MASTER

TIGHT UPPER AND LOWER BOUNDS FOR CORRELATION
OF BIVARIATE DISTRIBUTIONS ARISING
IN AIR POLLUTION MODELING

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TECHNICAL REPORT NO. 5
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BIVARIATE DISTRIBUTIONS ARISING IN AIR POLLUTION MODELING

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STUDY ON STATISTICS AND ENVIRONMENTAL
FACTORS IN HEALTH

PREPARED UNDER SUPPORT TO SIMS FROM
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1. Introduction.

It has long been known, but largely ignored that for any two random variables X with distribution F and Y and with distribution G there exist bivariate distributions H_0 and H_1 which are the minimal and maximal distributions in the sense that

$$(1.1) \quad H_0(x,y) \leq H(x,y) \leq H_1(x,y) \quad \text{for each } x,y \in \mathbb{R}.$$

for any other bivariate distribution H having F and G as margins.

The explicit form of H_0 and H_1 is known and can hence in theory be used to find the minimum and maximum correlations possible between X and Y . It is the object of section 2 to find various ways of discovering the functional relationships between X and Y which give rise to H_0 and H_1 and therefore also to the extreme correlations. Section 3 examines this question for the special case of two lognormal distributions where it is found that the range of possible correlations can be surprisingly smaller than $[-1,1]$. An example using real air pollution data from the California State Resources Board is given in section 4. Here it is found that because the range of correlations has shrunk, interpretations of "small" correlations can be seriously misleading. In a recent study of asthma aggravation by Chess [9], conclusions were reached by considering simple correlations between pollutants which are approximately log normally distributed and between pollutant and meteorological factors. From the results of this section it is clear that conclusions based on correlations from untransformed data are misleading at best. Section 5 discusses some ways of avoiding difficulties in interpreting these correlations and mentions further work to be done in this area.

2. Fréchet Distributions.

Consider any two random variables X and Y with distributions F and G respectively. Fréchet has shown [2] that there exist bivariate distributions $H_0(x,y)$ and $H_1(x,y)$ with margins F and G such that for any other bivariate distribution $H(x,y)$ with the same margins,

$$(2.1) \quad H_0(x,y) \leq H(x,y) \leq H_1(x,y)$$

$H_0(x,y) = \max\{0, F(x) + G(y) - 1\}$ is known as the Fréchet minimal distribution and $H_1(x,y) = \min\{F(x), G(y)\}$ is the Fréchet maximal distribution.

Hoeffding [4] has shown that if X and Y have joint distribution $H(x,y)$ then

$$\text{corr}(X,Y) = \frac{1}{\sqrt{\sigma_x^2 \sigma_y^2}} \iint [H(x,y) - F(x)G(y)] dx dy$$

Hence the correlation is maximized at $H(x,y) = H_1(x,y)$ and minimized at $H(x,y) = H_0(x,y)$.

Considering this problem for two identically distributed random variables $X \sim F$ and $Y \sim F$ we have

$$H_1(x,y) = \min\{F(x), F(y)\} = F(\min(x,y))$$

$$H_0(x,y) = \max\{0, F(x) + F(y) - 1\}$$

That is, the maximal distribution can be obtained by taking $P(X=Y) = 1$. This clearly leads to $\text{corr}(X,Y) = 1$. The exact form of the minimal distribution and the value of the minimal correlation however depend on the margins.

It is not always possible (as will be shown later) to give a joint distribution so that $\text{corr}(X,Y) = -1$. In fact, the lower bound is sometimes surprisingly near 0. The purpose of this section is to find various ways of finding the explicit functional relationship between X and Y resulting in the minimal distribution.

We have the following proposition:

Proposition 2.1. Let X and Y be identically distributed with distribution function F . Then $X = -Y$ gives the Fréchet minimal distribution if and only if F is symmetric (around 0). In this case $\rho_{\min} = -1$.

Proof. Since $H_0(x,y)$ is singular on $F(x)+F(y)-1 = 0$ [5], we must solve $F(x) = 1-F(y)$. If F is symmetric then $F(x) = 1-F(-x)$. Hence $Y = -X$ is a solution. Conversely if $Y = -X$ gives the minimal distribution then $F(x) = 1-F(-x)$ and F is symmetric. Also

$$\text{corr}(X,Y) = \frac{E(X)(-X) - EXE(-X)}{\text{Var } X} = -\frac{EX^2 + (EX)^2}{\text{Var } X} = -1.$$

In general, if F has a density f , let $Y = g(X)$ be a differentiable function of X giving the Fréchet minimal distribution. Then since

$$F(x) = 1-F(y)$$

differentiating we have

$$f(x) = -g'(x)f(g(x))$$

We can then try to solve this functional differential equation. An example will help to clarify.

