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AND TURBULENCE; ANALOGIES WITH PHASE-TRANSITION PHENOMENA

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FUNCTIONAL-INTEGRAL FORMULATIONS FOR
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ANALOGIES WITH PHASE-TRANSITION PHENOMENA

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ABSTRACT:

The formalism of Martin, Siggia and Rose is utilized to write a functional-integral representation for generating functionals in plasma transport theory, following Nakayama and Dawson. Parallel treatments of Navier-Stokes turbulence (attempted by Rosen) and of critical dynamics, by Kawasaki, are compared to illustrate the application of common field-theory techniques, such as the effective action. Quasi-classical methods for functional integrals are discussed.

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I. Introduction

A common mathematical formulation of basic problems in several related areas of physics allows the transfer of sophisticated techniques and qualitative insights from one such area to another. A recent example is provided by the functional methods studied in static critical phenomena and phase transition theory and in field theories of fundamental particles, which share the concepts of spontaneously broken symmetries, Goldstone modes, instantons, and scale invariances.

Application of field-theoretic methods to classical time-dependent, nonequilibrium many-body problems has been less fruitful to date, although there is great practical interest in nonequilibrium systems. As examples of such systems, we will consider finite amplitude plasma instabilities, hydrodynamic turbulence, and critical dynamics for order-disorder (or magnet) models. In this paper we will compare formulations of functional integrals (for cumulant generating functionals) for these three classes of problems, and discuss approximation methods suggested by previous studies of continuum mathematical models of phase transitions, and by field-theoretic "soliton" models of fundamental particles.

II. Generating Functionals

We consider the observables in a continuum classical statistical system to be a classical field, or set of classical fields, $\phi(x,t)$, which satisfy classical nonlinear equations of motion

$$\frac{\partial}{\partial t} = Q[\phi]$$

We describe the statistical properties of these fields, in a given ensemble, by the set of expectation values at a given time:

$$\langle \phi(X_1) \dots \phi(X_n) \rangle_t$$

An equivalent description of the ensemble at a given time is provided by a ^{time-dependent} generating functional $F_t[\xi]$ with the property

$$\left. \frac{\delta^n F_t[\xi]}{\delta \xi(X_1) \dots \delta \xi(X_n)} \right|_{\xi=0} = \langle \phi(X_1) \dots \phi(X_n) \rangle_t$$

This generating functional is determined in principle by its values at some initial time t_0 , together with equations of motion, which can be formally written:

$$\frac{\partial F_t[\xi]}{\partial t} = i \int dX \xi(X) Q\left[-\frac{\delta}{i\delta \xi}\right] F_t[\xi]$$

where the operators on the right are determined by the form of the equations of motion for ϕ , and have been derived by Hopf (1) for hydrodynamic systems, Nakayama and Dawson (2) for plasmas, and Kawasaki (3) for magnets.

In classical Navier-Stokes (incompressible) hydrodynamics, the fields ϕ are the fluid velocity field components; the phase space $\{X\}$ is three-dimensional coordinate space.

In plasma theory, ϕ is a scalar (Klimontovich or Vlasov) density in 6-dimensional $(\vec{r}, \vec{p})$ phase space.

In order-disorder (magnet) dynamics, ϕ is an order-parameter (e.g. spin) amplitude, and the phase-space is a lattice or continuum in coordinate space of d dimensions. If η are test functions with arguments (X, t) ,

A time-independent generating functional provides an alternative description.

$G[\eta]$ can be defined such that

$$\left. \frac{\delta^2 G[\eta]}{\delta \eta(X_1, t) \delta \eta(X_2, t)} \right|_{\eta=0} = \langle \phi(X_1) \phi(X_2) \rangle_t \quad (2.4)$$

The time-evolution of observables, such as $\langle \phi(X_1) \phi(X_2) \rangle$, can be most conveniently described by

the time-dependent generating

functional, F_t . This can be obtained from $G[\eta]$, and gives the approach of Hopf (1), Rosen (4) and Nakayama and Dawson (2). However, the field-theory techniques are easier to apply using $G[\eta]$.

