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SPACE CHARGE EQUATION CALCULATIONS FOR THE REFLEX TRIODE

J. W. Shearer

September 1976

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J. W. Shearer

MS. date: September, 1976

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SPACE CHARGE EQUATION CALCULATIONS FOR THE REFLEX TRIODE

Abstract

The Poisson equation solution of the space charge problem for the reflex triode is reviewed and illustrated with a number of examples. The numerical calculation technique for obtaining these results is briefly described. The results show that the characteristics of the triode are strong functions of the reflex-electron kinetic-energy spectrum, especially of the high-energy electrons.

Introduction

The pulsed high-voltage reflex triode¹ is an efficient source of ions of short duration (30 to 100 ns), high current (10 to 100 kA), and high voltage (0.8 to 1.8 MV) (Fig. 1). Anodes have been fabricated from wire grids,¹ from thin foils,^{2,3,4,5} from combination grid-foil meshes,⁶ and from plasmas.⁷ Recently the pulse duration has been extended to the 1- μ s range.⁸ The space charge equation theory of these devices has also been worked out.^{3,9}

In addition to the space charge theory of electron and ion motion in the anode-cathode gap, it is necessary to calculate the slowing down and scattering of the reflexing electrons by the anode. The two calculations are separable as long as the anode is sufficiently conducting to remain at constant potential. In the theories cited above, the space charge equation was solved exactly, but the reflex electron spectrum was approximated - either by arbitrary model spectra¹ or by simplifying approximations.⁹

In this report the previously developed space charge theory^{3,9} is reviewed and reformulated in a (hopefully) simple way. Then a small summation code is described for rapid calculation of specific cases of the reflex electron spectrum. Finally, a number of examples of these code calculations are presented and commented on.

In a later report it is intended to present Monte Carlo calculations of the electron transport through the anode foil, and to combine such calculations with these space charge equation calculations to formulate a useful design model for future experiments.

Fig. 1. Reflex triode (double diode) geometry and voltage profile (anode foil is thin compared to range of electrons accelerated across gap; electrons are scattered and slowed down by foil, but magnetic-field-induced spiraling motion inhibits transverse motion; ions are emitted in both directions).

The theory presented here is an equilibrium theory; transient effects¹⁰ are not included.

Space Charge Equation and Solutions

The one-dimensional Cartesian coordinate system is shown in Fig. 1; one needs only the left-hand side of the symmetrical triode ("double diode"). Then Poisson's equation is:

$$\epsilon_0 \frac{d^2V}{dx^2} = -Ze n_i + e (n_e + n_{ef}), \quad (1)$$

where n_i is the ion density, n_e is the density of full voltage electrons that have just been emitted by the cathode, and n_{ef} is the density of reflex electrons that have interacted with the anode one or more times.

Now neglect collisions and assume space-charge-limited emission of monoenergetic cathode electrons and anode ions. Then in the nonrelativistic approximation, one has simply:

$$n_i = f_i \left[\frac{M}{2Ze (V_0 - V)} \right]^{1/2}, \quad (2)$$

$$n_e = f_e \left(\frac{m}{2eV} \right)^{1/2}, \quad (3)$$

where f_i is the ion flux, f_e is the cathode electron flux (units of $\text{cm}^{-2} \text{sec}^{-1}$), and M is the ion mass.

The reflex electrons require a more extensive discussion because they are not monoenergetic; their energy spectrum is a function of scattering and slowing-down effects in the anode material. Such calculations will be discussed elsewhere. For the purpose of this report, it is sufficient to define a reflex-electron energy distribution function, $F_{||}$:

$$F_{||} \equiv \frac{1}{f_e} (df_r/dT_{||})_{V=V_0}, \quad (4)$$

where $eT_{||}$ is the kinetic energy at the anode corresponding to $v_{||}$ (the velocity parallel to the magnetic field),

$$eT_{||} = \frac{1}{2} m v_{||}^2, \quad (5)$$

and where $(df_r/dT_{||})$ is the number of electrons emitted per unit time per unit voltage interval at the anode. It is intuitively obvious that only the parallel velocity (and energy) are affected by the electric field; the perpendicular velocity (cyclotron motion) is unaffected.

