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INTRODUCTION

After the relevance of classical soliton solutions to the quantum theory has been established at the semi-classical level, the main problem remaining was to develop a systematic perturbation expansion about this solutions. The straightforward way of doing this which consists of shifting the field by the classical solution runs into serious difficulties. Namely, by the shift various symmetries (like translational invariance) get broken and consequently zero frequency modes appear leading to an infrared problem. The most effective way of eliminating this difficulty is given by the collective coordinate method² in which a proper treatment of symmetry degrees of freedom is achieved. In this approach it is then easy to develop a consistent set of Feynman rules and formulate a systematic perturbation expansion.³

In what follows we will review some further work on collective coordinates and their application to soliton quantization, most of which was done during the last year in collaboration with J. L. Gervais. First, in sec. I we give a brief summary of the canonical collective coordinate method. Then in sec. II a nontrivial application is presented on the example of soliton scattering. Finally in sec. III we describe an alternative, noncanonical treatment of zero frequency modes.

I. THE CANONICAL COLLECTIVE COORDINATE METHOD

In the initial works dealing with the one soliton sector perturbation expansion^{2, 4, 5} the center of mass position was treated as a collective

PATH INTEGRAL QUANTIZATION OF SOLITONS*

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ABSTRACT

A review of some recent work on the collective coordinate approach to perturbation expansion about soliton solutions is presented. Consistent formulation of the method in framework of the Feynman path integral quantization procedure is described. As an illustrative application we discuss in detail a perturbation expansion for scattering of solitons.

MASTER

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variable since it was necessary to preserve translational invariance when shifting by a space dependent classical solution. To treat more complicated soliton solutions we have formulated in collaboration with J. L. Gervais and B. Sakita a general method of collective coordinates in ref. (6). The basic idea consists in promoting the symmetry parameters X_α , which appear in certain classical solution $\phi_{cl}(x, t; X_\alpha)$ to dynamical variables (collective coordinates). Their conjugate variables $p_\alpha(t)$ are then precisely the generators $P_\alpha[\Pi, \phi]$ of the symmetry group. This new set of canonical variables is introduced into the path integral expression for transition amplitude

$$T_{fi} = \int \mathcal{D}\Pi \mathcal{D}\phi \psi_f^* \psi_i \exp \{ iA[\Pi, \phi] \} \quad (1.1)$$

by using the identities

$$\begin{aligned} \int \prod_\alpha \mathcal{D}p_\alpha \prod_t \delta(p_\alpha(t) - P_\alpha[\Pi, \phi]) &= 1 \\ \int \prod_\alpha \mathcal{D}X_\alpha \prod_t \delta(Q_\alpha[\phi|X(t)]) J &= 1 \\ J &= \det \left\| \frac{\partial Q_\alpha}{\partial X_\beta} \right\| \end{aligned} \quad (1.2)$$

Here $\phi|X$ denotes the transform of ϕ and the Q_α 's represent arbitrary symmetry fixing conditions. The collective coordinates can be transferred into the action by performing a canonical transformation

$$\begin{aligned} \phi(x, t) &\rightarrow \phi(x, t) & \Pi(x, t) &\rightarrow \Pi(x, t) \\ & & [-X(t)] & [-X(t)] \end{aligned} \quad (1.3)$$

Furthermore if the initial and final states represent eigenstates of the conserved quantities P_α it is possible to integrate out the collective variables and obtain

$$T_{fi} = \prod_\alpha (2\pi) \delta(p_\alpha^i - p_\alpha^f) \int \mathcal{D}\phi \psi_f^* \psi_i \prod_t \delta(p_\alpha(t) - P_\alpha) \delta(Q_\alpha) J e^{iA} \quad (1.4)$$

This result is formal since in deriving it we made a canonical transformation by formally changing variables in the path integral.