The Hopf functional F_t , for comparison, is defined on space-dependent (but not time-dependent) functions $\xi(X)$,

Then F_{t_0} specifies initial conditions, and F_t satisfies the equations of motion given earlier.

The relation between $F_t[\xi]$ and $G[\eta]$ is given formally by:

$$F_t[\xi] = \sum_n \int \xi(X_1) \dots \xi(X_n) \left. \frac{\delta^n G[\eta]}{\delta \eta(X_1, \tau) \dots \delta \eta(X_n, \tau)} \right|_{\eta=0} dX_1 \dots dX_n \quad (2.5)$$

To express the dynamics of nonequilibrium ensembles in field-theoretic language it is convenient to introduce adjoint fields $\tilde{\phi}$ which obey δ -fermion local commutation relations with the physical fields, thus playing a role like conjugate momentum variables to the fields. A discussion of this point is given by Martin, Siggia and Rose (5), who also give formal expressions for the equations of motion, expressed in terms of a pseudo-Hamiltonian $H(\phi, \tilde{\phi})$, defined such that:

$$\frac{\partial \phi}{\partial \tau} = [H, \phi] \quad \text{and} \quad \frac{\partial \tilde{\phi}}{\partial \tau} = [H, \tilde{\phi}] \quad (2.6)$$

From the functionals $H(\phi, \tilde{\phi})$ we can derive a pseudo-Lagrangian $L(\phi, \tilde{\phi})$ such that

$$L = -i\phi \sigma_2 \frac{\partial}{\partial \tau} \phi - H(\phi) \quad (2.7)$$

where ϕ is a two-component field $(\phi, \tilde{\phi})$ for each dynamical field ϕ , and σ_2 is the Pauli 2x2 matrix.

A path-integral representation for the appropriate generating functional (for time-ordered expectation values of products of ϕ and $\tilde{\phi}$) then can be written in terms of L .

The classical equations of motion, obtained by considering the stationary paths for the functional integral, serve as a check that the correct L has been obtained. We also examine time-independent stationary solutions, to find equilibrium ensembles, corresponding to stationary flows in

hydrodynamics, or Vlasov equilibria in plasma problems. Effective Lagrangians for small, finite perturbations around these equilibria can be explicitly obtained in some cases; the time-dependent Landau-Ginzburg Ansatz is obtained in critical dynamics, and the Graham(6)-Haken(7) free-energy Functional is obtained for the Benard convection instability. We expect a similar analysis carried out for plasmas will yield a corresponding analytic approach to finite-amplitude beam-plasma instabilities.

III. Functional-Integral Representations

The equations of motion for the three systems we discuss can be written explicitly as follows:

(1) MAGNETIC, ϕ is a spin density, and (3): $\frac{\partial \phi}{\partial t} = r_0 \phi + v^2 \phi - 4u_0 \phi^3$ (2.1)

(2) Incompressible
NAVIER-STOKES
HYDRODYNAMICS(NS): ϕ_i ($i=1,2,3$) are velocity-field components, and (4):

$$\frac{\partial \phi_i}{\partial t} = \nu \nabla^2 \phi_i - \int d^3 x^1 \Delta_{ijk} (x-x^1) \phi_j (x^1) \phi_k (x^1) \quad (2.2)$$

where ν is the viscosity, and

$$\Delta_{ijk}(X) \equiv 2\pi i \int d^3 q \left(\delta_{ij} - \frac{q_i q_j}{q^2} \right) \frac{1}{q} \exp(2\pi i q \cdot X) \quad (2.3)$$

(3) ELECTROSTATIC PLASMA, in Vlasov-ensemble (continuous fluid) formulation of Nakayama and Dawson (2) (VND): ϕ is density in $(\vec{r}, \vec{p})$ 6-dimensional phase space, and:

$$\frac{\partial \phi}{\partial t} = -\frac{\vec{p}}{m} \cdot \frac{\partial \phi}{\partial \vec{r}} + \frac{\partial \phi}{\partial \vec{p}} \cdot \int d^3 r' \frac{\partial V(|\vec{r}-\vec{r}'|)}{\partial \vec{r}} \phi(\vec{r}') \quad (2.4)$$

where $V(|\vec{r}|)$ is the interparticle (Coulomb) potential, and M is the mass of the particles constituting the plasma.