Example. Suppose X and Y are exponential, $f(x) = e^{-x}$. Then we have

$$e^{-x} = -g'(x)e^{-g(x)}$$

$$x = g(x) - \log(-g'(x)) \quad (g' < 0)$$

or taking derivatives

$$g'(x)^2 - g'(x) - g''(x) = 0$$

$$\Rightarrow g = -\log(1-e^{-x}) .$$

Hence $Y = -\log(1-e^{-X})$ gives the minimal distribution of two identically distributed exponential random variables.

The Fréchet distributions are invariant under changes of location and scale in the sense that if $X \approx F$, $Y \approx G$ are jointly distributed as H_0 (H_1) then if W and Z are the new random variables under the transformation, W and Z are still distributed with the minimal (maximal) distribution with the transformed margins. More precisely,

Proposition 2.2. Let X, Y be random variables distributed jointly as $H_1(x, y)$, the Fréchet maximal distribution with margins F and G , respectively. Let $W = f(X)$ and $Z = g(Y)$ be monotonically increasing transformations of X and Y . Then, W and Z are jointly distributed as the Fréchet maximal distribution $H_1^*(w, z)$ with margins F^* and G^* where

$$F^*(w) = F(f^{-1}(w)) \quad G^*(z) = G(g^{-1}(z)) .$$

Proof. Let H be any bivariate distribution with margins F^* and G^* .

Let U and V be random variables jointly distributed with $H(w,z)$.

Then

$$\begin{aligned} H(w,z) &= P(U \leq w, V \leq z) \\ &\leq \min(F^*(w), G^*(z)) \\ &= \min(F(f^{-1}(w)), G(g^{-1}(z))) \\ &= H_1(f^{-1}(w), g^{-1}(z)) \\ &= P(X \leq f^{-1}(w), Y \leq g^{-1}(z)) \\ &= P(W \leq w, Z \leq z) . \end{aligned}$$

Hence W and Z have the Fréchet maximal distribution.

Note: Under the same conditions the result is true for the minimal distribution H_0 .

Example. Lognormal

Let X and Y be normally distributed with means 0 and variances 1, although not necessarily bivariate normal. Let $X = e^X$ and $Z = e^Y$. For two normally distributed random variables we know by Proposition 2.1 that $Y = X$ and $Y = -X$ give the maximal and minimal distributions. Hence by Proposition 2.1 $W = e^X = Z$ and $W = e^X = \frac{1}{Z}$ give the maximal and minimal distributions for standard log normal random variables.

When $W = Z$, $\text{cov}(W, Z) = 1$. When $W = \frac{1}{Z}$, $\text{cov}(W, Z) =$

$$\begin{aligned}\text{cov}(W, Z) &= \frac{E W \frac{1}{W} - E W E \frac{1}{W}}{\text{Var } W} \\ &= \frac{1 - e}{e(e-1)} = -\frac{1}{e} \approx -.368.\end{aligned}$$

That is, two standard lognormals cannot have negative correlation larger than $-.368$.

3. The Lognormal Distribution.

Since the 1950's the lognormal has been the primary and until quite recently practically the only distribution used to model air quality data. Recently [3] and [6] for example, some work has been done with other distributions, but the lognormal will continue to be important in air pollution studies.

The results of the last section however, raise some serious questions as to the interpretation of correlation between lognormal random variables. It will be seen that for any two lognormal random variables with known mean and variance, the range of possible correlations is easily computed and is usually smaller than $[-1, 1]$. As we have seen, for two standard lognormals the range has already shrunk to $[-\frac{1}{e}, 1]$.

Let X and Y be normal random variables mean 0, variance 1. We have seen that $H_0(x, y)$ is obtained by letting $Y = -X$ and $H_1(x, y)$

by letting $Y = X$. Consider

$$\begin{aligned} W_1 &= e^{\sigma_1(X-\mu_1)} & Z_1 &= e^{\sigma_2(-X-\mu_2)} \\ W_2 &= e^{\sigma_1(X-\mu_1)} & Z_2 &= e^{\sigma_2(X-\mu_2)} \end{aligned}$$

Each transformation is monotone and hence by Prop. 22 W_1 and Z_1 have the Fréchet minimal dist while W_2 and Z_2 have the Fréchet maximal distribution for arbitrary lognormal random variables with means