In general it is not known how to perform canonical transformations in quantum theory. We have made in ref. (7) an investigation of point transformations in the path integral since they are the ones most often used in applications of the collective coordinate method. Let us consider a general system with n -degrees of freedom

$$L(\dot{q}, q) = \frac{1}{2} \sum_{a=1}^n \dot{q}_a^2 - V(q)_J$$

in presence of an external source J . The generating functional is then given by

$$Z[J] = \int \prod_{i=1}^n \mathcal{D}q_i \exp \{ i \int dt L(\dot{q}, q) \} \quad (1.5)$$

Performing a point transformation $q_a = F^a(Q)$ we obtain formally

$$\int \prod_{i=1}^n \mathcal{D}Q_i g(Q)^{\frac{1}{2}} \exp \{ i \int dt L(F_{,i} \dot{Q}_i, F(Q)) \} \quad (1.6)$$

where

$$\begin{aligned} F_{,i}^a &= \frac{\partial}{\partial Q_i} F^a(Q) \\ g_{ij} &= \sum_{a=1}^n F_{,i}^a F_{,j}^a \\ g &= \det g_{ij} \end{aligned} \quad (1.7)$$

However, by making use of the diagram technique one can show that this expression differs from the original generating functional (1.5). This means that a formal change of integration variables does not represent a correct step in the path integral.

To derive the correct answer we use a more precise form for the path integral

$$Z(J) = \lim_{N \rightarrow \infty} \int \prod_{a=1}^n \prod_{k=1}^{N-1} \frac{dq_a(k)}{(2\pi i\epsilon\hbar)^2} \prod_{k=0}^{N-1} \exp\left\{\frac{i}{\hbar} A(q(k+1), q(k))\right\} \quad (1.8)$$

obtained by deviding the time interval into N equal intervals $\epsilon = T/N$ and denoting $t_k = k\epsilon$, $q(k) = q(t_k)$. Then after the change of variables $q_a(k) = F^a(Q(k))$, the new short-time action becomes

$$\tilde{A}(Q(k+1), Q(k)) = \frac{1}{2\epsilon} \sum_{a=1}^n (F^a(Q(k+1)) - F^a(Q(k)))^2 - \epsilon V(F(Q(k))). \quad (1.9)$$

Now it is crucial to observe that it is not correct to approximate this expression by using

$$F^a(Q(k+1)) - F^a(Q(k)) \approx F^a_{,i} \Delta Q(k)$$

since in the path integral $\Delta Q = Q(\epsilon^{\frac{1}{2}})$ and not $Q(\epsilon)$. Deciding to expand about the midpoint $\bar{Q}(k) = \frac{1}{2} (Q(k+1) + Q(k))$ the correct approximation for the short-time action reads

$$\tilde{A} \approx \frac{1}{2\epsilon} \sum [g_{ij}(\bar{Q}) \Delta Q_i \Delta Q_j + \frac{1}{12} F^a_{,i} F^a_{,jlm} \Delta Q_i \Delta Q_j \Delta Q_l \Delta Q_m] - \epsilon V(F(\bar{Q})). \quad (1.10)$$

Similarly for the measure we obtain

$$g(\bar{Q})^{\frac{1}{2}} \left[1 + \frac{1}{16} (g^{ij}_{,lm} + g^{ij}_{,lm}) \Delta Q_l \Delta Q_m \right]. \quad (1.11)$$

Next it is possible to substitute the additional terms appearing in (1.10) and (1.11) by an equivalent potential term. The form of this additional potential is

$$\Delta V(\bar{Q}) = \frac{\hbar^2}{8} \Gamma^i_{jf}(\bar{Q}) \Gamma^j_{lm}(\bar{Q}) g^{lm}(\bar{Q}) \quad (1.12)$$

where g^{lm} represents the inverse of g_{ij} and Γ^i_{jf} are the Christoffel symbols.

To summarize, we have found that a careful treatment of point transformations in the path integral leads to

- (a) An additional set of vertices coming from $\Delta V(Q)$.
- (b) Specification of the otherwise ambiguous contractions at the same point as implied by the midpoint path integral.

