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Equations of Motion for Free-Flight Systems of Rotating-Translating Bodies Having Variable Mass and Configuration

Albert E. Hodapp, Jr.

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EQUATIONS OF MOTION FOR FREE-FLIGHT SYSTEMS
OF ROTATING-TRANSLATING BODIES
HAVING VARIABLE MASS AND CONFIGURATION

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ABSTRACT

General vector differential equations of motion are developed for a system of rotating-translating, unbalanced, variable mass, variable configuration bodies. Complete flexibility is provided in placement of the reference coordinates within the system of bodies and in placement of body fixed axes within each body. Examples are presented to demonstrate the ease with which these equations can be reduced to the equations of motion for systems of interest.

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Nomenclature

c_g	Center of gravity
$\frac{d(\)}{dt}$	Derivative of () with respect to time
F	Total force (external + gravitational) acting on the system of bodies $\left[\sum_j F_j, \text{ Eq. (73)} \right], \text{ lb or N}$
F_j	Total force (external + gravitational + interaction) acting on the <u>j</u> th body $\left[\sum_p F_{jp} + T_j, \text{ Eq. (39)} \right], \text{ lb or N}$
F_{jp}	Total force (external + gravitational + interaction) acting on particle "p" of the <u>j</u> th body [Eq. (9)], lb or N
F_X, F_Y, F_Z	Components of total external force along the X, Y, Z axes, respectively, lb or N
g_X, g_Y, g_Z	Components of gravitational acceleration along the X, Y, Z axes, respectively, ft/s ² or m/s ²
H_j	Apparent moment of momentum of the <u>j</u> th body about the origin of $X_j Y_j Z_j$ [Eq. (61)], ft-lb-s or N-m-s
i, j, k	Unit vectors along the X, Y, Z axes, respectively
i_j, j_j, k_j	Unit vectors along the X_j, Y_j, Z_j axes respectively
$I_{X_j}, I_{Y_j}, I_{Z_j}$	Moments of inertia of the <u>j</u> th body about the X_j, Y_j, Z_j axes, respectively [Eq. (62)], slug-ft ² or kg-m ²
$I_{X_j Y_j}, I_{Y_j Z_j}, I_{X_j Z_j}$	Products of inertia of the <u>j</u> th body relative to the $X_j Y_j Z_j$ axes [Eq. (62)], slug-ft ² or kg-m ²
m	Total system mass [Eq. (68)], slug or kg
m_j	Mass of the <u>j</u> th body [Eq. (12)], slug or kg
m_{jp}	Mass of particle "p" (Figure 1) in the <u>j</u> th body, slug or kg
M_o, M_{j_o}, M_{jp_o}	Moments of forces F, F_j, F_{jp} , respectively, about "o" (origin of XYZ axes), ft-lb or N-m
M_X, M_Y, M_Z	Components of the moment of external forces about "o" (origin of XYZ axes) relative to the X, Y, Z axes, respectively [Eqs. (84) and (109)] ft-lb or N-m
$M_{X_2}, M_{Y_2}, M_{Z_2}$	Components relative to the X, Y, Z axes, respectively, of the moment about "o" (origin of XYZ axes) of the external forces (not interactions) acting on Body 2 [Eq. (110)], ft-lb or N-m