In other words, the distribution function, $F_{||}$ - Eq. (4) - is the number of reflex electrons of parallel kinetic energy, $eT_{||}$, per unit voltage increment, $\Delta T_{||}$, per unit cathode electron. This distribution is defined at the anode $V = V_0$. It is also useful to define the parameter η , the average number of times each electron is incident on the anode; then

$$\eta - 1 = \int_0^{V_0} F_{||} dT_{||}, \quad (6)$$

where we have subtracted off the cathode electrons because $F_{||}$ is defined only in terms of the reflex electrons.

Note also that the reflex electron distribution changes as one moves away from the anode into regions of potential $V < V_0$. The low-energy electrons are reflected, and the high-energy electrons have a reduced energy and a lower velocity. Thus, the reflex electron density, n_{ef} , must be written down with care:

$$n_{ef} = 2 \int_{V_0-V}^{V_0} \left(\frac{df_r}{dT_{||}} \right)_{V=V_0} \left(\frac{1}{v_{ef}} \right)_V dT_{||}. \quad (7)$$

The factor 2 is necessary because all the reflex electrons are reflected somewhere in the gap ($T_{||} < V_0$); thus the space charge is doubled by the counterflowing reflex-electron fluxes. The velocity v_{ef} must be evaluated at the local voltage V :

$$v_{ef} = \left\{ \frac{2e}{m} [T_{||} - (V_0 - V)] \right\}^{1/2}. \quad (8)$$

The integration - Eq. (7) - is performed over the energy distribution at the anode, but the limits are chosen so that the velocity v_{ef} is always positive, thus accounting for all reflex electrons that are energetically capable of reaching the potential V . Combining Eqs. (4), (7), and (8), one obtains at last:

$$n_{ef} = 2f_e \left(\frac{m}{2e}\right)^{1/2} \int_{v_0-v}^{v_0} \frac{F_{\parallel} dT_{\parallel}}{[T_{\parallel} - (v_0 - v)]^{1/2}}. \quad (9)$$

A useful parameter in space charge diode theory is the ion-electron current ratio parameter α :

$$\alpha \equiv \frac{J_i}{J_e} \left(\frac{M}{mZ}\right)^{1/2} = Z \frac{f_i}{f_e} \left(\frac{M}{mZ}\right)^{1/2}, \quad (10)$$

where J_i is the ion current density ($J_i = Ze f_i$), and J_e is the electron current density ($J_e = e f_e$). Substituting this parameter α - Eq. (10) - and the particle densities - Eqs. (2), (3), and (9) - into Poisson's equation - Eq. (1) - one obtains:

$$\epsilon_0 \frac{d^2 V}{dx^2} = J_e \left(\frac{m}{2e}\right) \left\{ - \frac{\alpha}{(v_0 - v)^{1/2}} + \frac{1}{v^{1/2}} + 2 \int_{v_0-v}^v \frac{F_{\parallel} dT_{\parallel}}{[T_{\parallel} - (v_0 - v)]^{1/2}} \right\}. \quad (11)$$

This is the fundamental space charge equation for the reflex triode. It is a generalization of the well-known Child-Langmuir equations for a simple diode.¹¹ With the ion flow term (finite α , but not the reflex electrons), one obtains Langmuir's bipolar space charge equation.¹²

It is convenient to introduce dimensionless coordinates before proceeding further:

$$\zeta \equiv T_{\parallel} / v_0, \quad (12a)$$

$$u \equiv v/v_0, \quad (12b)$$

$$\lambda \equiv x/x_0, \quad (12c)$$

$$f_{\parallel} \equiv F_{\parallel} / (v_0)(n-1). \quad (12d)$$

Note that for the last case, one has:

$$F_{\parallel} dT_{\parallel} = (n-1) \int_0^1 f_{\parallel} ds. \quad (13)$$

In addition, the Child-Langmuir current¹⁰ J_0 for the diode is useful:

$$J_0 = \frac{4}{9} \epsilon_0 \left(\frac{2e}{m} \right)^{1/2} \left(V_0^{3/2} / x_0^2 \right). \quad (14)$$

Now, after substituting Eqs. (12) through (14) into Eq. (11) and after some algebraic manipulation, one obtains:

$$\frac{d^2 u}{d\lambda^2} = \frac{4}{9} \frac{J_e}{J_0} \left[- \frac{\alpha}{(1-u)^{1/2}} + \frac{1}{u^{1/2}} + 2(\eta-1) \int_{1-u}^1 \frac{f_{\parallel} d\zeta}{(\zeta-1+u)^{1/2}} \right]. \quad (15)$$

The Poisson equation for the space charge solution for the reflex triode is now ready to be integrated.