Using this general result it is then simple to derive the correct Feynman rules in the path integral collective coordinate method. We can also mention that it is not difficult to demonstrate the equivalence of this method with the parallel operator approach.⁸ However there are certain advantages to the path integral approach, since first of all it leads directly to the Feynman rules and furthermore it appears to be more powerful in discussing more complicated processes like for example scattering of solitons.

II. AN ILLUSTRATIVE APPLICATION: PERTURBATION EXPANSION FOR SOLITON SCATTERING

By now there is an extensive literature on applications of the collective coordinate method to develop perturbation expansions in the one soliton sector for various field theories. In contrast, the multi-soliton sector appears to be much less studied since it is considerably more difficult to treat. So instead of going over the various one soliton sector examples we choose to discuss in this section the problem of soliton scattering.⁹ It is here that the full power of the path integral approach becomes evident.

We consider the sine-Gordon model

$$L(\phi) = \frac{1}{2} (\partial \phi)^2 - V(\phi)$$

$$V(\phi) = \frac{1}{\gamma} (1 - \cos(\sqrt{\gamma} \phi))$$

since for this case exact multi-soliton solutions are known. The static soliton and antisoliton solutions read

$$\phi_s(x) = \frac{4}{\sqrt{\gamma}} \operatorname{tg}^{-1} e^x$$

$$\phi_a(x) = \frac{4}{\sqrt{\gamma}} \operatorname{tg}^{-1} e^{-x}$$

and their mass is $M_0 = 8/\gamma$. Our treatment of the soliton-antisoliton scattering is based on the exact classical solution $\phi_{sa}(x, t; u_1, u_2)$, which describes the transition of soliton and the antisoliton through each other. It has the following asymptotic behavior

$$\lim_{t \rightarrow \pm \infty} \phi_{sa}(x, t) = \phi_s\left(\frac{x - u_1 t - X_1^\pm}{(1 - u_1^2)^{\frac{1}{2}}}\right) + \phi_a\left(\frac{x - u_2 t - X_2^\pm}{(1 - u_2^2)^{\frac{1}{2}}}\right) \quad (2.1)$$

so that after the collision only the positions become changed

$$\begin{aligned} \Delta X_1 &= X_1^+ - X_1^- = (1 - u_1^2)^{\frac{1}{2}} \ln \left(1 - \frac{4M_0^2}{s}\right) \\ \Delta X_2 &= X_2^+ - X_2^- = (1 - u_2^2)^{\frac{1}{2}} \ln \left(1 - \frac{4M_0^2}{s}\right) \end{aligned} \quad (2.2)$$

where s is the usual Mandelstam variable.

We now consider the s - a scattering amplitude written in the path integral representation

$$S_{sa} = \lim_{\substack{t_i \rightarrow -\infty \\ t_f \rightarrow +\infty}} e^{i(E_f t_f - E_i t_i)} \int \mathcal{D}\phi \mathcal{D}\psi_{P_1 P_2}(t_f) \psi_{P_1 P_2}^*(t_i) e^{iA[\pi, \phi]} \quad (2.3)$$

with the initial and final wave functionals defined by

$$\psi_{P_1 P_2}[\phi] = \langle \phi | P_1, P_2 \rangle$$

This two-particle wave functional is simply obtained from separate soliton and antisoliton functionals. The latter are easily derived if we use the fact that for the one soliton sector

$$\hat{\phi}(x) = \phi_s\left(\frac{x - \hat{X}}{(1 - \hat{u}^2)^{\frac{1}{2}}}\right) + \hat{\eta}(x)$$

$$\int dx \phi_s' \hat{\eta} = 0$$

where the hats denote operators in the Schrödinger picture. Now

$$\psi_P[\phi] = \langle X, \eta | P \rangle = e^{ipX} \psi_0[\eta] \left| \int dx \phi_s' \eta = 0 \right.$$

where ψ_0 stands for a Gaussian wave functional representing the ground state. From this we conclude that

$$\psi_{P_1 P_2} = \frac{2}{i\pi} \int dX_i e^{ip_i X_i} \delta\left(\int dx \phi_s' \left(\frac{x - X_i}{(1 - u_i^2)^{\frac{1}{2}}}\right) \phi(x)\right) \psi_0[\phi - \phi_s - \phi_a]. \quad (2.4)$$

