13
1010111011111111111111111111111101010001                         t=14
100001010100000000000000000000001000010101                       t=15
10100100000100000000000000000000101001000001                     t=16
1000110100010100000000000000000010001101000101                   t=17
101011100101000100000000000000001010111001010001                 t=18
10000101110001010100000000000000100001011100010101               t=19
1010010010110100000100000000000010100100101101000001             t=20
100011011001100100010100000000001000110110011001000101           t=21
10101110111111110101000100000000101011101111111101010001         t=22
1000010101000000100001010100000010000101010000001000010101       t=23
101001000001000010100100000100001010010000010000101001000001     t=24
10001101000101001000110100010100100011010001010010001101000101   t=25
```

Figure 3b.

```
                         10100100001                              t=0
                        1000110100101                             t=1
                       101010000110001                            t=2
                      10000010010010101                           t=3
                     1010001011011000001                          t=4
                    100010100000001000101                         t=5
                   10101000100000101010001                        t=6
                  1000001010100010000010101                       t=7
                 101000100000101010001000001                      t=8
                10001010100010000010101000101                     t=9
               1010100000101010001000001010001                    t=10
              100000100010000010101000100010101                   t=11
             10100010101010001000001010101000001                  t=12
            1000101000000010101000100000001000101                 t=13
           101010001000001000001010100000101010001                t=14
          10000010101000101000100000100010000010101               t=15
         1010001000001010001010100010101010001000001              t=16
        100010101000100010100000101000000010101000101             t=17
       10101000001010101000100010001000001000001010001            t=18
      1000001000100000001010101010100010100010001010 1            t=19
     101000101010100000100000000000000101000101010100000 1        t=20
    10001010000000100010100000000000100010100000001000101         t=21
   1010100010000010101000100000000010101000100000101010001        t=22
  100000101010001000001010100000001000001010100010000010101       t=23
 10100010000010101000100000100000101000100000101010001000001      t=24
1000101010001000001010100010100010001010100010000010101000101     t=25
```

Figure 3c.

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