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HYDRODYNAMIC STABILITY OF LITHIUM FLOWS ON POROUS SPHERICAL SURFACES

by

Roy A. Axford

ABSTRACT

Lithium flows on the inside surface of a porous spherical shell are analyzed to determine their stability characteristics against small perturbations in (1) the film thickness, (2) the colatitude component of the lithium velocity, and (3) the rate of mass supply of lithium through the porous wall. A linearized stability analysis is developed, and it is concluded that lithium film flows are stable against these types of perturbations.

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I. INTRODUCTION

Bohachevsky, Booth, and Hafer¹ have developed a steady-state model that describes the flow of lithium in a thin film on the inside surface of a porous spherical shell. A class of stability problems associated with such flows is analyzed in this study. Specifically, we follow the time evolution of small perturbations in (1) the film thickness and (2) the colatitude component of the lithium velocity when the rate of mass supply of lithium through the porous wall is perturbed, and when nonzero initial value perturbations are introduced into the film thickness and lithium speed. We conclude on the basis of linear stability analysis that the steady-state flows analyzed by Bohachevsky, Booth, and Hafer¹ are stable against small perturbations in (1) the film thickness, (2) the colatitude component of the lithium velocity, and (3) the rate of mass supply of lithium through the porous wall.

II. DERIVATION OF THE TRANSIENT MASS AND MOMENTUM BALANCES

The governing equations that express conservation of mass and momentum for time-dependent motions of a thin lithium film on the inner surface of a porous spherical shell can be derived as follows.

We consider a control volume that is defined by the following relations. Let A_t be the cross section of the control volume that is perpendicular to the tangential direction to the inner wall of the shell, and let A_n be the cross section perpendicular to the normal direction to the wall. If $\delta(\theta, t)$ is the film thickness at colatitude angle θ at time t , then

$$A_t = R \sin\theta \, d\phi \, \delta, \quad (1)$$

and

$$A_n = R^2 \sin\theta \, d\phi \, d\theta, \quad (2)$$

where ϕ is the azimuthal angle, and R is the radius of the inner surface of the shell. If we set

$$h = R - R \cos\theta, \quad (3)$$

then we can write the differential volume element $d\tau$ of the control volume in the following alternative ways;

$$d\tau = A_t \frac{dh}{\sin\theta} = A_t R d\theta = R^2 \delta \sin\theta \, d\theta \, d\phi, \quad (4)$$

and

$$d\tau = A_n \delta = R^2 \delta \sin\theta \, d\theta \, d\phi. \quad (5)$$

To derive the transient mass balance, let $m(\theta, t)$ be the rate of mass supply per unit area through the porous wall. Alternatively, we can interpret $m(\theta, t)$ as the difference between the rate of mass supply per unit area and the ablation rate per unit area on the inner surface of the lithium film. For example,

$$m(\theta, t) = m(\theta) + M(\theta, t) - E(\theta, t), \quad (6)$$

in which $m(\theta)$ is the steady-state rate of mass supply per unit area, $M(\theta, t)$ is a perturbation in the rate of mass supply per unit area, and $E(\theta, t)$ is the ablation rate per unit area on the inner surface of the film.

Let $v(\theta, t)$ be the colatitude component of the mean lithium velocity. A statement of conservation of mass is based upon the following facts:

1. $\rho A_n \delta$ = mass in the control volume at time t ;
2. $m A_n \Delta t$ = mass entering the control volume during the time interval Δt as a result of mass supply, or mass supply minus loss due to ablation;

3. $\rho v A_t \Delta t$ = mass entering the control volume through its upper surface during the time interval Δt ;
4. $\rho v A_t \Delta t + (\partial/\partial\theta)(\rho v A_t \Delta t) d\theta$ = mass leaving the control volume through its lower surface during the time interval Δt ;
5. $\rho A_n \delta + (\partial/\partial t)(\rho A_n \delta) \Delta t$ = mass in the control volume at the time $t + \Delta t$.

Accordingly, conservation of mass may first be expressed in the form

$$\frac{\partial}{\partial t} (A_n \delta) = \frac{m}{\rho} A_n - \frac{\partial}{\partial \theta} (v A_t) d\theta . \quad (7)$$

By combining Eqs. (1), (2), and (7) we obtain

$$R \frac{\partial \delta}{\partial t} = \frac{mR}{\rho} - \frac{\partial}{\partial \theta} (v \delta) - \cot \theta (v \delta) \quad (8)$$

as the fundamental continuity equation.