As the next step we apply the collective coordinate method to separate and integrate out the variables corresponding to space and time translations. The result reads

$$\begin{aligned} S_{sa} &= (2\pi)^2 \mathcal{Q}_{P_1 P_2} \delta(E_f - E_i) \lim_{t_i \rightarrow -\infty} e^{i(E_f t_f - E_i t_i)} \int \mathcal{D}\pi \mathcal{D}\phi \psi_{P_1 P_2}^* \psi_{P_1 P_2} \\ &\cdot \delta(C) \delta(Q) \delta(P_i - P) \delta(E_i - H) e^{iA[\pi, \phi]} \end{aligned} \quad (2.5)$$

Here P and H denote the total momentum and the hamiltonian while Q and $\bar{Q}$ represent arbitrary gauge conditions. We make the choice

$$\begin{aligned} Q[\phi] &= \int dx \phi_s' (\phi - \phi_{sa}) \\ \bar{Q}[\phi] &= \int dx \phi_{sa}' (\phi - \phi_{sa}) \end{aligned} \quad (2.6)$$

which leads to an elegant elimination of zero frequency modes which are associated with space and time translations. At the initial and final times the conditions (2.6) imply

$$\int dx \phi'_t \left(\frac{x - u_k t_k - X_k^+}{(1 - u_k^2)^{\frac{1}{2}}} \right) \phi(x, t) = 0 \quad t = 1, 2.$$

Therefore only the values

$$\begin{aligned} X_k^+ &= u_k t_k + X_k^- \\ X_k^- &= u_k t_k + X_k^+ \end{aligned} \quad k = 1, 2 \quad (2.7)$$

contribute to the integrals which appear in the initial and final wave functionals.

At this point we are ready to do perturbation about the stationary point ϕ_{sa} by making the shift

$$\begin{aligned} \phi &= \phi_{sa} + \eta \\ \Pi &= \dot{\phi}_{sa} + \Pi_\eta \end{aligned}$$

We will be concerned only with the first quantum correction to the classical answer and in this approximation the path integral expression for the s - a scattering phase shift reads

$$e^{2i\delta_{sa}} = \exp \left\{ i \sum_{k=1}^2 (E(p_k) T \cdot p_k u_k T \cdot p_k \Delta X_k) + i A_{sa} \right\} \cdot \int \mathcal{D}\phi^+ \mathcal{D}\phi \delta(Q) \delta(\bar{Q}) \delta \left(i \int dx \phi^+ I_{sa} \phi \right) \delta \left(i \int dx \phi^+ I_{sa} \dot{\phi} \right) e^{iA^{(2)}} \quad (2.8)$$

with the notation

$$\phi = \begin{pmatrix} \eta \\ \pi \end{pmatrix}, \quad \phi_{sa} = \begin{pmatrix} \phi_{sa} \\ \dot{\phi}_{sa} \end{pmatrix}, \quad I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and the quadratic action given by

$$A^{(2)} = \frac{1}{2} \int dt dx \phi^+ H_2 \phi$$

$$H_2 = i \begin{pmatrix} \partial/\partial t & -1 \\ -\frac{\partial^2}{\partial x^2} + V^{(2)}(\phi_{sa}) & \partial/\partial t \end{pmatrix} \quad (2.9)$$

From the expression (2.8) we see that the leading contribution reads

$$2\delta_{sa}^{cl} = \sum_{k=1}^2 \left(TM_0 (1 - u_k^2)^{\frac{1}{2}} - \frac{M_0 u_k}{(1 - u_k^2)^{\frac{1}{2}}} \Delta X_k \right) + A_{sa} \quad (2.10)$$

which is the classical phase shift (4, 5, 10). We point out here that the precise specification of initial and final wave functionals appeared to be necessary in order to obtain this answer. Otherwise one would have to supply the first two terms in (2.10) essentially by hand.