P_{jp}	Linear momentum of the variable mass particle "p" (Figure 1) of the <u>j</u> th body [Eq. (5)], lb-s or N-s
r_{cg}	Radius vector from "o" (origin of XYZ axes) to the cg of the system of bodies, Figure 1 [Eq. (72)], ft or m
r_j	Radius vector from "o" (origin of XYZ axes) to the origin of the $X_j Y_j Z_j$ axes, Figure 1, ft or m
R_o	Radius vector from origin of $X_i Y_i Z_i$ axes to "o" (origin of XYZ axes), Figure 1, ft or m
t	Time, s
T_j	Total thrust acting on <u>j</u> th body [Eq. (36), external force], lb or N
u_j	Velocity of ejected or swallowed mass (velocity of "o", Figure 1) relative to the reference point at the exit from or entrance to the <u>j</u> th body [Eq. (20)], ft/s or m/s
V	Velocity of "o" (origin of XYZ axes, Figure 1) relative to inertial space, ft/s or m/s
XYZ	Reference coordinates for system of bodies (arbitrary location and motion), Figure 1
$X_i Y_i Z_i$	Inertially fixed coordinates, Figure 1
$X_j Y_j Z_j$	Coordinates fixed in the <u>j</u> th body (arbitrary location in body), Figure 1
$\Delta()$	Difference or change in ()
ρ_{jcg}	Radius vector from origin of $X_j Y_j Z_j$ axes to the cg of the <u>j</u> th body [Eq. (16)], Figure 1, ft or m
ρ_{je}	Radius vector from origin of $X_j Y_j Z_j$ axes to the point in the <u>j</u> th body either at the center of the nozzle exit plane or inlet entrance plane, Figure 1, ft or m
ρ_{jm}	Radius vector from origin of $X_j Y_j Z_j$ axes to the cg (point "c") of the mass slug incoming to or outgoing from the <u>j</u> th body, Figure 1, ft or m
ρ_{jp}	Radius vector from origin of $X_j Y_j Z_j$ axes to the particle "p" of the <u>j</u> th body, Figure 1, ft or m
ρ_{jpm}	Radius vector from "p" to the mass particle incoming to or outgoing from "p", Figure 1 ft or m
$\rho_{jp\Delta m}$	Radius vector from "c" to a particle of the mass slug, Figure 1, ft or m
$\sum_p \sum_j$	Summations for all particles "p" of the <u>j</u> th body and for all <u>j</u> bodies of the system, respectively

ψ, θ	Yaw, pitch sequence of angular rotations respectively, to define the position of $X_2Y_2Z_2$ relative to $X_1Y_1Z_1$ (i. e., Body 2 relative to Body 1, Figure 4), rad or deg
ω_j	Angular velocity of the j th body relative to inertial space, rad/s
Ω	Angular velocity of the reference XYZ axes relative to inertial space, rad/s
$(-), (\rightarrow), (\hat{})$	Denotes vectors resolved relative to the $X_iY_iZ_i$, XYZ, and $X_jY_jZ_j$ axes, respectively
$\left\{ \right\}$	Vector, column matrix
$(\dot{}), (\ddot{})$	First and second derivatives with respect to time

EQUATIONS OF MOTION FOR FREE-FLIGHT SYSTEMS OF ROTATING-TRANSLATING BODIES HAVING VARIABLE MASS AND CONFIGURATION

Introduction and Summary

Often when dealing with the flight dynamics and control problems associated with missiles, reentry vehicles, aircraft, spacecraft, bombs, and shells the need arises for describing the motion of systems of varying-mass, varying-configuration bodies. Examples of systems of this type are thrusting missiles, thrusting aircraft, inflatable structures, etc., having moving control surfaces and/or moving internal components. Because of the large number of possible variations in the configurations of multiple body systems, it is desirable to have a general set of equations that can be easily reduced to describe dynamic systems of interest. For the aeroballistician it is particularly desirable to have a general set of equations that can easily implement the use of a nonrolling reference coordinate frame. Equations of motion for multiple body systems can be found in the literature; however, they are usually derived for rigid bodies (no mass or configuration variation) and do not provide for easy implementation of nonrolling reference coordinates; e.g., Reference 1. Unfortunately, even for single bodies, many incomplete general derivations of the variable mass equations appear in the literature; e.g., the incomplete force and moment equations of References 2 and 3.

Beginning with the development of variable mass particle dynamics, vector differential equations are derived herein that provide a complete description for the motion of a system of rotating-translating, variable-mass, variable configuration bodies. With respect to placement of coordinate frames, these general equations of motion, like the earlier rigid body equations derived by the author,⁴ are different from previous derivations in that the reference coordinate frame for the system is not required to be fixed in any of the bodies. The angular velocities of the individual bodies and the angular velocity of the reference coordinate frame can all differ. In addition, the coordinate frame in each body of the system is not required to have its origin fixed at the body cg. This flexibility in placement of the reference coordinate frame and each of the body fixed coordinate frames can be of great advantage in simplifying the equations for specific applications. For example, the requirement for nonrolling reference axes is easily implemented, as is the simulation of unbalanced bodies.