The first integral is obtained by multiplying both sides by 2 $(du/d\lambda)$ and integrating from $u = 0$ to $u = \phi$. Note also that the electric field $E(u)$ is:

$$E(u) = \frac{V_0}{x_0} \left(\frac{du}{d\lambda} \right) \equiv \bar{E} \frac{du}{d\lambda}. \quad (16)$$

Combining the results of the integration with Eq. (16):

$$\left[\frac{d\phi}{d\lambda} \right]^2 \left[\frac{E(\phi)}{\bar{E}} \right]^2 = \frac{16}{9} \frac{J_e}{J_0} G(\phi), \quad (17)$$

where the electric field function $G(\phi)$ is defined:

$$G(\phi) \equiv \phi^{1/2} - \alpha \left[1 - (1-\phi)^{1/2} \right] + (\eta-1) S(\phi), \quad (18)$$

where the "shape function" $S(\phi)$ is defined as the double integral:

$$S(\phi) \equiv \int_0^\phi du \int_{1-u}^1 \frac{f_{\parallel} d\zeta}{(\zeta-1+u)^{1/2}}. \quad (19)$$

In deriving Eq. (17) use was made of the space-charge-limited emission condition at the cathode by setting $E(0) = 0$. In addition, when the ion emission at the anode is space-charge-limited, one also has the condition: $E(1) = 0$, or $G(1) = 0$, from which one solves for the parameter α :

$$\eta = 1 + (\eta - 1) S(1). \quad (20)$$

A useful form for the function $G(\phi)$ is then obtained by substituting Eq. (20) into Eq. (18):

$$G(\phi) = \phi^{1/2} - 1 + (1 - \phi)^{1/2} - (\eta - 1) S(1) \left[1 - (1 - \phi)^{1/2} - \frac{S(\phi)}{S(1)} \right]. \quad (21)$$

The potential distribution across the diode gap is found from the second integral, which is derived from Eq. (17):

$$\chi = \frac{3}{4} \left(\frac{J_0}{J_e} \right)^{1/2} \int_0^\phi \frac{d\phi'}{[G(\phi')]^{1/2}}. \quad (22)$$

This integral completes the exposition of the solution of the space charge equation; these results were previously obtained (with slightly different algebra) at Physics International.³

An important property of these solutions is that they are not always real. A simple way to understand this is to examine the structure of the function $G(\phi)$ in Eq. (21). When no reflex electrons are present, the last term is zero, and $G(\phi)$ is always positive. For sufficiently large values of $\eta - 1$ (large numbers of reflex electrons), $G(\phi)$ will be negative at some values of ϕ . This result implies an imaginary solution for the electric field E , which is not physically possible.

Because electron range increases with voltage, it is obvious that $\eta - 1$ is an increasing function of voltage V_0 . In addition, in future reports it will be found that for many anode designs the shape function $S(\phi)$ does not vary strongly with voltage V_0 , and can be approximated by the same function over a considerable range of the parameter $\eta - 1$. This constant shape approximation is not exact, but is sufficiently good to permit considerable insight into the behavior of most anodes, particularly over small voltage ranges $\Delta V/V$.

Now consider in more detail the behavior of the constant shape approximation as one increases the voltage and corresponding reflex parameter η (remember that in this approximation $S(\phi)$ does not change). Eventually, η will

approach a critical value n_c for which $G(\phi_c) = 0$ at some particular value of $\phi = \phi_c$. Physically, this result means that the electric field E approaches zero at some particular critical value of voltage $V = V_c = \phi_c V_0$. Furthermore, the contribution to the second integral - Eq. (22) - grows quite large in this region of ϕ space, implying a large contribution to the total distance across the gap ($\Delta x = x_0 \Delta \lambda$). In other words, this solution implies that there will be a voltage plateau somewhere in the gap, and that this plateau will grow flatter and flatter as $n \rightarrow n_c$.