To compute the first quantum correction we need to evaluate the Gaussian functional integral with constraints. This is done by finding the normal modes as follows. First, we solve for the complete set of solutions to the equation of motion $H_2 \phi = 0$, which consists of the discrete modes ϕ_{sa}^+ , $\dot{\phi}_{sa}$ and the continuum $\{\phi_k\}$. Imposing periodic boundary conditions we calculate the stability angles¹¹ ν_n and obtain a set of orthonormal eigenfunctions

$$f_n(x, t) = e^{i \frac{\nu_n}{2T} t} \phi_{k_n}(x, t)$$

$$H_2 f_n = -\frac{\nu_n}{2T} f_n \quad (2.11)$$

with the inner product defined by

$$(f, g) = \int dx f^+(x) I g(x)$$

Now we still have to consider the zero frequency modes which in fact

have zero norms. However, in order to satisfy the first two subsidiary conditions in (2.8) we need to define a new basis by making linear combinations

$$\tilde{\gamma}_n = (\phi_n - \alpha_n \phi'_{sa} - \beta_n \phi''_{sa}) e^{i \frac{\nu_n t}{2T}} \quad (2.12)$$

and furthermore the zero frequency modes give identically zero.

Next, by expanding the field

$$\Phi(x, t) = \sum_n (\tilde{\gamma}_n(x, t) a_n(t) + c.c.)$$

we reduce the action to a set of decoupled harmonic oscillators

$$A^{(2)} = \int_{t_1}^{t_2} dt \sum_n \left[\frac{1}{2i} (\dot{a}_n^* a_n - a_n^* \dot{a}_n) - \frac{\nu_n}{2T} a_n^* a_n \right] \quad (2.13)$$

while the initial and final wave functionals give precisely the ground state of this oscillator. Consequently the expression for the first quantum correction reads

$$\delta \phi_{sa}^{(1)} = M^{(1)} T \sum_{n=1}^{\infty} (1-u^2)^{\frac{1}{2}} \cdot \sum_n \frac{1}{4} \nu_n \quad (2.14)$$

where $M^{(1)}$ denotes the quantum correction to the soliton mass. After renormalization and some calculation we obtain in the C.M. frame the following answer

$$\delta \phi_{sa}^{(1)} = -\frac{2}{\pi} \int_0^u dz \frac{\ln z}{1-z^2} + \frac{2u}{\pi} \ln u + \frac{1}{4} \pi \quad (2.15)$$

For comparison, the classical phase shift is given by

$$\delta^{cl} = \frac{16}{Y} \int_0^u dz \frac{\ln z}{1-z^2} + \frac{4\pi^2}{Y} \quad (2.16)$$

An interesting conjecture concerning the exact soliton-antisoliton scattering amplitude has been made by Faddeev, Korepin and Kulish. Namely, knowing the classical result (2.16) and also

the bound state spectrum this author's conjectured that for special values of the coupling constant

$$\gamma' = \frac{8\pi}{N}, \quad N = 1, 2, \dots \quad (2.17)$$

in this theory there is no reflection and that

$$S_{sa} = \prod_{n=1}^{N-1} \left(-\frac{\xi + e^{i \frac{\pi n}{N}}}{1 + \xi e^{i \frac{\pi n}{N}}} \right) \quad (2.18)$$

where $\xi = \frac{s-2M^2 + \sqrt{s(s-4M^2)}}{2M^2}$. Expanding now this expression in the coupling constant γ one indeed obtains for the first and second terms precisely (2.16) and (2.15).