A quantitative statement of conservation of momentum comes out of the following facts:

1. $\rho v A_n \delta$ = momentum of the fluid in the control volume at time t ;
2. $\rho v^2 A_t \Delta t$ = momentum of the fluid entering the control volume through its upper surface during the time interval Δt ;
3. $\rho v A_n \delta + (\partial/\partial t)(\rho v A_n \delta) \Delta t$ = momentum of the fluid in the control volume at time $t + \Delta t$;
4. $\rho v^2 A_t \Delta t + (\partial/\partial \theta)(\rho v^2 A_t) d\theta \Delta t$ = momentum of the fluid leaving the control volume through its lower surface during the time interval Δt ;
5. $(\partial/\partial t)(\rho v A_n \delta) + (\partial/\partial \theta)(\rho v^2 A_t) d\theta$ = rate of change of momentum of the fluid in the control volume = (body force per unit volume) $d\tau$ - (viscous drag force per unit volume) $d\tau$;
6. $\rho g \sin \theta$ = body force per unit volume (parallel to surface of wall), where g is the acceleration due to gravity;
7. $\mu(\partial^2 v / \partial y^2)$ = viscous drag force per unit volume.

To compute an estimate of the viscous drag force per unit volume, it is necessary to assume a dependence of the lithium speed upon the distance y from the wall. We assume a velocity profile of the form

$$v(y) = \begin{cases} v_{\max} \left[1 - \left(1 - \frac{y}{f\delta} \right)^2 \right] , & \text{for } 0 \leq y \leq f\delta , \\ v_{\max} , & \text{for } f\delta \leq y \leq \delta , \end{cases} \quad (9)$$

in which f is defined as the fraction of the film thickness which is a laminar sublayer with an assumed parabolic shape. The mean lithium speed is given by

$$v\delta = \delta(1 - f)v_{\max} + \int_0^{f\delta} v(y)dy, \quad (10)$$

from which it follows that

$$v_{\max} = \frac{v}{1 - \frac{f}{3}}. \quad (11)$$

Therefore, the viscous drag force per unit volume is estimated to be

$$\mu \frac{\partial^2 v}{\partial y^2}(y) = \frac{2v\mu}{(1 - \frac{f}{3})f^2\delta^2}. \quad (12)$$

If $f = 0.10$, then the factor $1 - f/3$ in the denominator may be replaced by unity without introducing significant error.

Combining the above facts produces the first form for momentum conservation, namely,

$$\frac{\partial}{\partial t}(\rho v \delta A_n) + \frac{\partial}{\partial \theta}(\rho v^2 A_t) d\theta = \rho g \sin\theta d\tau - \frac{2\mu v A_n \delta}{f^2 \delta^2}. \quad (13)$$

With Eqs. (1), (2), (5), and (13), we find that conservation of momentum takes the form,

$$R \frac{\partial}{\partial t}(v\delta) + v\delta \frac{\partial v}{\partial \theta} + v \frac{\partial}{\partial \theta}(v\delta) + \cot\theta v^2 \delta = gR\delta \sin\theta - \frac{2\nu Rv}{f^2 \delta}, \quad (14)$$

where ν is the kinematic viscosity of the lithium. By using the conservation of mass relation of Eq. (8) to eliminate $(\partial/\partial\theta)(v\delta)$ from Eq. (14), it follows that the transient momentum balance can be simplified to the form

$$R\delta \frac{\partial v}{\partial t} + v\delta \frac{\partial v}{\partial \theta} = gR\delta \sin\theta - \frac{v\mu R}{\rho} - \frac{2\nu Rv}{f^2 \delta}. \quad (15)$$

The conservation of mass and momentum relations that provide the starting point for stability analysis will be taken as Eq. (8) and Eq. (15), respectively.

III. LINEARIZED PERTURBATION EQUATIONS

The instantaneous film thickness and colatitude component of the lithium velocity were shown in the previous section to be governed by a set of two nonlinear, first-order partial differential equations, namely, Eqs. (8) and (15). The steady-state counterpart of the mass balance is

$$\frac{\partial}{\partial \theta} (v\delta) + \cot \theta v\delta = \frac{mR}{\rho}, \quad (16)$$

and that of the momentum balance is

$$v\delta \frac{\partial v}{\partial \theta} = gR\delta \sin \theta - \frac{vmR}{\rho} - \frac{2vRv}{f^2\delta}. \quad (17)$$

In these last two equations the lithium film thickness, speed, and rate of mass supply are functions of only the colatitude angle.