Furthermore, as $n \rightarrow n_c$ and the integral gets larger, one notes from Eq. (22) that the ratio J_e/J_0 must also increase to large values, because at $\lambda = 1$, $\phi = 1$, one obtains from Eq. (22):

$$\frac{J_e}{J_0} = \frac{9}{16} \left\{ \int_0^1 \frac{d\phi}{[G(\phi)]^{1/2}} \right\}^2. \quad (23)$$

Also, the ion current density, J_i , will increase, as obtained from Eqs. (10) and (20):

$$\frac{J_i}{J_0} = \frac{J_e}{J_0} \left(\frac{mZ}{M} \right)^{1/2} [1 + (\eta - 1) S(1)]. \quad (24)$$

This divergent behavior of J_e and J_i as $n \rightarrow n_c$, plus the nonexistence of equilibrium solutions for $n > n_c$, has led many workers to describe these solutions as "resonant."

Numerical Integration of Space Charge Equation

A short code is available to calculate numerical solutions of these equations. The voltage scale V_0 is subdivided into equal increment subintervals $\Delta V = V_0/N$ (usually $N = 200$). The various integrals are replaced by summations over the appropriate ranges. Consider first the combination of Eqs. (13) and (19):

$$(\eta - 1) S(\phi) = (\eta - 1) \int_{1-u}^1 \frac{f_{||} d\zeta}{(\zeta - 1 + u)^{1/2}}. \quad (25)$$

The integral over the variable ζ is replaced by:

$$\text{NSUM}(K) = \text{FRAC} \sum_{J=N-K+1}^{J=N-1} [\text{FLUX}(J)] \left[\frac{N}{J + K - N - \frac{1}{2}} \right]^{1/2}, \quad (26)$$

where FRAC is an adjustable input constant to vary $\eta - 1$, where FLUX (J) corresponds to $F \left| \frac{d\zeta}{d\lambda} \right|$, and where the dimensionless variables of the theory have been replaced by appropriate summation indices: ($\zeta \rightarrow J$, $u \rightarrow K$). The parameter $1/2$ is inserted to center each velocity in the middle of its appropriate subinterval.

The values of NSUM(K) are used in two ways. First, the parameter α is obtained by the replacement of Eq. (20):

$$\text{ALPHA} = 1 + \frac{1}{N} \sum_{K=1}^N \text{NSUM}(K). \quad (27)$$

Second, a zone-centered average is found:

$$\text{HAF}(K) = \frac{1}{2} [\text{NSUM}(K) + \text{NSUM}(K-1)]. \quad (28)$$

This zone-centered parameter is then summed over the index K, which approximates the integration over the variable u in Eq. (25):

$$\text{ESUM}(L) \equiv \frac{1}{N} \sum_{K=1}^{K=L} \text{HAF}(K), \quad (29)$$

where L becomes the summation index for the next summation.

The next step is to calculate $G(\phi)$ from Eq. (18); this is written:

$$\text{GEE}(L) = [\text{PHIL}(L)]^{1/2} - \text{ALPHA} \left\{ 1 - [1 - \text{PHI}(L)]^{1/2} \right\} + \text{ESUM}(L). \quad (30)$$

The code now tests for the algebraic sign of GEE(L) at all values of L ($0 < L \leq N$). If one or more negative values are detected, the calculation stops here, and the results to this point are printed out. If all values of GEE(L) are positive, the calculation proceeds through the remaining steps. The physical reason for this procedure has been described in the foregoing section of this report.

Equation (22) is next approximated by the sum,

$$VSUM(M) = \frac{1}{N} \sum_{L=1}^{L=M} \left[\frac{1}{GEE(L)} \right]^{1/2}, \quad (31)$$

and the following output parameters are then determined by comparison with Eqs. (10), (22), (23), and (24):

$$LAMDA(M) = VSUM(M)/VSUM(N), \quad (32a)$$

$$ECUR = (9/16)[VSUM(N)]^2 \quad (32b)$$

$$ICUR = (ECUR)(ALPHA)(M/mz)^{1/2}, \quad (32c)$$

where the parameter M/mz is an input constant determined by the ion species in the triode. Usually, the calculations are done for deuterium ions.

Another parameter of interest is the net space charge per unit volume, as given by the right-hand side of Eq. (1). Let the dimensionless space charge ρ be defined:

$$\rho \equiv \frac{V_0}{J_e} \left(\frac{2e}{m} \right)^{1/2} \left(\epsilon_0 \frac{d^2 V}{dx^2} \right). \quad (33)$$

Then from Eqs. (11) and (12):

$$\rho = \frac{\alpha}{(1 - \phi)^{1/2}} - \frac{1}{\phi^{1/2}} - 2 \int_{1-u}^1 \frac{F_{||} dT_{||}}{(1 - \zeta - 1 + u)^{1/2}}. \quad (34)$$

The corresponding expression from the numerical calculation is written:

$$SPCH(K) = \frac{ALPHA}{[1 - PHI(K)]^{1/2}} - \left[\frac{1}{PHI(K)} \right]^{1/2} - 2 HAF(K). \quad (35)$$

This expression can have either sign, according to whether the ion or electron space charge is dominant.