At the end of this section a comment is in order concerning the fact that up to now we were discussing only the transition part of the full s-a amplitude. The question is whether the reflection part can be treated in a similar fashion. One may doubt this since at the classical level there exists no solution describing reflection and it is a purely quantum effect (like tunnelling in quantum mechanics). However, it has been shown by Korepin in ref. (12) that by employing complex time trajectories in the path integral it is possible to give a semiclassical treatment of reflection. Namely, if in the usual soliton-antisoliton classical solution

$$\phi_{sa}(z, t) = \frac{4}{Y} \operatorname{tg}^{-1} \left(\frac{\frac{1}{u} \frac{\operatorname{sh} \frac{ut}{(1-u^2)^{1/2}}}{\operatorname{ch} \frac{x}{(1-u^2)^{1/2}}}} \right) \quad (2.19)$$

we chose the following time contour

$$\begin{aligned} \text{Im } t &= 0 & t < 0 \\ \text{Im } t &= \frac{\pi(1-u^2)^{1/2}}{v} & t > 0 \end{aligned} \quad (2.20)$$

than this solution describes reflection. Taking into account the two possible ways of approaching the turning point $t_0 = i\pi(1-u^2)^{1/2}/2v$ one derives an expression for the reflection part of the s - a scattering amplitude. Interestingly enough, for $\gamma = 8\pi/N$ this amplitude instead vanishes as conjectured before.

III. NONCANONICAL COLLECTIVE COORDINATES

We will now describe an alternative treatment of zero frequency modes¹³ in which the symmetry parameters are not promoted to new canonical variables, but are rather treated as ordinary integration variables in the path integral. Consequently this method is not canonical but it appears to be simpler than the one described in section I.

For illustration we consider a two-dimensional scalar field theory

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - V(\phi)$$

possessing a static classical solution $\phi_0(x)$. After the shift $\phi(x, t) = \phi_0(x) + \eta(x, t)$ the quadratic part of the action reads

$$\begin{aligned} A^{(2)}[\eta] &= \int dt dx \frac{1}{2} \eta \hat{M} \eta \\ \hat{M} &= -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - V^{(2)}(\phi_0(x)) \end{aligned} \quad (3.1)$$

The equation satisfied by the "naive" propagator is then

$$\hat{M} D(x, t; x', t') = i\delta(t-t') \delta(x-x') \quad (3.2)$$

and the corresponding eigenvalue problem in the box

$$\hat{M} f_{mn}(x, t) = \lambda_{mn} f_{mn}(x, t)$$

is solved by

$$\begin{aligned} f_{mn}(x, t) &= \frac{e^{i\nu_m t}}{\sqrt{T}} \psi_n(x) \\ \lambda_{mn} &= \nu_m^2 - \omega_n^2, \quad \nu_m = \frac{2\pi m}{T} \end{aligned} \quad (3.3)$$

Here $\{\psi_n\}$ represents a complete set of eigenfunctions of

$$\left[-\frac{d^2}{dx^2} + V^{(2)}(\phi_0) \right] \psi_n(x) = \omega_n^2 \psi_n(x) \quad (3.4)$$

and we assume that it consists of the translational zero frequency

mode $\psi_0(x) = \frac{1}{\sqrt{N}} \phi'_0(x)$, $\omega_0 = 0$ and the continuum $\{\psi_k(x)\}$.

Now the "naive" expression for the propagator which reads

$$\sum_{m,n} f_{mn}(x, t) \frac{1}{\lambda_{mn} - i\epsilon} f_{mn}^*(x', t') \quad (3.5)$$

contains an infrared problem due to the $m=0, n=0$ mode, since then $\lambda_{00} = 0$. However, in the path integral it is possible to preserve the translational symmetry and introduce a subsidiary condition of the form

$$\int dt dx f_{00}(x, t) \eta(x, t) = 0 \quad (3.6)$$

by the argument due to Faddeev and Popov which we describe in detail later. Consequently we have the more sensible expression for the propagator

$$D(x,t;x',t') = \sum_k \int \frac{dv}{2\pi} e^{iv(t-t')} \frac{i}{v^2 - \omega(k)^2 + i\epsilon} \psi_k(x) \psi_k^*(x') \\ + \sum_{n \neq 0} \frac{iv_n(t-t')}{T} \cdot \frac{i}{v_n^2} \psi_0(x) \psi_0(x') \quad (3.7)$$