$e^{\mu_i + \sigma_i^2/2}$ and variances $e^{2\mu_i + \sigma_i^2} \sqrt{e^{\sigma_i^2} - 1}$ ($i = 1, 2$) respectively. To compute the correlation we use the fact that $\sigma_1(X-\mu_1)$ and $\sigma_2(-X-\mu_2)$ are bivariate normal ($\rho = -1$) and $\sigma_1(X-\mu_1)$ and $\sigma_2(X-\mu_2)$ are bivariate normal ($\rho = +1$). From the appendix we see that for two lognormals W, Z whose logs are bivariate normal mean $\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ and covariance matrix

$$\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

$$\text{corr}(W, Z) = \frac{e^{\rho\sigma_1\sigma_2} - 1}{\sqrt{\begin{pmatrix} \sigma_1^2 & \sigma_1^2 - 1 \\ e^{\sigma_1^2} - 1 \end{pmatrix} \begin{pmatrix} \sigma_2^2 & \sigma_2^2 - 1 \\ e^{\sigma_2^2} - 1 \end{pmatrix}}}$$

Hence

$$\text{corr}_{\max}(W, Z) = \frac{e^{\sigma_1\sigma_2} - 1}{\sqrt{\begin{pmatrix} \sigma_1^2 & \sigma_1^2 - 1 \\ e^{\sigma_1^2} - 1 \end{pmatrix} \begin{pmatrix} \sigma_2^2 & \sigma_2^2 - 1 \\ e^{\sigma_2^2} - 1 \end{pmatrix}}}$$

when $\rho = 1$

and

$$\text{corr}_{\min}(W, Z) = \frac{e^{-\sigma_1^2 \sigma_2^2 - 1}}{\sqrt{\left(\begin{vmatrix} \sigma_1^2 & \\ e & -1 \end{vmatrix} \begin{vmatrix} \sigma_2^2 & \\ e & -1 \end{vmatrix} \right)}}$$

when $\rho = -1$, generalizing a result of Paul Switzer (7) for the autocorrelation between two lognormals.

This formula yields some striking results. Note that if $\sigma_1 = 2$ $\sigma_2 = 5$ the correlation is constrained to lie in the interval $[-.012, +.012]$. Note that σ_1 and σ_2 refer to the bivariate normal from which the maximal and minimal distributions were derived. Since

$$EW = e^{\mu_1 + \frac{\sigma_1^2}{2}} \quad \text{Var } W = e^{2\mu_1 + \sigma_1^2} (e^{\sigma_1^2} - 1)$$

to convert to the parameters of the lognormal random variable we have

$$\frac{\sqrt{\text{Var } W}}{EW} = \sqrt{e^{\sigma_1^2} - 1}$$

or

$$\gamma_1^2 = e^{\sigma_1^2} - 1$$

Here

$$\gamma(W) = \frac{\sqrt{\text{Var } W}}{EW}$$

is the coefficient of variation. This yields

$$\sigma_1^2 = \log(1 + \gamma_1^2)$$

Before looking at an example of real data let us examine some of the properties of the correlation function,

$$\frac{e^{\rho\sigma_1\sigma_2} - 1}{\sqrt{\left(e^{\sigma_1^2} - 1\right)\left(e^{\sigma_2^2} - 1\right)}}.$$

When $\sigma_1 = \sigma_2$ it reduces to

$$\text{corr}(W, Z) = \frac{e^{\rho\sigma^2} - 1}{e^{\sigma^2} - 1}$$

so when $\rho = 1$ it is always 1. But if $\rho = -1$

$$\text{corr}_{\min}(W, Z) = \frac{e^{-\sigma^2} - 1}{e^{\sigma^2} - 1} \rightarrow 0 \text{ as } \sigma \rightarrow \infty.$$

If $\rho = 0$, for any values of σ_1 and σ_2 , $\text{corr}(W, Z) = 0$, so uncorrelated variables are preserved under the transformation.

If $\sigma_1 \neq \sigma_2$ then

$$\text{corr}_{\min}(W, Z) = \frac{e^{-\sigma_1\sigma_2} - 1}{\sqrt{\left(e^{\sigma_1^2} - 1\right)\left(e^{\sigma_2^2} - 1\right)}} \rightarrow 0$$

for fixed σ_1 as $\sigma_2 \rightarrow \infty$. Also

$$\text{corr}_{\max}(W, Z) = \frac{e^{\sigma_1\sigma_2} - 1}{\sqrt{\left(e^{\sigma_1^2} - 1\right)\left(e^{\sigma_2^2} - 1\right)}} \rightarrow 0$$

as $\sigma_2 \rightarrow \infty$ for fixed σ_1 . Hence for either σ_1 or σ_2 large, the range of correlations shrinks toward 0.