(For simplicity, in this paper, as in Nakayama and Dawson (2), we consider only single-component nonrelativistic plasmas with no magnetic fields. The formalism clearly can be extended to a wider class of plasma problems at the expense of complexity.)

Following MSR (5), we obtain the pseudo-Hamiltonians H for the three systems:

$$\underline{M}: H(\phi, \tilde{\phi}) = \frac{V_0}{2} \phi^2 - \frac{1}{2} (\nabla \phi)^2 + \frac{U_0 \phi \phi^3}{6} + \frac{\pi}{2} \tilde{\phi} - \frac{1}{2} (\nabla \tilde{\phi})^2 \quad (3.5)$$

$$\underline{NS}: H(\phi_i, \tilde{\phi}_i) = v \phi_i \nabla^2 \phi_i + \tilde{\phi}_i(X) \int d^3 X \Delta_{ijk}(X-X^1) \phi_j(X^1) \phi_k(X^1) \\ + \phi_i(X) \int d^3 X^1 \Delta_{ijk}(X-X^1) \tilde{\phi}_j(X^1) \phi_k(X^1) + v \tilde{\phi}_i \nabla^2 \tilde{\phi}_i \quad (3.6)$$

$$\underline{VND}: H(\phi, \tilde{\phi}) = -\frac{1}{2} \left(\phi \frac{\vec{p}}{m} \cdot \frac{\partial \phi}{\partial \vec{r}} \right) - \frac{1}{2} \left(\tilde{\phi} \frac{\vec{p}}{m} \cdot \frac{\partial \tilde{\phi}}{\partial \vec{r}} \right) + \tilde{\phi} \frac{\partial \phi}{\partial \vec{p}} \cdot \int d^3 r^1 \frac{\partial^2 V(r-r^1)}{\partial \vec{r}^2} \phi(r^1) \\ + \phi \frac{\partial \tilde{\phi}}{\partial \vec{p}} \cdot \int d^3 r^1 \frac{\partial V(r-r^1)}{\partial \vec{r}^2} \phi(r^1) \quad (3.7)$$

Now the functionals L can be constructed as outlined in the previous sections. The result for M ^{can be} obtained from the work of Kawasaki (3),

BJW (8), and others. For NS , the result is essentially that of Rosen (4); while for VND, the result has not been previously written down.

Following Rosen (4), we can write the time-evolution of the Hopf functional

$$as: \quad F_t [\xi] = \int d\xi^1 \quad K_{t,t_0} [\xi, \xi^1] \quad F_{t_0} [\xi^1] \quad (3.8)$$

where K , the propagator, has a functional-integral representation:

$$K_{t,t_0} [\xi, \xi^1] = \int_{\xi''(t_0) = \xi^1}^{\xi(t) = \xi} \int d\xi'' \quad T \exp \left\{ -i \int_{t_0}^t dt'' \int dx'' \quad L[\xi'', \xi''] \right\} \quad (3.9)$$

where $\tilde{\xi}$ is the field conjugate to ξ .

If we now consider to first approximation that the dominant contributions to K are given by the stationary-phase paths ("quasi-classical" approximation), we obtain functionals F_t which are functional δ -fns., with arguments determined by solutions of the field equations of motion with initial conditions weighted by F_{t_0} . Expanding L around such "stationary-flow" functions, we obtain fluctuation corrections, which include "tunnelling" transitions between "nearby" stationary flows or Vlasov equilibria. In Benard convection, the resulting L_{eff} has been obtained by Graham (6) and Haken (7), based on the detailed study of Newell and Whitehead (9) of the "classical" equations, and has the form of a Landau-Ginzburg functional, as in M.