By introducing restraints (e.g., limiting the number of bodies, limiting their degrees of freedom, defining the dynamic coupling of the bodies, etc.), the general equations presented herein can be reduced in form to describe the equations of motion for many multiple body or single body dynamic systems of interest. These systems can contain rigid bodies as well as bodies of varying

mass and/or configuration. To demonstrate the flexibility and usefulness of these equations, two simplified cases are considered. For the first, equations of motion are developed relative to nonrolling coordinates for a spinning-thrusting body containing an internal rotating mass. The second example involves the development of equations of motion to describe the effects of a flexible joint on the dynamics of a spinning-thrusting combination of bodies. Examples for rigid body systems are presented in Reference 4. These include equations of motion relative to nonrolling coordinates for a spinning body having mass asymmetries, and equations of motion for a spinning body having an internal flexible vibrating member. The equations resulting from these examples are useful for describing the free-flight motions of both aerodynamically stabilized and gyroscopically stabilized spinning vehicles.

Theoretical Analysis

In Figure 1, the positions of the j bodies ($j = 1, 2, \dots, n$), which make up the system of interest, are described relative to both inertial space ($X_i Y_i Z_i$) and the reference coordinate system (XYZ). The center of mass or center of gravity (cg) for the system of bodies is not required to be coincident with the origin "o" of the reference coordinates. These bodies translate and rotate relative to one another as well as relative to the rotating reference frame XYZ . The body fixed coordinate system $X_j Y_j Z_j$ can be located arbitrarily in the j th body; therefore, as shown in Figure 1, the bodies can have cg offsets.

Figure 1. Coordinate Systems and Particle Locations

For the following development, the symbol above a vector quantity denotes the coordinate frame in which the vector is resolved and also the coordinate frame in which derivatives of the vector are taken. Examples are $\bar{\mathbf{r}}_j$ and $\dot{\bar{\mathbf{r}}}_j$, $\bar{\mathbf{r}}_j$ and $\dot{\mathbf{r}}_j$, or $\hat{\mathbf{r}}_j$ and $\dot{\hat{\mathbf{r}}}_j$. These respective groups indicate vectors resolved relative to, and derivatives taken with respect to, the $X_1Y_1Z_1$ inertial coordinates, the XYZ reference coordinates, and the $X_jY_jZ_j$ body-fixed coordinates.

Particle Dynamics

As shown in Figure 1, the radius vectors $\bar{\mathbf{R}}_{jp}$, from the origin of the $X_1Y_1Z_1$ inertial reference system to the particle "p" of the j th body, and $\bar{\mathbf{R}}_{jpm}$, from the origin of the $X_1Y_1Z_1$ system to the mass particle incoming to or outgoing from "p", are described by

$$\bar{\mathbf{R}}_{jp} = \bar{\mathbf{R}}_o + \bar{\mathbf{r}}_j + \bar{\rho}_{jp} \quad (1)$$

$$\bar{\mathbf{R}}_{jpm} = \bar{\mathbf{R}}_o + \bar{\mathbf{r}}_j + \bar{\rho}_{jpm} + \bar{\rho}_{jp}\Delta m = \bar{\mathbf{R}}_{jp} + \bar{\rho}_{jpm} \quad (2)$$

The force acting on the variable mass particle "p" and the moment of that force about the origin "o" of the reference XYZ coordinate frame are given as

$$\bar{\mathbf{F}}_{jp} = \frac{d}{dt} (\bar{\mathbf{P}}_{jp}) \quad (3)$$

$$\bar{\mathbf{M}}_{jpo} = (\bar{\mathbf{r}}_j + \bar{\rho}_{jp}) \times \bar{\mathbf{F}}_{jp} \quad (4)$$

where the instantaneous linear momentum of the particle "p" is given as

$$\bar{\mathbf{P}}_{jp} = m_{jp} \dot{\bar{\mathbf{R}}}_{jp} \quad (5)$$

Equation (3) is intended to apply only to a system of definite mass. As indicated by Greenwood,¹ Thomson,² Meriam,⁵ and others, the equation can be applied to a system of varying mass provided the same mass is examined for the change in momentum at two instances of time. The example used to depict this (Figure 2) is of a particle gaining mass.