To perform a stability analysis of the initial steady state, the film thickness, the lithium speed, and the rate of mass supply in the transient mass and momentum balances, Eqs. (8) and (15), are replaced by

$$\delta \rightarrow \delta(\theta) + \Delta(\theta, t), \quad (18)$$

$$v \rightarrow v(\theta) + V(\theta, t), \quad (19)$$

and

$$m \rightarrow m(\theta) + M(\theta, t), \quad (20)$$

respectively. The first terms on the right hand sides of Eqs. (18) through (20) are the initial steady-state values obtained by solving the set of two nonlinear, first-order partial differential equations contained in Eqs. (16) and (17). The second terms on the right hand sides of Eqs. (18) through (20) are small perturbations whose products are regarded as negligible. We are interested in analyzing the stability of the steady-state flow when the perturbation $M(\theta, t)$ in the rate of mass supply is specified, and when the initial values $\Delta(\theta, 0)$ and $V(\theta, 0)$ of the perturbations in the film thickness and the lithium speed, respectively, are specified.

The transient mass and momentum balances can be linearized by substituting Eqs. (18) through (20) into Eqs. (8) and (15) and by neglecting terms that contain products of the perturbations. When this is done, it is found that the perturbations in the lithium film thickness and speed are governed by the following set of two linear first-order partial differential equations. The

transient mass balance linearizes to

$$R \frac{\partial \Delta}{\partial t} = \frac{MR}{\rho} - \frac{\partial}{\partial \theta} (v\Delta) - \frac{\partial}{\partial \theta} (\delta V) - \cot \theta v\Delta - \cot \theta \delta V, \quad (21)$$

and the transient momentum balance linearizes to

$$\begin{aligned} R\delta \frac{\partial v}{\partial t} + v\delta \frac{\partial v}{\partial \theta} + v\Delta \frac{\partial v}{\partial \theta} + \delta V \frac{\partial v}{\partial \theta} \\ = gR\Delta \sin \theta - \frac{mRV}{\rho} - \frac{vRM}{\rho} - \frac{2\nu RV}{f^2 \delta} + \frac{2\nu Rv\Delta}{f^2 \delta^2}. \end{aligned} \quad (22)$$

It is emphasized that in Eqs. (21) and (22) $v = v(\theta)$ is the steady-state solution for the lithium speed and that $\delta = \delta(\theta)$ is the steady-state solution for the lithium film thickness as obtained by solving Eqs. (16) and (17) for a specified rate of mass supply, i.e., for a given $m = m(\theta)$. Consequently, even though Eqs. (21) and (22) are linear in the perturbed quantities, these two equations contain coefficients that are complex functions of the colatitude angle that satisfy a set of nonlinear first-order partial differential equations whose exact solutions are not known. Although Eq. (16) can be solved exactly to give an expression for the steady-state film thickness in terms of the steady-state speed, the subsequential elimination of the film thickness from Eq. (17) then yields a form of Abel's equation of the first kind for the steady-state lithium speed. An exact solution of this particular example of a nonlinear first-order ordinary differential equation of Abel's type has not been obtained.

IV. ORDINARY DIFFERENTIAL EQUATIONS SATISFIED BY THE LAPLACE TRANSFORMS OF THE FILM THICKNESS AND LITHIUM SPEED PERTURBATIONS

The time evolutions of small perturbations in the lithium film thickness and speed are governed by Eqs. (21) and (22) when the perturbation $M(\theta, t)$ of the rate of mass supply is regarded as a specified function. To incorporate specified initial values of the perturbations in the lithium film thickness and speed, we take the Laplace transforms with respect to the time coordinate of the linearized transient mass and momentum balances.

