SPIN FROM ISOSPIN IN A GAUGE THEORY*

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*This work is supported in part through funds provided by
ERDA under Contract AT(11-1)-3069

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February, 1976

Center for Theoretical Physics Publication #524

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ABSTRACT

We show that in an $SU(2)$ gauge theory, with isospin symmetry broken by a triplet of scalar mesons, isospinor degrees of freedom are converted into spin degrees of freedom, in the field of a magnetic monopole.

Classical solutions to gauge theories¹ mix spatial and internal symmetry indices; for example an iso-triplet vector field $A_a^i(\vec{r})$ is proportional to $\epsilon_{aij} \hat{r}^j$. We now know that such solutions play a role in the corresponding quantum theory; in a weak coupling limit they approximate matrix elements of the quantum field between soliton-monopole states². It is natural to ask: Does the conjunction of spin and internal symmetry persist in the quantum theory; and, when internal symmetry is spontaneously broken, do internal degrees of freedom reappear as spin degrees of freedom?

An affirmative answer to this question is indicated by our investigation³ of the interaction of an iso-spinor, spin 1/2 fermion with a monopole, arising in an SU(2) Yang-Mills-Higgs model with spontaneously broken iso-spin symmetry⁴. We found in the quantum theory a tower of spinless states, describing dyons with magnetic charge $Q_m = \pm \frac{1}{e}$ and electric charge $Q_e = \pm \frac{N}{2}e$ [e is the gauge coupling constant, N is an integer or zero]. Moreover, the quantum Fermi field Ψ effects a transition between them, $\langle Q_m, Q_e - \frac{e}{2} | \Psi | Q_m, Q_e \rangle \neq 0$. This matrix element is related to the static, zero-energy solution of a c-number Dirac equation in the monopole field. The solution ψ_{vn} [v refers to Dirac indices; n to iso-spin indices], has vanishing lower components [$v = 3, 4$], while the upper components have the form

$$\psi_{vn} = f(r) \frac{1}{\sqrt{2}} [s_v^+ s_n^- - s_v^- s_n^+] \quad (1)$$

$v, n = 1, 2$

where s^+ [s^-] is a Pauli up [down] bispinor, and f is a normalized wavefunction.

In the transition form-factor (1), spin and iso-spin form an anti-symmetric singlet. This explains mathematically the curious circumstance that a spinor field should possess non-vanishing matrix elements between spinless states. Evidently degrees of freedom of the spontaneously broken iso-spin symmetry survive as spin degrees of freedom, and couple to Dirac spin. For physical clarification, consider the non-vanishing crossed matrix element $\langle Q_m, Q_e - \frac{e}{2}; -Q_m, -Q_e | \Psi | 0 \rangle$, which describes the creation of two spinless dyons by a spin 1/2 field. Angular momentum is conserved, since as is well-known, the total angular momentum of magnetic systems contains contributions from magnetic monopoles. In the above configuration, the electromagnetic angular momentum has the magnitude $(-Q_m)(Q_e - \frac{e}{2}) - (Q_m)(-Q_e) = \frac{Q_m e}{2} = \frac{1}{2}$ and provides the requisite half-integer. Thus we suggest that in the soliton-monopole sector of the quantum theory total angular momentum $\vec{J}$ is the sum of conventional $\vec{M}$ [orbital plus spin] and $\vec{I}$ [iso-spin].

$$\vec{J} = \vec{M} + \vec{I} \quad (2)$$

In this note we exhibit another, more dramatic, example of the conversion of iso-spin to spin, and derive Eq. (2). We consider a spinless iso-spinor field U , coupling to the usual $SU(2)$ Yang-Mills-Higgs theory, with a classical solution $U = u(r)s$, where s is an arbitrary, position-independent iso-spinor and u vanishes as $r \rightarrow \infty$. The solution is degenerate; the degeneracy is a three-parameter iso-spin rotation of s . In the quantum theory, the multiplicity of classical solutions leads to a tower of monopole-soliton states labeled by a

new degree of freedom, interpreted as spin. We confirm our interpretation by introducing collective coordinates to account for the classical degeneracy and by verifying that the canonically conjugate collective variable is indeed an angular momentum $\vec{J}$ which coincides with iso-spin $\vec{I}$.

The model Lagrangian is

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} F_a^{\mu\nu} F_{a\mu\nu} + \frac{1}{2} (D^\mu \Phi)_a (D_\mu \Phi)_a - \frac{1}{e^2} V_1(e^2 \Phi^2) \\ & + (D^\mu U)^\dagger (D_\mu U) - \frac{1}{e^2} V_2(e^2 U^\dagger U, e^2 \Phi^2) \end{aligned}$$