Figure 2. Particle Mass Variation

At time t , the momentum of the particle and approaching mass are given as

$$\bar{P}_{jp} - \Delta \bar{P}_{jp} = m_{jp} \dot{\bar{R}}_{jp} + \Delta m_{jp} \dot{\bar{R}}_{jpm} \quad (6)$$

At time $t + \Delta t$, the two masses have joined. Then the momentum of the combined particle becomes

$$\bar{P}_{jp} = (m_{jp} + \Delta m_{jp}) (\dot{\bar{R}}_{jp} + \Delta \dot{\bar{R}}_{jp}) \quad (7)$$

The rate of change of linear momentum for the varying mass particle "p" is obtained by combining Eqs. (6) and (7) into difference form and taking the limit as $\Delta t \rightarrow 0$.

$$\dot{\bar{P}}_{jp} = \lim_{\Delta t \rightarrow 0} \left(\frac{\Delta \bar{P}_{jp}}{\Delta t} \right) = m_{jp} \ddot{\bar{R}}_{jp} + \dot{m}_{jp} \left(\dot{\bar{R}}_{jp} - \dot{\bar{R}}_{jpm} \right) \quad (8)$$

Substituting Eq. (8) into Eq. (3), the force acting on the variable mass particle "p" is given as

$$\bar{F}_{jp} = m_{jp} \ddot{\bar{R}}_{jp} + \dot{m}_{jp} \left(\dot{\bar{R}}_{jp} - \dot{\bar{R}}_{jpm} \right) \quad (9)$$

Equation (9) indicates that the force acting on a variable mass particle is instantaneously equal to the particle mass times acceleration plus the product of the particle mass rate of change and the velocity relative to the particle of the swallowed or exhausted mass. This fundamental equation for varying mass systems, often called the "rocket equation," can be found in many dynamics texts; e.g., References 1 and 2.

At this point in many derivations, such as those in References 1 and 2, the last term in Eq. (9) is interpreted to be thrust, an external force. Because it then appears instantaneously that, even for a varying mass system, force equals mass times acceleration, m_{jp} is often thereafter treated as a constant; e.g., the general derivations of References 2 and 3. When these interpretations are used, many important terms in the force and moment equations are lost.

Thrust is determined by the product of the mass rate of change and the velocity of the exhausted or swallowed mass relative to the point on the body from which the mass was ejected or ingested. Since we are dealing at this point in the derivation with a particle and not with a body, the concept of thrust is premature. For a general derivation, the term m_{jp} , like the remaining terms in Eq. (9), must be treated as functions of time.

Differentiating Eqs. (1) and (2) and then substituting the result into Eq. (9), the force acting on the particle "p" can be written as

$$\bar{F}_{jp} = m_{jp} \left(\ddot{\bar{R}}_o + \ddot{\bar{R}}_j + \ddot{\bar{p}}_{jp} \right) + \dot{m}_{jp} \left[\dot{\bar{p}}_{jp} - \left(\dot{\bar{p}}_{jm} + \dot{\bar{p}}_{jp} \Delta m \right) \right] \quad (10)$$

Substituting Eq. (10) into Eq. (4), the moment of the force $\bar{F}_{jp}$ about the origin of the reference XYZ coordinate system becomes

$$\bar{M}_{jp_o} = m_{jp} (\bar{r}_j + \bar{p}_{jp}) \times (\ddot{\bar{R}}_o + \ddot{\bar{r}}_j + \ddot{\bar{p}}_{jp}) + \dot{m}_{jp} (\bar{r}_j + \bar{p}_{jp}) \times \left[\dot{\bar{p}}_{jp} - (\dot{\bar{p}}_{jm} + \dot{\bar{p}}_{jp} \Delta m) \right]. \quad (11)$$

Note that if the thrust interpretation had been used in Eq. (9), the last terms of Eqs. (10) and (11) would be missing. It will become obvious later in the derivation that not only are thrust induced forces and moments derived from these terms, but so are the jet damping forces and moments and the contributions of varying mass to rate of change of moments and products of inertia.

jth Body Dynamics

Summing for all of the particles "p" of the jth body, the mass of the body is given as

$$\sum_p m_{jp} = m_j, \quad (12)$$

while the mass of an element or slug of the continuous incoming or outgoing mass stream (Figure 1) is

$$\sum_p \Delta m_{jp} = \Delta m_j.$$

Differentiating Eq. (12), the rate of change of the jth body's mass is given as

$$\sum_p \dot{m}_{jp} = \dot{m}_j. \quad (13)$$

The sign of Eq. (13) determines if mass is ejected from or swallowed by the jth body.