4. An Example.

Finally let us consider the following data from the State of California Air Resources Board [1]. Suppose we want to see if NO and Hydrocarbons are correlated for a certain station in a given year. Taking the data (of over 16000 points) from the Redding-Market Street Station in Sacramento for 1972, we have

	No. obs.	mean	st. dev.	γ
NO	7812	0.6	1.7	2.83
Hydro- carbons	8761	2.9	0.4	.138

Hence

$$\sigma_{11} = \sqrt{\log(1 + \gamma_1^2)} = 1.483$$

$$\sigma_{22} = \sqrt{\log(1 + \gamma_2^2)} = .137$$

giving

$$\text{corr}_{\max} = \frac{e^{\sigma_1 \sigma_2} - 1}{\sqrt{\left(\begin{matrix} \sigma_1^2 \\ e^{\sigma_1^2} - 1 \end{matrix} \right) \left(\begin{matrix} \sigma_2^2 \\ e^{\sigma_2^2} - 1 \end{matrix} \right)}} = .58$$

$$\text{corr}_{\min} = \frac{e^{-\sigma_1 \sigma_2} - 1}{\sqrt{\left(\begin{matrix} \sigma_1^2 \\ e^{\sigma_1^2} - 1 \end{matrix} \right) \left(\begin{matrix} \sigma_2^2 \\ e^{\sigma_2^2} - 1 \end{matrix} \right)}} = -.47$$

Clearly any interpretation of correlations must be considered in light of the fact that the range of possible correlations is almost half that expected.

In a study of asthma aggravation by Chess 1970-1971, [9] simple correlations were taken between variables which are usually approximately lognormal and between these and other variables, some discrete, some continuous. They have labeled many of these as significant at $p < .05$ or $p < .001$ presumably based on the assumption that the distribution theory of the bivariate normal was applicable. As we have seen, without transforming back to normal variables the simple correlations obtained are next to meaningless and at best misleading.

5. Concluding Remarks.

In view of the results of the preceeding sections, it seems clear that in working with data assumed to be lognormal the data must be transformed back to normal data before correlations are taken. As we have seen the correlations otherwise are meaningless. A study of how other measures of correlation in a broader sense (e.g. Spearman's ρ or Kendall's τ) are affected in the same circumstances would be of use and is planned.

Not discussed in this paper is the statistical question of how to determine the range of correlations. Clearly the most naive estimates of the relevant quantities were taken in section 4.

Finally, is the shrinking of the correlation range a property of skewed distributions in general? Should all skewed distributions be transformed before correlation studies are made? Is there a way of expressing the correlation range as a function of skewness and/or kurtosis? I hope to look at these and other related problems in further reports.

6. Appendix.

Let X and Y be bivariate normal

$$N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right)$$

Let $W = e^X$ and $Z = e^Y$ be lognormal. Then

$$\begin{aligned} EW^k &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} e^{kx} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \sigma_1} e^{-\frac{x^2 - 2\mu_1 x + \mu_1^2 - 2\sigma_1^2 kx}{2\sigma_1^2}} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \sigma_1} e^{-\left\{ \frac{x^2 - 2x(\mu_1 + k\sigma_1^2) + (\mu_1 + k\sigma_1^2)^2}{2\sigma_1^2} + \frac{\mu_1^2}{2\sigma_1^2} - \frac{(\mu_1 + k\sigma_1^2)^2}{2\sigma_1^2} \right\}} dx \\ &= e^{\frac{-1}{2\sigma_1^2} (-\mu_1^2 + \mu_1^2 + 2k\sigma_1^2 \mu_1 + k^2 \sigma_1^4)} \\ &= e^{k\mu_1 + \frac{k^2}{2} \sigma_1^2} \end{aligned}$$

Hence

$$EW = e^{\mu_1 + \frac{\sigma_1^2}{2}} \quad \text{Var } W = e^{2\mu_1 + \sigma_1^2} (e^{\sigma_1^2} - 1)$$

Similarly

$$EZ = e^{\mu_2 + \frac{\sigma_2^2}{2}} \quad \text{Var } Z = e^{2\mu_2 + \sigma_2^2} (e^{\sigma_2^2} - 1)$$