Let the Laplace transforms of the perturbations in the lithium film thickness, lithium speed, and rate of mass supply be defined by

$$\bar{\Delta} = \bar{\Delta}(\theta, p) = \int_0^{\infty} dt e^{-pt} \Delta(\theta, t), \quad (23)$$

$$\bar{V} = \bar{V}(\theta, p) = \int_0^{\infty} dt e^{-pt} V(\theta, t), \quad (24)$$

and

$$\bar{M} = \bar{M}(\theta, p) = \int_0^{\infty} dt e^{-pt} M(\theta, t), \quad (25)$$

respectively. Taking the Laplace transform of the linearized transient mass balance, Eq. (21), produces

$$\frac{\partial}{\partial \theta} (v \bar{\Delta}) + \frac{\partial}{\partial \theta} (\delta \bar{V}) + v \cot \theta \bar{\Delta} + \delta \cot \theta \bar{V} + R p \bar{\Delta} = \Delta_0 R + \bar{M} \frac{R}{\rho}, \quad (26)$$

where $\Delta_0 = \Delta(\theta, 0)$ is the initial value of the perturbation in the film thickness. The Laplace transform of the linearized transient momentum balance, Eq. (22), is

$$\begin{aligned} \delta \frac{\partial}{\partial \theta} (v \bar{V}) + R \delta p \bar{V} + v \frac{\partial v}{\partial \theta} \bar{\Delta} - g R \sin \theta \bar{\Delta} - \frac{2\nu R v \bar{\Delta}}{f^2 \delta^2} + \frac{m R \bar{V}}{\rho} + \frac{2\nu R \bar{V}}{f^2 \delta} \\ = R \delta V_0 - \frac{v R \bar{M}}{\rho}, \end{aligned} \quad (27)$$

where $V_0 = V(\theta, 0)$ is the initial value of the perturbation in the lithium speed. The last two equations, Eqs. (26) and (27), provide a coupled set of first-order linear ordinary differential equations for the Laplace transforms of the perturbations in the lithium film thickness and the lithium speed. The coefficients in these two equations are complicated functions of the colatitude angle because they contain the steady-state solutions for the lithium film thickness and speed.

As shown in the next section it is not necessary to obtain the complete solutions of Eqs. (26) and (27) in order to draw conclusions relative to the stability of the steady-state flow. Such conclusions can be obtained by considering the limit of Eqs. (26) and (27) as the colatitude angle approaches zero.

V. VALUES OF THE LAPLACE TRANSFORMS OF THE FILM THICKNESS AND LITHIUM SPEED DERIVATIVE PERTURBATIONS AT $\theta = 0$

Without a knowledge of exact solutions for the steady-state lithium film thickness and speed as governed by the nonlinear set, Eqs. (16) and (17), the

possibility of obtaining exact solutions for the Laplace transforms of the perturbations in the lithium film thickness and speed by solving the linear set, Eqs. (26) and (27), exactly appears to be somewhat remote. However, by taking the limits of Eqs. (26) and (27) as the colatitude angle approaches zero, it is possible to generate a set of two linear inhomogeneous algebraic equations whose solution will provide the values of the Laplace transforms of the perturbations in the lithium film thickness and first derivative of the lithium speed, with respect to the colatitude angle evaluated at $\theta = 0$, i.e., at the top of the inner surface of the spherical shell. The first derivative of the perturbation of the lithium speed evaluated at $\theta = 0$ enters in because the lithium speed itself together with its perturbation vanishes at $\theta = 0$.

To perceive the $\theta = 0$ limit of Eqs. (26) and (27) more easily, we first rewrite Eq. (26) in the alternative form

$$v \frac{\partial \bar{\Delta}}{\partial \theta} + \bar{\Delta} \frac{\partial v}{\partial \theta} + \bar{v} \frac{\partial \delta}{\partial \theta} + \delta \frac{\partial \bar{v}}{\partial \theta} + \cos \theta \bar{\Delta} \frac{v}{\sin \theta} + \cos \theta \delta \frac{\bar{v}}{\sin \theta} + R \rho \bar{\Delta} = \Delta_0 R + \frac{\bar{M} R}{\rho}, \quad (28)$$

and Eq. (27) in the alternative form

$$\begin{aligned} & \delta \frac{\partial \bar{v}}{\partial \theta} \frac{v}{\sin \theta} + \delta \frac{\partial v}{\partial \theta} \frac{\bar{v}}{\sin \theta} + R \delta \rho \frac{\bar{v}}{\sin \theta} + \frac{v}{\sin \theta} \frac{\partial v}{\partial \theta} \bar{\Delta} - g R \bar{\Delta} - \frac{2 \nu R \bar{\Delta}}{f^2 \delta^2} \frac{v}{\sin \theta} \\ & + \frac{m R}{\rho} \frac{\bar{v}}{\sin \theta} + \frac{2 \nu R}{f^2 \delta} \frac{\bar{v}}{\sin \theta} = \frac{R \delta v_0}{\sin \theta} - \frac{R \bar{M} v}{\rho \sin \theta}. \end{aligned} \quad (29)$$