The instantaneous cg location of the jth body, obtained by summing particle mass moments about the origin of the $X_i Y_i Z_i$ inertial reference system, can be written as

$$\sum_p m_{jp} \bar{R}_{jp} = m_j \bar{R}_{jcg} \quad (14)$$

where

$$\bar{R}_{jcg} = \bar{R}_o + \bar{r}_j + \bar{p}_{jcg}. \quad (15)$$

Using Eqs. (1), (12), and (15), Eq. (14) can be reduced to

$$\sum_p m_{jp} \bar{p}_{jp} = m_j \bar{p}_{jcg}. \quad (16)$$

Equation (16) is important because it allows the summations of the unknown particle mass moments to be expressed in terms of easily determined parameters: the mass m_j and cg location $\bar{p}_{jcg}$ of the j th body. Note that $\bar{p}_{jcg}$ is a variable because of the motion of the j th body and also because of the effects of varying mass and varying configuration.

Letting "c" in Δm_j (Figure 1) be the cg of the mass slug, the summation of particle mass moments about "c" yields

$$\sum \Delta m_{jp} \bar{p}_{jp} \Delta m = 0 \quad (17)$$

The vector $\bar{R}_{jm}$ then describes the position of the mass slug's cg relative to the origin of the $X_i Y_i Z_i$ inertially fixed axes:

$$\bar{R}_{jm} = \bar{R}_o + \bar{r}_j + \bar{p}_{jm} \quad (18)$$

It is assumed for this development that the cg's of the exhausted or swallowed mass slugs enter or exit the j th body at the point located by $\bar{R}_{je}$. As shown in Figure 1 and as indicated below, $\bar{R}_{je}$ is the distance from the origin of the $X_i Y_i Z_i$ inertial reference system to the point at the center of either the nozzle exit plane or inlet entrance plane:

$$\bar{R}_{je} = \bar{R}_o + \bar{r}_j + \bar{p}_{je} \quad (19)$$

The velocity of the ejected or swallowed mass slug (velocity of point "c") relative to the reference points at the entrances to or exits from the j th body is given as

$$\bar{u}_j = \dot{\bar{R}}_{jm} - \dot{\bar{R}}_{je} \quad (20)$$

Substituting the derivatives of Eqs. (18) and (19) into Eq. (20) and then rearranging the result gives

$$\dot{\bar{p}}_{jm} = \bar{u}_j + \dot{\bar{p}}_{je} \quad (21)$$

The effects of varying mass on the motion of the j th body are caused by the mass slug instantaneously entering or exiting the body. Therefore, for these equations, the parameters $\bar{p}_{jm}$ and $\dot{\bar{p}}_{jp} \Delta m$ must take on the values that would exist at the entrance to or exit from the body; i.e.,

$$\bar{p}_{jm} \rightarrow \bar{p}_{jc} \quad (22)$$

and

$$\dot{\bar{p}}_{jp} \Delta m \rightarrow \dot{\bar{p}}_{jp} \Delta e \quad (23)$$

Summing Eqs. (10) and (11) for all particles "p" of the jth body and using Eqs. (21) and (23), the force acting on the jth body and the moment of that force about "o" (Figure 1) are given as

$$\begin{aligned}\bar{\mathbf{F}}_j = \sum_p \left[\bar{\mathbf{F}}_{jp} + \dot{m}_{jp} \bar{\mathbf{u}}_j \right] &= \left(\sum_p m_{jp} \right) \left(\ddot{\mathbf{R}}_o + \ddot{\mathbf{r}}_j \right) - \left(\sum_p \dot{m}_{jp} \right) \dot{\bar{\rho}}_{je} + \left(\sum_p m_{jp} \ddot{\bar{\rho}}_{jp} \right) \\ &+ \left(\sum_p \dot{m}_{jp} \dot{\bar{\rho}}_{jp} \right) - \left(\sum_p \dot{m}_{jp} \dot{\bar{\rho}}_{jp} \Delta e \right) \quad (24)\end{aligned}$$

$$\begin{aligned}\bar{\mathbf{M}}_{j_o} = \sum_p \left[\bar{\mathbf{M}}_{jp_o} + \bar{\mathbf{r}}_j \times \dot{m}_{jp} \bar{\mathbf{u}}_j + \dot{m}_{jp} \bar{\rho}_{jp} \times \left(\bar{\mathbf{u}}_j + \dot{\bar{\rho}}_{jp} \Delta e \right) \right] &= \left(\sum_p m_{jp} \bar{\rho}_{jp} \times \ddot{\bar{\rho}}_{jp} \right) \\ &+ \left(\sum_p \dot{m}_{jp} \bar{\rho}_{jp} \times \dot{\bar{\rho}}_{jp} \right) + \left[\left(\sum_p m_{jp} \right) \bar{\mathbf{r}}_j + \left(\sum_p m_{jp} \bar{\rho}_{jp} \right) \right] \times \left(\ddot{\mathbf{R}}_o + \ddot{\mathbf{r}}_j \right) \\ &+ \bar{\mathbf{r}}_j \times \left[\left(\sum_p m_{jp} \ddot{\bar{\rho}}_{jp} \right) + \left(\sum_p \dot{m}_{jp} \dot{\bar{\rho}}_{jp} \right) - \left(\sum_p \dot{m}_{jp} \dot{\bar{\rho}}_{jp} \Delta e \right) \right] \\ &- \left[\left(\sum_p \dot{m}_{jp} \right) \bar{\mathbf{r}}_j + \left(\sum_p \dot{m}_{jp} \bar{\rho}_{jp} \right) \right] \times \dot{\bar{\rho}}_{je} \quad (25)\end{aligned}$$