Two output graphs are prepared by the numerical code: dimensionless voltage $\phi = PHI$ vs dimensionless gap distance λ , and the dimensionless space charge $\rho = SPCH$ vs the same parameter λ .

Numerical Results for Some Model Spectra

The first numerical problem is a calculation of Langmuir's bipolar solution.¹² In this case there are no reflex electrons, but there is space-charge-limited emission of both electrons and ions. Figure 2 presents the calculated voltage and space charge profiles, showing a symmetrical distribution with an excess of negative electrons near the cathode and an excess of positive ions near the anode. Table 1 is a comparison of the computed values of electron current enhancement by the ions for the code vs the references from this table. One notes that the present numerical computation is a closer approximation to the exact elliptic integral solution than the early numerical work of Langmuir. Similar accuracies were also found for a numerical comparison at four points on the voltage profile in Fig. 2. This close check with earlier work on the bipolar flow problem gives one confidence in the correctness of the code.

Table 1. Bipolar electron current enhancement.

Reference	Method	J_e/J_0
Langmuir ¹² (1929)	Numerical (desk)	1.8605
Müller-Lübeck ¹² (1951)	Elliptic integrals	1.85324
Shearer (1976)	Numerical (computer)	1.8537

The next series of numerical calculations used the model spectra, which were hand-calculated in the early work.³ These models were linear fits to the reflex-electron flux function, $f_{||}$ -- Eqs. (12) and (13):

$$f_{||} \text{ (model spectra)} \equiv f_{\text{MOD}} \equiv a + b\zeta, \quad (36)$$

where a and b are input constants subject to the unity normalization condition:

$$\int_0^1 f_{\text{MOD}} d\zeta = 1 = a + b/2. \quad (37)$$

Fig. 2. Langmuir's bipolar solution: left graph is a plot of dimensionless gap voltage $\phi = V/V_0$ vs dimensionless gap distance $\alpha = x/x_0$; right graph is space charge profile - Eq. (35) - plotted vs same distance parameter α . Positive space charge is indicated by plus (+) sign; negative space charge is indicated by zero (0).

For this function the integrals in the theory can be carried out analytically. The first integral in Eq. (19) becomes:

$$\int_{1-u}^1 \frac{(a+b\zeta) d\zeta}{(\zeta-1+u)^{1/2}} = \int_0^u \frac{a+b(1-u+v)}{v^{1/2}} dv = 2(a+b)u^{1/2} + \frac{4}{3}bu^{3/2}. \quad (38)$$

Carrying out the second integral of Eq. (19), one obtains:

$$S(\phi) = \frac{4}{3}(a+b)\phi^{3/2} - \frac{8}{15}b\phi^{5/2}, \quad (39)$$

and the ion-electron current ratio parameter α is found from Eq. (20):

$$\alpha = 1 + \frac{4}{3}(\eta-1)\left(a + \frac{3}{5}b\right). \quad (40)$$

The analytical determination of the critical reflex parameter, η_c , is more difficult, because Eq. (21) for $G(\phi) = 0$ at some unknown value of ϕ . The

following method is the best hand calculation procedure for this purpose that the author could devise. At every ϕ there is some maximum value $\eta_0(\phi)$ at which $G(\phi) = 0$. In that case one can solve for $\eta_0(\phi)$:

$$\eta_0(\phi) = 1 + \frac{\phi^{1/2} - 1 + (1 - \phi)^{1/2}}{S(1) \left[1 - (1 - \phi)^{1/2} - \frac{S(\phi)}{S(1)} \right]}. \quad (41)$$

Then the critical η_c is the minimum value of $\eta_0(\phi)$. In principle, one could find this minimum by calculating the derivative $d\eta_0/d\phi$, setting it equal to zero, and solving for ϕ . However, in practice this becomes a gargantuan insoluble equation for ϕ . Instead, one substitutes trial values of ϕ into Eq. (41), and searches for the minimum of $\eta_0(\phi)$ by successive approximation. The method is not elegant, but with moderate effort it converges within a few percent of the correct value.