The following limits are required when taking the $\theta = 0$ limit of Eqs. (28) and (29);

$$\lim_{\theta \rightarrow 0} \frac{\partial v}{\partial \theta} = v_1, \quad (30)$$

$$\lim_{\theta \rightarrow 0} \frac{\partial \bar{v}}{\partial \theta} = \bar{v}_1, \quad (31)$$

$$\lim_{\theta \rightarrow 0} \frac{v}{\sin \theta} = v_1, \quad (32)$$

$$\lim_{\theta \rightarrow 0} \frac{\bar{v}}{\sin \theta} = \bar{v}_1, \quad (33)$$

and

$$\lim_{\theta \rightarrow 0} \frac{V_0}{\sin \theta} = V_{10}. \quad (34)$$

In Eq. (34) we define V_{10} to be the initial value of the perturbation $(\partial V / \partial \theta)$ at $\theta = 0$. Because of the above limits and the initial conditions at $\theta = 0$,

$$v(\theta) \xrightarrow{\theta \rightarrow 0} 0, \quad (35)$$

and

$$\bar{V}(\theta, p) \xrightarrow{\theta \rightarrow 0} 0, \quad (36)$$

it can be seen that taking the $\theta = 0$ limit of Eq. (28) yields

$$(2v_1 + R\rho)\bar{\Delta}(0, p) + 2\delta_0 \bar{V}_1(p) = \Delta_{00}R + \frac{R\bar{M}(0, p)}{\rho}, \quad (37)$$

where δ_0 is the steady-state film thickness evaluated at $\theta = 0$, and Δ_{00} is the initial value of the film thickness perturbation at $\theta = 0$. The $\theta = 0$ limit of Eq. (29) provides

$$\begin{aligned} & \left(v_1^2 - gR - \frac{2\nu R v_1}{f^2 \delta_0^2} \right) \bar{\Delta}(0, p) + \left(R\delta_0 p + 2\delta_0 v_1 + \frac{m_0 R}{\rho} + \frac{2\nu R}{f^2 \delta_0} \right) \bar{V}_1(p) \\ &= R\delta_0 V_{10} - \frac{v_1 R \bar{M}(0, p)}{\rho}, \end{aligned} \quad (38)$$

where m_0 is the value of the unperturbed rate of mass supply evaluated at $\theta = 0$.

The Laplace transform $\bar{\Delta}(0, p)$ of the film thickness perturbation evaluated at $\theta = 0$ and the Laplace transform $\bar{V}_1(p)$ of the first derivative of the lithium speed perturbation with respect to the colatitude angle at $\theta = 0$ are obtained by solving the linear algebraic set of Eqs. (37) and (38). It is found that

$$\bar{\Delta}(0, p) = \frac{1}{D_S} \left\{ \left(p + \frac{2v_1}{R} + \frac{m_0}{\rho \delta_0} + \frac{2\nu}{f^2 \delta_0^2} \right) \left(\Delta_{00} + \frac{\bar{M}(0, p)}{\rho} \right) - \frac{2\delta_0}{R} \left(v_{10} - \frac{v_1 \bar{M}(0, p)}{\delta_0 \rho} \right) \right\}, \quad (39)$$

and

$$\bar{V}_1(p) = \frac{1}{D_S} \left\{ \left(p + \frac{2v_1}{R} \right) \left(v_{10} - \frac{v_1 \bar{M}(0, p)}{\delta_0 \rho} \right) - \frac{1}{R\delta_0} \left(v_1^2 - gR - \frac{2\nu R v_1}{f^2 \delta_0^2} \right) \left(\Delta_{00} + \frac{\bar{M}(0, p)}{\rho} \right) \right\}. \quad (40)$$

In Eqs. (39) and (40) the denominator is given by

$$D_S = p^2 + \left(\frac{4v_1}{R} + \frac{m_o}{\rho\delta_o} + \frac{2\nu}{f^2\delta_o^2} \right) p + \frac{2v_1^2}{R^2} + \frac{2g}{R} + \frac{2v_1 m_o}{R\rho\delta_o} + \frac{8\nu v_1}{Rf^2\delta_o^2} . \quad (41)$$

The nature of the transient response is determined by the character of the two roots of the quadratic equation, $D_S = 0$.