Similar reasoning applies to the functional integral for G .

To provide comparisons with other field-theoretic work in transport theory, and also allow application of "effective-action" techniques, it is convenient to introduce, following Kawasaki, generating functionals for products of fields including the adjoint fields $\tilde{\phi}$, as well as composite fields

$$D_{\alpha\beta}(x,t) \equiv \phi_{\alpha}(x,t) \cdot \phi_{\beta}(x,t) \quad (3.10)$$

and the adjoint field

$$\tilde{D}_{\alpha\beta}(x,t) \equiv \tilde{\phi}_{\alpha}(x,t) \cdot \tilde{\phi}_{\beta}(x,t) \quad (3.11)$$

The generating functional arguments then include sources for these fields, which we denote by $\tilde{\eta}$, $\tilde{\xi}$, and $\tilde{\zeta}$ respectively; for example,

$$\left. \frac{\delta F_t[\eta, \tilde{\eta}, \xi, \tilde{\xi}]}{\delta \eta(x_1) \tilde{\eta}(x_2)} \right|_{S=0} = \langle \phi(x_1) \tilde{\phi}(x_2) \rangle \quad (3.12)$$

and

$$\left. \frac{\delta F_t[\eta, \tilde{\eta}, \xi, \tilde{\xi}]}{\delta \xi_{\alpha\beta}(x)} \right|_{S=0} = \langle D_{\alpha\beta}(x) \rangle \quad (3.13)$$

For brevity, we have deviated the set of sources $(\eta, \tilde{\eta}, \xi, \tilde{\xi})$ by S .

The introduction of $\tilde{\phi}$ as a dynamical variable, as discussed in MSR (5), allows a closed system of equations for renormalized vertex functions and propagators (the Schwinger-Dyson equations) to be derived; these have been explicitly studied in weakly interacting classical fluids by MSR, and Forster and Martin (10), and comparisons are possible with the work of Krommes and Oberman (11).

The generalization of G including (space and time dependent) sources for $\tilde{\phi}$, D , and $\tilde{D}$ can be written (following Kawasaki (3)) as:

$$G[S] = \int \delta\phi \delta\tilde{\phi} T \exp \left\{ -i \int dt \int dX [L(\phi, \tilde{\phi}) + \eta\phi + \tilde{\eta}\tilde{\phi} + \xi D + \tilde{\xi}\tilde{D}] \right\} \quad (3.14)$$

where $(\xi, \tilde{\xi})$ and $(D, \tilde{D})$ have tensor indices in the NS problem.

The introduction of D and $\tilde{D}$ allows application of the techniques for effective action of composite operators, a generalization of the Gorkov method (12) for deriving effective functionals for long-wavelength and long-time fluctuations. The time-dependent Hartree method from quantum mechanics is easily applied also in this formalism, in the functional formalism of Kerman and Koonin (13).

Composite operators such as D were also introduced to form a self-consistent dynamical basis of NS turbulence theory by Edwards and McComb (14), and they appear in Krammes and Oberman's discussion of convection cells in a 2-dimensional plasma (11).

Many of the difficulties in formulating satisfactory theoretical methods for long-wavelength phenomena in fluids can be recognised

as common to field theories with strong coupling, "infra-red" divergences (i.e., long-range interactions), and spontaneously broken symmetries (inequivalent vacua). In the NS and plasma problems, the long-range interactions are evident in the basic "Lagrangian" (microscopic functional weight); while in the M system, it is necessary to look at critical phenomena, corresponding to a particular choice of parameters, to obtain effective Lagrangians with long-range correlations.

In this sense, the basic theoretical problem of quantum chromodynamics, as a spontaneously-broken gauge theory of massless elementary particles, common to turbulence and plasma instability problems, can be stated as obtaining the effective action for long-wavelength phenomena. This similarity has been recognised previously for critical phenomena.