Here the first part represents the contribution of continuum modes and it is the usual propagator of the canonical collective coordinate method denoted by $G(x,t;x',t')$. But, now in addition we have the contribution coming from the zero mode which can be summed up to read

$$G_0(x,t;x',t') = i \left[\frac{T}{12} - \frac{1}{2} |t-t'| + \frac{1}{2T} (t-t')^2 \right] \psi_0(x) \psi_0(x') \quad (3.8)$$

Now, we show how the subsidiary condition (3.6) is introduced.

Namely, into the path integral expression for the one soliton sector transition amplitude

$$\langle p' | e^{-iHT} | p \rangle = \int \mathcal{D}\phi \, \psi_{p'}[\phi(T)] \psi_p[\phi(0)] e^{iA[\phi]} \quad (3.9)$$

we insert the identity

$$\int da \, \delta \left(\int dt dx \, f(x,t) \phi(x-a,t) \right) J = 1 \\ J = \int dt dx \, f(x,t) \frac{\partial}{\partial a} \phi(x-a,t) \quad (3.10)$$

Next, changing variables $\phi(x,t) \rightarrow \phi(x+a,t)$ what implies

$$\psi_p[\phi] \rightarrow e^{ipa} \psi_p[\phi], \quad \psi_{p'}[\phi] \rightarrow e^{-ip'a} \psi_{p'}[\phi]$$

and performing the integration over a we obtain

$$\langle p' | e^{-iHT} | p \rangle = (2\pi) \delta(p'-p) F(p)$$

$$F(p) = \int \mathcal{D}\phi \, \psi_p^* \psi_p \left(\int dt dx \, f(x,t) \phi(x,t) \right) J e^{iA[\phi]} \quad (3.11)$$

Here $f(x,t)$ is arbitrary and observe that the subsidiary condition in this path integral is not canonical. Now for the zero momentum case one simply chooses $f(x,t) = f_{00}(x,t)$. In general for arbitrary momenta we have to use the time dependent solution instead of the static one. Furthermore it is possible to introduce even a more general translation invariance fixing term than the δ -function in (3.11). By the well-known argument one derives for example

$$F(0) = \int \mathcal{D}\eta \, J \exp \{ iA[\phi_0 + \eta] + \frac{1}{2\alpha} \left(\int dt dx \, \phi_0(x) \eta(x,t) \right)^2 \} \quad (3.12)$$

and then the zero mode part of the propagator reads

$$G_{0\alpha}(x,t;x',t') = i \left[\alpha - \frac{1}{2} |t-t'| \right] \psi_0(x) \psi_0(x') \quad (3.13)$$

Here α is arbitrary and the dependence on this parameter cancels out in the calculation.⁽¹⁴⁾ The Feynman rules are completed with the vertices coming from the higher than quadratic part of the action and also the Jacobian.

The advantage of the method described above is in its simplicity.

First of all, it appears that the transformation which was performed to introduce the collective coordinate does not require any special treatment in contrast to the case of canonical transformations discussed in Section I. This virtue is especially useful in developing for example the perturbation expansion for soliton scattering. Applying the method of this section to that case one easily derives a simple set of Feynman rules (now the propagator contains the contribution of two zero modes), which can then be used to perform even higher loop calculations. (15)

Furthermore, being noncanonical this method is also appropriate for treating the zero-frequency modes which appear in perturbation expansion about instanton solutions. For example, let us consider the SU(2) invariant Euclidean Yang-Mills field theory