The values of the steady-state film thickness δ_o , the colatitude derivative of the lithium speed v_1 , and the rate of mass supply m_o at $\theta = 0$ that appear in the results for the Laplace transforms of the perturbations in Eqs. (39)-(41) are determined from relations obtained by taking the $\theta = 0$ limit of the steady-state mass and momentum balances, Eqs. (16) and (17). The $\theta = 0$ limit of Eq. (16) is

$$\delta_o v_1 = \frac{m_o R}{2\rho} , \quad (42)$$

and that of Eq. (16) is

$$v_1 = \frac{gR}{v_1} - \frac{m_o R}{\rho\delta_o} - \frac{2\nu R}{f^2\delta_o^2} . \quad (43)$$

Eliminating v_1 from Eq. (43) with Eq. (42) yields

$$\frac{3}{4} m_o^2 + \frac{\nu\rho m_o}{f^2\delta_o^2} - \frac{g\rho^2\delta_o^2}{R} = 0 . \quad (44)$$

If the steady-state film thickness δ_o at $\theta = 0$ is regarded as specified, then Eq. (44) gives the rate of mass supply at $\theta = 0$ in the form

$$m_o = \frac{2\nu\rho}{3f^2\delta_o^2} \left[-1 + \sqrt{1 + \frac{3gf^4\delta_o^4}{R\nu^2}} \right] . \quad (45)$$

When

$$\frac{3gf^4\delta_o^4}{R\nu^2} < < 1 , \quad (46)$$

we see that Eq. (45) reduces to the approximate relation

$$m_o = \frac{\rho g f^2 \delta_o^3}{R \nu} . \quad (47)$$

If the rate of mass supply m_o at $\theta = 0$ is regarded as given, then the corresponding steady-state film thickness at $\theta = 0$ can be calculated from Eq. (47) provided that the inequality of Eq. (46) is satisfied. Also, Eq. (47) can provide a first estimate of δ_o which can be improved by solving Eq. (44) iteratively, for example, by Newton's method. Once corresponding values of the steady-state film thickness and rate of mass supply at $\theta = 0$ have been found, the value of the colatitude derivative of the lithium speed at $\theta = 0$ can be determined from Eq. (42).

VI. TRANSIENT RESPONSE AND STABILITY RESULTS

A. Response to an Initial Film Thickness Perturbation

To demonstrate the stability of the film thickness we consider the result for the Laplace transform of the perturbation in the film thickness contained in Eq. (39) together with the final value theorem for Laplace transforms, namely,

$$\lim_{t \rightarrow \infty} \Delta(0, t) = \lim_{p \rightarrow 0} p \bar{\Delta}(0, p) . \quad (48)$$

Suppose that we introduce a nonzero initial perturbation Δ_{00} in the film thickness at $\theta = 0$ and set all other perturbations equal to zero, i.e., $V_{10} = 0$ and $\bar{M}(0, p) = 0$. Then Eq. (39) becomes

$$\bar{\Delta}(0, p) = \frac{\Delta_{00}}{D_S} \left(p + \frac{2\nu_1}{R} + \frac{m_o}{\rho \delta_o} + \frac{2\nu}{f^2 \delta_o^2} \right) . \quad (49)$$

Combining Eqs. (48) and (49) and evaluating the limit as p goes to zero show that the time-dependent perturbation in the film thickness due to introducing a nonzero initial value in the film thickness perturbation goes to zero as time approaches infinity. In this case the actual time response of the film thickness perturbation is given by

$$\begin{aligned} \Delta(0, t) = & \frac{\Delta_{00} e^{p_1 t}}{(p_1 - p_2)} \left(p_1 + \frac{2\nu_1}{R} + \frac{m_o}{\rho \delta_o} + \frac{2\nu}{f^2 \delta_o^2} \right) \\ & + \frac{\Delta_{00} e^{p_2 t}}{(p_2 - p_1)} \left(p_2 + \frac{2\nu_1}{R} + \frac{m_o}{\rho \delta_o} + \frac{2\nu}{f^2 \delta_o^2} \right) , \end{aligned} \quad (50)$$