The numerical determination of the critical reflex parameter, η_c , is also a convergence method. For each spectrum one runs the numerical integration code for different trial values of the input constant FRAC - Eq. (26) - which corresponds to the parameter $\eta - 1$. The GEE(L) output numbers are searched for the minimum value of GEE(L). Linear interpolation is used to make the best estimate of η_c where the minimum value of GEE(L) = 0. This is the method used in the succeeding examples; only a few iterations suffice to find η_c to better than 1% accuracy. The ultimate accuracy is difficult to estimate, because it depends on the number of integration steps N.

Table 2 summarizes the results for three different model reflex-electron spectra, utilizing the analytical and numerical results just described. Spectrum A contains predominantly low-energy electrons, and in that case only a low number of electron reflections are possible in steady state. The corresponding ion-electron current ratio is low, implying poor efficiency for ion production. On the other hand, the comparatively high-energy spectrum C has a large average number of electron reflections and better ion current efficiency. The achievable current densities for such production will be discussed later.

Consider now additional details of the numerical results. Figure 3 shows the results for spectrum A at a value of η that is close to η_c . A comparison of Fig. 3 with Fig. 2 shows that the reflex electron have considerably changed the net voltage and charge distributions in the gap.

Table 2. Parameters for Model spectra.³

spectrum name	Eq. (36) Constants		Eq. (39) Shape function $S(\phi)$	Numerical determination η_c	Eq. (40) ratio parameter α at η_c
	a	b			
A	2.	02.	$\frac{16}{15} \phi^{5/2}$	4.136	4.345
B	1.	0.	$\frac{4}{3} \phi^{3/2}$	10.716	13.95
C	0.	2.	$\frac{8}{3} \phi^{3/2} (1 - \frac{2}{5} \phi)$	15.4705	24.15

 Fig. 3. Spectrum A: $\eta = 4.135$; $(J_e/J_0) = 28.84$; $(J_i/J_0) = 2.013$ (deuterium ions).

Note also that the ion and electron current densities have both been considerably enhanced above their bipolar solution values. The space charge profile (Fig. 3) shows that the reflex electrons have created a negative charge cloud near the anode. This charge cloud increases the anode electric field E , which pulls an increased ion current from the anode. This ion current, in turn, creates a net positive charge cloud near the cathode, thus increasing the cathode electric field and pulling out more electron current.

In this way, the two currents "bootstrap" each other to higher values. In the middle of the gap, the electric field and the charge density are at low values. The counter streaming electron and ion currents can be thought of as a kind of plasma, since their space charges almost cancel.

Figures 4, 5, and 6 are profiles of a sequence of calculations done for spectrum B (see Table 2) at three different values of the reflex parameter η . When these figures are compared with Fig. 2, one observes a transitional sequence from the bipolar solution up to nearly $\eta = \eta_c$. A comparison of Figs. 3 and 6 reveals that the plateau voltage near η_c is lower for the higher average electron energy spectrum B than for the softer electron energy spectrum A. Also, this sequence shows how the ion current ratio J_1/J_0 increases rapidly as the voltage gradient near the anode steepens with increasing η . The higher energy electron spectrum C (Table 2) is shown in Fig. 7 for a reflex parameter η very close to η_c . The plateau voltage is even lower than the previous examples, continuing the same trends.

Since there is interest in even higher ion-electron current ratios, steeper higher energy spectra were investigated analytically. The most interesting was the quartic spectrum:

$$f_{\parallel} = f_Q \equiv 5\zeta^4. \quad (42)$$

Carrying out the integrations of Eq. (19) involves some tedious algebra, which is omitted here. The result for the quartic spectrum is:

$$S_Q(\phi) = 4\phi^{3/2} \left(\frac{5}{3} - \frac{8}{3}\phi + \frac{16}{7}\phi^2 - \frac{64}{63}\phi^3 + \frac{128}{693}\phi^4 \right). \quad (43)$$

Equation (20) for alpha then becomes:

$$\alpha_Q = 1 + \frac{20}{11} (\eta - 1), \quad (44)$$

for the quartic spectrum. With the numerical method described above, the critical value of η was found to be $\eta_c = 37.366$; whence the maximum $\alpha_Q = 67.12$. Figure 8 shows profiles for one numerical run with this spectrum. The plateau potential is so close to the cathode voltage that the accuracy of the calculation may not be particularly high. Nevertheless, one does find the expected increase in ion-electron current ratio for this very energetic reflex-electron spectrum model.