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + e \epsilon_{abc} A_b^\mu A_c^\nu$$

$$(D^\mu \Phi)_a = \partial^\mu \Phi_a + e \epsilon_{abc} A_b^\mu \Phi_c$$

$$(D^\mu U) = \partial^\mu U - ie \frac{\tau^a}{2} A_a^\mu U$$

$$\Phi^2 = \Phi_a \Phi_a, \quad a = 1, 2, 3$$

$$V_1(\Phi^2) = \frac{\lambda^2}{2} \left[\frac{\mu^2}{\lambda^2} - \Phi^2 \right]^2$$

$$V_2(U^\dagger U, \Phi^2) = m^2 U^\dagger U + g^2 (U^\dagger U)^2 - h^2 U^\dagger U \left(\frac{\mu^2}{\lambda^2} - \Phi^2 \right) \quad (3)$$

For a range of values of m^2 , g^2 and h^2 ,⁵ the solution in the vacuum sector is $\Phi^2 = \frac{\mu^2}{\lambda^2}$, $U = 0$; while in the soliton-monopole sector one may decrease the energy by a non-vanishing position dependent U . Thus we expect that the following static, degenerate solution exists.

$$A_a^0 = 0, \quad A_a^i = \epsilon_{aij} \hat{r}^j A(r)$$

$$\phi_a = \hat{r}^a \phi(r), \quad U = u(r) \exp[i\vec{\alpha} \cdot \frac{\vec{\tau}}{2}]s \quad (4)$$

At $r = 0$, ϕ, A vanish, and u is a constant. For large r , ϕ, U approach exponentially their vacuum values, and A tends to $-\frac{1}{er}$. The arbitrary triplet $\vec{\alpha}$ parametrizes the degeneracy. The solutions are order e^{-1} , hence large for weak coupling.

Quantization is performed in the unitary gauge⁶ where ϕ_a points in the third iso-spin direction. Classical solutions in this gauge, obtained from (4) by an iso-rotation, are

$$\vec{A}_a = \hat{n}_a [A(r) + \frac{1}{er}] \quad a = 1, 2 \quad (5a)$$

$$\vec{A}_3 = \vec{A}_D(\vec{r}) \quad (5b)$$

$$U = u(r) R \exp[i\vec{\alpha} \cdot \frac{\vec{\tau}}{2}]s \quad (5c)$$

where R is an iso-rotation matrix effecting the desired transformation. $\vec{A}_D$ is a Dirac vector potential for a monopole of strength $1/e$: $\vec{\nabla} \times \vec{A}_D = \frac{1}{e} \frac{\hat{r}}{r^2}$, $\vec{\nabla} \cdot \vec{A}_D = 0$, and $\hat{n}_{1,2}$ is a doublet of ortho-normal unit vectors, orthogonal to $\hat{r}$. The quantities R , $\vec{A}_D$, and $\hat{n}_a$ are to a large extent arbitrary, reflecting the remaining electromagnetic gauge freedom; we shall not need their explicit forms.

In the quantum theory we must allow for a multiplicity of monopole-soliton states so that the matrix elements of U , which are proportional to (5c), reflect the classical degeneracy parametrized by $\vec{\alpha}$. The spinorial form of (5c) strongly suggests this degree of freedom to be spin; we confirm this by calculating the angular momentum operator $\vec{J}$, and showing that it generates rotations of $\vec{\alpha}$.

For the Lagrangian (3), $\vec{J}$ is given by

$$\vec{J} = \int d^3r \vec{r} \times \{ \Pi_a^i \vec{\nabla} A_a^i + \Pi_a \vec{\nabla} \phi_a + (\Pi_U^\dagger \vec{\nabla} U + \text{h.c.}) \} + \int d^3r \vec{\Pi}_a \times \vec{A}_a$$

$a = 1, 2, 3$ (6)

The canonical momenta are $\Pi_a^i = F_a^{0i}$, $\Pi_a = (D^0 \phi)_a$, $\Pi_U = (D^0 U)$. In the unitary gauge, only $\phi_3 \equiv \phi$, $\Pi_3 \equiv \Pi$ survive. We redefine the charged, massive vector fields by $\vec{A}_a \equiv \vec{W}_a$, $a = 1, 2$; and the massless photon field by $\vec{A}_3 \equiv \vec{A}$, $\vec{\Pi}_3 \equiv \vec{\Pi} = \vec{\Pi}_T + \vec{\Pi}_L$. The longitudinal part $\vec{\Pi}_L$ satisfies a constraint equation.

$$\vec{\nabla} \cdot \vec{\Pi}_L = -e \epsilon_{ab} \vec{\Pi}_a \cdot \vec{W}_b - e(i \Pi_U^\dagger \frac{\tau^3}{2} U + \text{h.c.}) \equiv -ej^0; \quad a = 1, 2$$

An integration by parts in (6) eliminates $\vec{\Pi}_L$ and $\vec{J}$ is expressed by unconstrained variables in the unitary gauge.

$$\begin{aligned} \vec{J} = & \int d^3r \vec{r} \times \{ \Pi_a^i [\vec{\nabla} \delta_{ab} + e \epsilon_{ab} \vec{A}] W_b^i + \Pi \vec{\nabla} \phi \\ & + (\Pi_U^\dagger [\vec{\nabla} + ie \frac{\tau^3}{2} \vec{A}] U + \text{h.c.}) + (\vec{\Pi}_T - \vec{\nabla} \nabla^{-2} ej^0) \times \vec{B} \} \\ & + \int d^3r \vec{\Pi}_a \times \vec{W}_a \\ \vec{B} = & \vec{\nabla} \times \vec{A}, \quad \vec{\nabla} \cdot \vec{A} = 0, \quad a = 1, 2 \end{aligned}$$