where p_1 and p_2 are the two roots of the quadratic equation $D_S = 0$, and D_S is given by Eq. (41); alternatively, we have

$$D_S = (p - p_1)(p - p_2) . \quad (51)$$

The two roots p_1 and p_2 have negative real parts, and, in fact, their values which correspond to steady-state flows computed by Bohachevsky, Booth, and Hafer¹ are both negative real numbers. This fact shows that the time response of the film thickness in Eq. (50) is nonoscillatory when an initial film thickness perturbation is introduced into the steady-state flows computed in Ref. 1.

Actual numerical values of the roots p_1 and p_2 can be calculated more easily by simplifying Eq. (41) with the use of Eq. (43). That is, we have

$$\frac{4v_1}{R} + \frac{m_0}{\rho\delta_0} + \frac{2\nu}{f^2\delta_0^2} = \frac{3v_1}{R} + \frac{g}{v_1} , \quad (52)$$

and

$$\frac{2v_1^2}{R^2} + \frac{2g}{R} + \frac{2v_1 m_0}{R\rho\delta_0} + \frac{8\nu v_1}{Rf^2\delta_0^2} = \frac{4v_1}{R} \left(\frac{g}{v_1} + \frac{\nu}{f^2\delta_0^2} \right) . \quad (53)$$

Accordingly, the roots p_1 and p_2 satisfy

$$p^2 + \left(\frac{3v_1}{R} + \frac{g}{v_1} \right) p + \frac{4g}{R} + \frac{4\nu v_1}{Rf^2\delta_0^2} = 0 , \quad (54)$$

so

$$p_{1,2} = -\frac{1}{2} \left(\frac{3v_1}{R} + \frac{g}{v_1} \right) \pm \frac{1}{2} \sqrt{\left(\frac{3v_1}{R} + \frac{g}{v_1} \right)^2 - 16 \left(\frac{g}{R} + \frac{\nu v_1}{Rf^2\delta_0^2} \right)} . \quad (55)$$

When the inequality

$$16 \left(\frac{g}{R} + \frac{\nu v_1}{Rf^2\delta_0^2} \right) \ll \left(\frac{3v_1}{R} + \frac{g}{v_1} \right)^2 \quad (56)$$

is satisfied, the roots p_1 and p_2 are given approximately by

$$p_1 = - \frac{\frac{4g}{R} + \frac{4\nu v_1}{Rf^2 \delta_o^2}}{\frac{3v_1}{R} + \frac{g}{v_1}}, \quad (57)$$

and

$$p_2 = - p_1 - \left(\frac{3v_1}{R} + \frac{g}{v_1} \right). \quad (58)$$

When the rate of mass supply is a step function

$$m_s(\theta) = \begin{cases} m_{os}, & \text{for } 0 \leq \theta \leq \theta_f \leq \theta_m \\ 0, & \text{for } \theta_f < \theta \leq \theta_m, \end{cases} \quad (59)$$

the total mass supply is normalized to 2000 g/s, the laminar sublayer thickness is $f = 0.10$, the ablation rate is $m_{ev} = 3.64 \times 10^{-3} \text{ g/cm}^2\text{-s}$ (see p. 9, Ref. 1), the cavity radius is $R = 150 \text{ cm}$, the lithium density is $\rho = 0.50 \text{ g/cm}^3$, and the lithium kinematic viscosity is $\nu = \mu/\rho = 0.009 \text{ cm}^2/\text{s}$, we find that δ_o as computed from Eq. (47) and v_1 as computed from Eq. (42) are as shown in Table I, in which $m_o = m_{os} - m_{ev}$. The corresponding roots of the quadratic equation $D_S = 0$ as computed with Eqs. (57) and (58) are given in Table II. As seen in Table II, the roots p_1 and p_2 are both real and negative, from which we can also conclude that film thickness perturbation time response function of Eq. (50) goes to zero at infinite time.