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a},$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon_{abc} A_\mu^b A_\nu^c, \quad (3.14)$$

possessing an exact classical solution

$$A_\mu^a(x; \rho, z) = \frac{2}{g} \frac{\eta_{\mu\nu} (x^\nu - z^\nu)}{(x-z)^2 + (1+\rho)^2} \quad (3.15)$$

found by Belavin et al. (17) Here z^ν and ρ are arbitrary constants associated with translation and scale invariance. The corresponding zero-frequency eigenfunctions now read

$$\phi_\mu^a(x) = \frac{\partial}{\partial \rho} A_\mu^a(x; \rho, z) \Big|_{\rho=0, z=0}$$

$$\phi_\mu^a(x; \nu) = \frac{\partial}{\partial x_\nu} A_\mu^a(x; \rho, z) \Big|_{\rho=0, z=0} \quad (3.16)$$

The zero-frequency modes associated with the SU(2) internal symmetry read

$$\psi_\mu^a(x; \alpha) = \frac{\partial A_\mu^a(x; \lambda)}{\partial \lambda_\alpha} \Big|_{\lambda=0} \quad (3.17)$$

where we denoted

$$A_\mu(x; \lambda) = U^{-1}(\lambda) A_\mu(x) U(\lambda), \quad U(\lambda) = \exp(i g \lambda^a \tau^a / 2),$$

$$A_\mu(x) = A_\mu^a \tau^a / 2. \quad (3.18)$$

The gauge is specified by $C^a(x) = \partial^\mu A_\mu^a(x)$ and the vacuum to vacuum amplitude which we consider then has the following path integral form:

$$\langle 0 | 0 \rangle_{in} = \int \prod_{x,a} dA_\mu^a(x) \Delta_{FP} \exp\{i \int d^4x \mathcal{L}(A(x))\}. \quad (3.19)$$

To proceed we need to introduce the notion of invariant SU(2) group integration

$$\int dU(\lambda) = \int U(g\lambda) \prod_{a=1}^3 (gd\lambda_a), \quad (3.20)$$

and the following condensed notation for the zero-frequency modes

$$\phi_\mu^a(x) = f_\mu^a(x; 1),$$

$$\phi_\mu^a(x; \nu) = f_\mu^a(x; \nu+1) \quad \nu = 1, 2, 3, 4,$$

$$\psi_\mu^a(x; \alpha) = f_\mu^a(x; \alpha+5) \quad \alpha = 1, 2, 3, \quad (3.21)$$

To eliminate this modes we insert the identity

$$\int d\rho \prod_{v=1}^4 dz_v \int dU(\Lambda) \prod_{i=1}^8 \delta\left(\frac{1}{\sqrt{N_1}} \int d^4 x f_{\mu}^a(x; i) A_{\mu}^a(x; \rho, z, \Lambda)\right) J=1, \quad (3.22)$$

into the path integral. Now changing the field integration variables $A_{\mu}^a(x; \rho, z, \Lambda) \rightarrow A_{\mu}^a(x)$ we obtain

$$\text{out} \langle 0 | 0 \rangle_{\text{in}} = \int d\rho \prod_{v=1}^4 dz_v \int dU(\Lambda) \int dA \prod_{i=1}^8 \delta(\square^a(x)) \Delta_{FP} J \prod_{i=1}^8 \delta\left(\frac{1}{\sqrt{N_1}} \int d^4 x f_{\mu}^a(x; i) A_{\mu}^a(x)\right) \exp\left[i \int d^4 x \mathcal{L}\right] \quad (3.23)$$

Here the Jacobian J is to be evaluated from (3.22) using infinitesimal symmetry transformations and is given by

$$J = \frac{1}{g^3} \det M_{ij}, \quad M_{ij} = \frac{1}{\sqrt{N_1}} \int d^4 x f_{\mu}^a(x; i) \frac{\partial A_{\mu}^a(x; a)}{\partial a_j} \Big|_{a=0} \quad (3.24)$$

where $\bar{a} \equiv (\rho, z^v, \Lambda_a)$. For explicit one-loop calculations the reader is referred to ref. (16). We also mention that it is possible to give a canonical description of vacuum tunnelling⁽¹⁸⁾ in theories with instantons by employing the canonical collective coordinate method discussed in Sect.I.

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