Fig. 4. Spectrum B: $\eta = 7$; $(J_e/J_0) = 1.3013$; $(J_1/J_0) = 0.1884$ (deuterium ions).

Fig. 5. Spectrum B: $\eta = 10$; $(J_e/J_0) = 2.0197$; $(J_1/J_0) = 0.4219$ (deuterium ions).

Fig. 6. Spectrum B: $\eta = 10.715$; $(J_e/J_0) = 10.730$; $(J_i/J_0) = 2.405$ (deuterium ions).

Fig. 7. Spectrum C: $\eta = 15.4700$; $(J_e/J_0) = 7.0388$; $(J_i/J_0) = 2.739$ (deuterium ions).

Fig. 8. Quartic spectrum; $\eta = 37.35$; $(J_e/J_0) = 0.899$; $(J_1/J_0) = 0.965$ (deuterium ions).

Overall Results and Summary

Further insight into these results can be obtained by plotting the electron and deuterium ion current ratios vs the reflex parameter η , as is done in Fig. 9. The model spectra A, B, and C (Table 2), the quartic spectrum Q, and the open grid anode spectrum derived at Cornell¹ are all plotted vs η . At low values of η , the ion current increases with η for most spectra, while the electron current usually decreases. Near $\eta = \eta_c$, both currents increase. In order to attain both a high current and a good ion current efficiency, one wants to have a high-energy reflex-electron spectrum and to be able to sustain high currents at the voltage that corresponds to the critical reflex parameter, η_c .

Such high-energy spectra as C and Q may be unrealistic because they neglect the initial absorption of the cathode beam in the anode foil before it becomes a reflex electron. Thus, it is unlikely that there will ever be a high reflex-electron flux, $f_{||}$, as $\zeta \rightarrow 1$. To test the effect of such high-energy electrons, spectrum C was modified by multiplying it by an additional function:

$$f_{||}(C^*) = b^* \zeta \left[1 - \exp \left(- \left(\frac{1 - \zeta}{0.1} \right)^2 \right) \right] \quad (45)$$

Fig. 9. Electron and ion current ratios (for various spectrum models described in the text) vs the parameter η (deuterium ions are assumed).

The constant b^* was adjusted numerically so that the flux function $f(C^*)$ was normalized to unity - Eq. (13). This modified spectrum C^* was compared with the unmodified spectrum C in several numerical runs. The results (Fig. 10) show that this small change in the number of high-energy electrons in the spectrum produces a large change in the current characteristics of the solution. This result is intuitively reasonable because the high-energy reflex electrons travel across most of the gap before they are reflected; therefore, they have a greater influence on the potential distribution and on the currents than the low-energy electrons do. Thus, it is important to have a good knowledge of the high-energy spectrum shape.¹³ It is intended that the calculation of such spectra be discussed in future reports.

The efficiency ϵ of ion current production is defined:

$$\epsilon \equiv \frac{J_i}{J_e + J_i} = \frac{1}{1 + \frac{1}{\alpha} \left(\frac{M}{mZ} \right)^{1/2}} \quad (46)$$

From the numerical calculation the profile of efficiency vs deuterium ion current (Fig. 11) is plotted for each model spectrum previously discussed. It is noted that for each spectrum the efficiency drops at low ion current, corresponding to low values of n/n_c . Also, high efficiencies correlate with high-energy reflex spectra, as one expects from the previously plotted results.

Finally, some other workers⁶ have asserted that these results are inconsistent with Poisson's equation. Their argument depends on an assumption that no reflex electrons are present in the voltage plateau region of the gap. The numerical calculations just described, however, do not agree with this assumption; for all model spectra considered there is a non-negligible fraction of reflex electrons moving back and forth through the voltage plateau region. Thus the above argument⁶ cannot be applied to the problems considered here. Indeed it would be surprising to find any inconsistency; the whole derivation and calculation are simply a straightforward application of Poisson's equation itself.

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Fig. 10. Comparison of model spectra C and C* (deuterium ions are assumed).

Fig. 11. Efficiency ϵ vs ion current for deuterium ions and various model spectra.

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