TABLE I
STEADY-STATE PARAMETERS FOR UNIFORM MASS SUPPLY

θ_f	$m_o \text{ (g/cm}^2\text{-s)}$	$\delta_o \text{ (cm)}$	$v_1 \text{ (cm/s-rad)}$
$\pi/2$	0.01051	0.1425	11.06
$3\pi/4$	0.00465	0.1086	6.42
$15\pi/16$	0.00350	0.0988	5.32

TABLE II

ROOTS OF THE QUADRATIC EQUATION $D_S = 0$

θ_f	$p_1 s^{-1}$	$p_2 s^{-1}$
$\pi/2$	-0.441	- 88.4
$3\pi/4$	-0.257	-152.5
$15\pi/16$	-0.213	-184.2

B. Response to a Step Change in the Rate of Mass Supply

Suppose that a small step change,

$$M(0,t) = \epsilon m_0, \quad (60)$$

where $|\epsilon| \ll 1$, is made in the rate of mass supply and that all other initial perturbations vanish so $\Delta_{00} = 0$ and $V_{10} = 0$. Then the Laplace transform of the film thickness perturbation at $\theta = 0$ from Eq. (39) becomes

$$\bar{\Delta}(0,p) = \frac{\epsilon m_0}{D_S \rho p} \left(p + \frac{4v_1}{R} + \frac{m_0}{\rho \delta_0} + \frac{2\nu}{f^2 \delta_0^2} \right). \quad (61)$$

Entering Eq. (61) into the final value theorem for Laplace transforms, Eq. (48), and carrying out the limit as $p \rightarrow 0$ produces

$$\lim_{t \rightarrow \infty} \Delta(0,t) = \frac{\epsilon m_0}{\rho D_S(p=0)} \left(\frac{4v_1}{R} + \frac{m_0}{\rho \delta_0} + \frac{2\nu}{f^2 \delta_0^2} \right). \quad (62)$$

This result shows that, if a small step change is made in the rate of mass supply, then the resulting perturbation in the film thickness approaches a finite, nonzero value at infinite time. For this case the actual time response function for the perturbation in the film thickness is

$$\begin{aligned} \Delta(0,t) &= \frac{\epsilon m_0}{\rho D_S(p=0)} \left(\frac{4v_1}{R} + \frac{m_0}{\rho \delta_0} + \frac{2\nu}{f^2 \delta_0^2} \right) \\ &+ \frac{\epsilon m_0 e^{p_1 t}}{p_1(p_1 - p_2)\rho} \left(p_1 + \frac{4v_1}{R} + \frac{m_0}{\rho \delta_0} + \frac{2\nu}{f^2 \delta_0^2} \right) \end{aligned}$$

$$+ \frac{\epsilon m_o e^{p_2 t}}{p_2(p_2 - p_1)\rho} \left(p_2 + \frac{4v_1}{R} + \frac{m_o}{\rho \delta_o} + \frac{2\nu}{f^2 \delta_o^2} \right), \quad (63)$$

where the roots are given by Eq. (55).

The ultimate value in the film thickness perturbation at $\theta = 0$ simplifies to

$$\lim_{t \rightarrow \infty} \Delta(0,t) = \frac{\epsilon m_o}{4\rho} \left[\frac{\frac{3v_1}{R} + \frac{g}{v_1}}{\frac{g}{R} + \frac{\nu v_1}{f^2 \delta_o^2 R}} \right], \quad (64)$$

when Eq. (62) is combined with Eqs. (52) and (53). Numerical values of this limit for the same steady-state parameters as those in and upon which Table I is based are included in Table III for the case $\epsilon = 0.10$. The third column in Table III indicates that the ultimate value of the film thickness perturbation at $\theta = 0$ that results when the rate of mass supply is given a step perturbation equal to 10% of its initial steady-state value is 3.34% of the initial steady-state value of the film thickness.

TABLE III
ULTIMATE VALUE OF FILM THICKNESS PERTURBATION DUE TO
A STEP CHANGE IN THE RATE OF MASS SUPPLY

θ_f	$\lim_{t \rightarrow \infty} \Delta(0,t), \text{ cm}$	$\frac{1}{\delta_o} \lim_{t \rightarrow \infty} \Delta(0,t)$
$\pi/2$	0.00476	0.0334
$3\pi/4$	0.00362	0.0334
$15\pi/16$	0.00330	0.0334

REFERENCES

1. I. O. Bohachevsky, L. A. Booth, and J. F. Hafer, "Lithium Flow on the Inside of a Spherical Fusion-Reactor Cavity," Los Alamos Scientific Laboratory report LA-6362-MS (May 1976).