IV. Effective Actions for Large-Scale Phenomena

The practical success of a many-body theory depends on methods to reduce a very large number of degrees of freedom to a small, tractable set of dynamical (collective) variables which accurately describe a limited set of observables.

In problems of hydrodynamic stability and convection problems, and in beam-plasma instability, we may typically consider a few mode amplitudes which are slowly varying in time and space compared to the fast, small-scale motions of the fluid (collision or plasma-oscillation frequencies). The time development of relatively slow, large-scale fluctuations should be described through an effective action for long-wavelength field components. One would expect such an effective action to be of Landau-Ginzburg form, as any free-energy functional should be expandable in a functional power series in the fields and their gradients. For the original fields, there may not be a finite LG limiting form for long

wavelengths, however; as in the renormalization-group treatment of critical phenomena, a limiting form-invariance may be obtained (scaling laws) instead. Such scaling behavior is well known in turbulence theory (Kolmogoroff scaling).

Systematic procedures for constructing effective actions, given a field theory Lagrangian, have been worked out by Coleman and Weinberg (15), and applied to N-component field theories (of the M type) by Coleman, Jackiw and Politzer (16), by Gross and Neveu (17), and by Dashen, Ma and Rajaraman (18). The latter authors have extensively used the composite field action formalism; they showed how the Landau-Ginzburg form follows, in the one-loop approximation to the effective action for variables such as D and $\tilde{D}$. Further systematic investigations of composite-operator effective actions have been carried out by Cornwall, Jackiw and Tomboulis (19). A similar method should be applicable to the pseudo-Lagrangians we have written down for (NS) turbulence and for (VND) plasmas.

The qualitatively important features in our problems will be associated with the appearance of structures: solitons, instantons, convective cells, or domain walls, depending on the physics. These structures, their properties, and their motions, appear when the parameters of the LG forms (more generally, the effective actions) fall into regions of instability or spontaneously broken symmetry.

Quantitative treatment of model systems with saturating instabilities can be obtained if we can follow the methods of Graham (6) and Haken (7), based on the finite-amplitude analysis of Newell and Whitehead (9) for Benard convection dynamics; we may approximate the resulting time-

dependent Landau-Ginzburg dynamics by a time-dependent Ising model, following Weigel's treatment (20) of condensation in the Van der Waals fluid.

We plan to apply this method to the estimation of beam-plasma instabilities with finite amplitudes in a future work, in an effort to augment partial numerical simulation (21, 22). We also expect in some circumstances to make contact with "soliton turbulence" theory (23), through consideration of plasmas far from equilibrium. Expanding L around Vlasov equilibria, we obtain the usual Landau solutions for time-dependent saddle-points. Selecting the oscillatory modes which are lightly damped or growing with time, roughly corresponding to the "neutral solutions" of hydrodynamic instability theory, we expand L in terms of the amplitudes of these modes, assumed slowly varying in time and space compared to the fast plasma periods. Then, as in the Graham-Haken-Newell-Whitehead analysis, we should obtain an effective action (free energy functional) describing finite-amplitude saturation processes of the instabilities (assuming the saturation levels are small); the squares of the mode amplitudes play a role as composite operators.

This method should be applicable to any beam instabilities which follow from perturbing the Vlasov equation around steady (Vlasov equilibrium) solutions, including not only two-stream and parametric, but also resistive instabilities such as the hose, which has been comprehensively analysed in linear approximation by Weinberg (24). It is of course necessary to include a sufficient number ^{of} modes to provide saturation mechanisms; this may result in a rather complex effective action in terms of the mode amplitudes, and very complex time-dependences, as known in models

of hydrodynamic turbulence (25).

The particular virtue of the functional-integral formalism in such problems is the potential of transferring quasi-analytic methods from other problems formulated in the same language, such as superconductivity (26) and dynamical critical phenomena (3, 8). Applicability to physical problems will depend crucially on success in first modelling them with a few important mode amplitudes.

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