(7)

To calculate $\vec{J}$ in the soliton-monopole sector, the quantum fields are shifted by the classical solutions, with all degeneracy parameters promoted to collective coordinates, i.e. time-dependent quantum operators⁷. We ignore the coordinates relevant to translations and charge rotations, and concentrate on the three-parameter degeneracy of (5c). To lowest order in the coupling constant, we need keep only the large, classical solutions. Thus we evaluate (7) with⁸

$$\Phi = \phi(r), \quad \Pi = 0$$

$$\vec{W}_a = \hat{n}_a [A(r) + \frac{1}{er}], \quad \vec{\Pi}_a = 0, \quad a = 1, 2$$

$$\vec{A} = \vec{A}_D(\vec{r}), \quad \vec{\Pi}_T = 0$$

$$U = u(r) R \exp[i\vec{\alpha}(t) \cdot \frac{\vec{\tau}}{2}] \equiv RU(r, t)$$

$$\Pi_U^\dagger = \dot{U}^\dagger(r, t) R^\dagger \equiv \Pi_U^\dagger(r, t) R^\dagger \quad (8)$$

Π_U is non-vanishing since U acquires time dependence from its collective coordinate $\vec{\alpha}(t)$. The other fields, not containing collective coordinates, are time-independent and their conjugate momenta vanish.

Now $\vec{J}$ becomes

$$\begin{aligned} \vec{J} &= \int d^3r \vec{r} \times \Pi_U^\dagger(r, t) R^\dagger [\vec{V} + ie \frac{\tau^3}{2} \vec{A}_D(\vec{r})] R U(r, t) \\ &+ \int d^3r \vec{r} \times \left\{ [\vec{V} \int \frac{d^3r'}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|} ie \Pi_U^\dagger(r', t) R^\dagger \frac{\tau^3}{2} R U(r', t)] \times \frac{\hat{r}}{er^2} \right\} + h.c. \\ &= \int d^3r \vec{r} \times \Pi_U^\dagger(r, t) R^\dagger [\vec{V} + ie \frac{\tau^3}{2} \vec{A}_D(\vec{r})] R U(r, t) \\ &+ \int d^3r \hat{r} \Pi_U^\dagger(r, t) R^\dagger \frac{i\tau^3}{2} R U(r, t) + h.c. \end{aligned} \quad (9a)$$

The rotation indicated by R is evaluated by comparison with (4) and (5) -- if we set $A(r) = -\frac{1}{er}$ in those equations, we recognize that (9a) is equal to

$$\begin{aligned} \vec{J} &= \int d^3r \vec{r} \times \Pi_U^\dagger(r, t) [\vec{V} + ie \frac{\vec{\tau}}{2} \times \frac{\hat{r}}{er}] U(r, t) \\ &+ \int d^3r \hat{r} \Pi_U^\dagger(r, t) i \frac{\vec{\tau} \cdot \hat{r}}{2} U(r, t) + h.c. \\ &= \int d^3r \Pi_U^\dagger(r, t) [\vec{r} \times \vec{V} + i \frac{\vec{\tau}}{2}] U(r, t) + h.c. \end{aligned} \quad (9b)$$

The first term in the bracket is the orbital angular momentum of U ; for spherically symmetric fields, as in (8), it vanishes. The remainder is exactly the iso-spin generator $\vec{I}$. When terms that we

have ignored are kept, $\vec{J}$ will of course also acquire conventional orbital and spin contributions. Thus we arrive at the promised derivation of (2).

For $\vec{\alpha}(t)$ pointing in a fixed direction $\hat{n}$, we have

$$U = u(r) R \exp[i\alpha(t) \frac{\tau^n}{2}] S$$

$$\dot{U} = u(r) R i\dot{\alpha}(t) \frac{\tau^n}{2} \exp[i\alpha(t) \frac{\tau^n}{2}] S$$

$$L = \int d^3r \mathcal{L} = \frac{\dot{\alpha}^2(t)}{4} \int d^3r u^2(r) + \dots$$

The momentum conjugate to α is $\frac{\dot{\alpha}}{2} \int d^3r u^2(r)$, which also is $I^n = J^n$. What appears to be an iso-rotation, is in fact a spin rotation-- in the quantum theory spin has been created from iso-spin!⁹

To go beyond the details of our example, it is clear that spin will always emerge from iso-spin whenever there is an iso-spin degenerate, classical solution in the field of a monopole. In future investigations we hope to explore further questions that are raised by these remarkable results: What are the statistics of our particles; how do Lorentz transformations act on them?

One of us (R.J) is happy to acknowledge conversations with P. W. Anderson, who insisted on the possibility of constructing non-integer angular momentum in gauge theories. Both of us benefited from discussions with S. Coleman and J. Goldstone.

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