And

$$\begin{aligned} EWZ &= \iint \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2} \right] \right\} \\ &\quad e^{xy} dx dy \\ &= \iint \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[\frac{x^2 - 2x\mu_1 + \frac{2\rho\mu_2\sigma_1}{\sigma_2} x - 2(1-\rho^2)\sigma_1^2 x + \mu_1^2}{\sigma_1^2} \right. \right. \\ &\quad \left. \left. - \frac{y^2 - 2y\mu_2 + \frac{2\rho\mu_1\sigma_2}{\sigma_1} y - 2(1-\rho^2)\sigma_2^2 y + \mu_2^2}{\sigma_2^2} \right] - \frac{2\rho}{\sigma_1\sigma_2} (xy + \mu_1\mu_2) \right\} dx dy \\ &= \iint \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[\frac{(x-\mu_1^*)^2}{\sigma_1^2} - \frac{2\rho(x-\mu_1^*)(y-\mu_2^*)}{\sigma_1\sigma_2} + \frac{(y-\mu_2^*)^2}{\sigma_2^2} \right] \right\} \\ &\quad \exp \left\{ +\frac{1}{2(1-\rho^2)} \left[\frac{1}{\sigma_1^2} (\rho^2\sigma_1^2\sigma_2^2 + \sigma_1^4 + 2\mu_1\rho\sigma_1\sigma_2 + 2\mu_1\sigma_1^2 + 2\rho\sigma_1^3\sigma_2 \right. \right. \\ &\quad \left. \left. - \frac{2\rho}{\sigma_1\sigma_2} (\rho\mu_1\sigma_1\sigma_2 + \mu_1\sigma_2^2 + \mu_2\rho\sigma_1\sigma_2 + \rho^2\sigma_1^2\sigma_2^2 + \rho\sigma_1^3\sigma_2^2 + \mu_2\sigma_1^2) \right] \right\} \end{aligned}$$

$$+ \rho \sigma_1^3 \sigma_2 + \sigma_1^2 \sigma_2^2) +$$

$$\frac{1}{\sigma_2^2} (\rho^2 \sigma_2^2 \sigma_1^2 + \sigma_2^4 + 2\mu_2 \rho \sigma_1 \sigma_2 + 2\mu_2 \sigma_2^2 + 2\rho \sigma_2^3 \sigma_1) \Big] \Big\} dx dy$$

where

$$\mu_1^* = \mu_1 + \rho \sigma_1 \sigma_2 + \sigma_1^2$$

$$\mu_2^* = \mu_2 + \rho \sigma_1 \sigma_2 + \sigma_2^2$$

$$\begin{aligned} &= \exp \left\{ \frac{1}{2(1-\rho^2)} \left[\rho^2 \sigma_2^2 + \sigma_1^2 + 2\mu_1 \rho \frac{\sigma_2}{\sigma_1} + 2\mu_1 + 2\rho \sigma_1 \sigma_2 - 2\rho^2 \mu_1 - 2\rho \mu_1 \frac{\sigma_1}{\sigma_2} \right. \right. \\ &\quad \left. \left. - 2\rho^2 \mu_2 - 2\rho^3 \sigma_1 \sigma_2 - 2\rho^2 \sigma_2^2 - 2\rho \mu_2 \frac{\sigma_1}{\sigma_2} - 2\rho^2 \sigma_1^2 - 2\rho \sigma_1 \sigma_2 \right. \right. \\ &\quad \left. \left. + \rho^2 \sigma_1^2 + \sigma_2^2 + 2\mu_2 \rho \frac{\sigma_1}{\sigma_2} + 2\mu_2 + 2\rho \sigma_1 \sigma_2 \right] \right\} \\ &= \exp \left(\frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \mu_1 + \mu_2 + \rho \sigma_1 \sigma_2 \right) . \end{aligned}$$

Hence

$$\text{Corr}(W, Z) = \frac{EWZ - EWEZ}{\sqrt{\text{Var } W \text{ Var } Z}} = \frac{e^{\left(\frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \mu_1 + \mu_2 + \rho \sigma_1 \sigma_2\right)} - e^{\mu_1 + \mu_2 + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2}}}{e^{\mu_1 + \frac{\sigma_1^2}{2}} \sqrt{e^{\sigma_1^2} - 1} e^{\mu_2 + \frac{\sigma_2^2}{2}} \sqrt{e^{\sigma_2^2} - 1}}$$

$$= \frac{e^{\left(\frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \mu_1 + \mu_2\right) \rho \sigma_1 \sigma_2 - 1}}{e^{\left(\frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \mu_1 + \mu_2\right) \sqrt{(\sigma_1^2 - 1)(\sigma_2^2 - 1)}}$$

$$= \frac{e^{\rho \sigma_1 \sigma_2 - 1}}{\sqrt{(\sigma_1^2 - 1)(\sigma_2^2 - 1)}} \quad \parallel$$

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