Not all of the summations in Eqs. (24) and (25) can be satisfied by Eqs. (12), (13), and (16). Derivatives of Eq. (16) are required to satisfy many. In the following development, an approach described by Meriam⁵ will be used to obtain the necessary derivatives.

At time t the cg location of the jth body is given by Eq. (14), while at $t + \Delta t$ the following relation can be used

$$m_j \bar{\mathbf{R}}_{jcg} + \Delta(m_j \bar{\mathbf{R}}_{jcg}) = \sum_p m_{jp} (\bar{\mathbf{R}}_{jp} + \Delta \bar{\mathbf{R}}_{jp}) + \sum_p \Delta m_{jp} (\bar{\mathbf{R}}_{jpm} + \Delta \bar{\mathbf{R}}_{jpm}) \quad (26)$$

The first term on the right-hand side of Eq. (26) accounts for the effect on mass moment of variation in position (configuration) of the mass particles which make up the jth body. The second term accounts for the change in mass moment due to the addition or deletion of mass to or from the particles. Combining Eqs. (14) and (26) into difference form, using Eqs. (2), (13), (17), and (18), and taking the limit as $\Delta t \rightarrow 0$,

$$\frac{d}{dt} (m_j \bar{\mathbf{R}}_{jcg}) = \lim_{\Delta t \rightarrow 0} \left[\frac{\Delta(m_j \bar{\mathbf{R}}_{jcg})}{\Delta t} \right] = \sum_p m_{jp} \dot{\bar{\mathbf{R}}}_{jp} + \dot{m}_j \bar{\mathbf{R}}_{jcm} \quad (27)$$

Expanding Eq. (27) and rearranging the result:

$$\sum_p m_{jp} \dot{\bar{R}}_{jp} = m_j \dot{\bar{R}}_{jcg} + \dot{m}_j (\bar{R}_{jcg} - \bar{R}_{jcm}) . \quad (28)$$

Combining Eq. (28) with the derivative of Eq. (14):

$$\sum_p \dot{m}_{jp} \bar{R}_{jp} = \dot{m}_j \bar{R}_{jcm} . \quad (29)$$

Equations (28) and (29) describe the elements of the particle mass moment summation first derivative. Equation (28) is necessary in order to obtain descriptions of mass moment summation related second derivatives, while Eq. (29), as shown later, has direct application in Eq. (25). If m_{jp} had been assumed to be instantaneously constant, the first derivative of the particle mass moment summation, like that of Reference 2, would have consisted only of the first term on the right-hand side of Eq. (28).

To obtain the second derivatives, Eq. (28) and the j th body's linear momentum and rate of change of linear momentum are used. Summing Eqs. (5) and (8) for all particles "p" of the j th body and using Eqs. (1), (2), (13), and (18)

$$\bar{P}_j = \sum_p \bar{P}_{jp} = \sum_p m_{jp} \dot{\bar{R}}_{jp} \quad (30)$$

$$\dot{\bar{P}}_j = \sum_p \dot{\bar{P}}_{jp} = \sum_p m_{jp} \ddot{\bar{R}}_{jp} + \sum_p \dot{m}_{jp} \dot{\bar{R}}_{jp} - \sum_p \dot{m}_{jp} \dot{\bar{p}}_{jp} \Delta m - \dot{m}_j \dot{\bar{R}}_{jcm} . \quad (31)$$

Combining Eqs. (28) and (30), the linear momentum of the j th body can be written as

$$\bar{P}_j = m_j \dot{\bar{R}}_{jcg} + \dot{m}_j (\bar{R}_{jcg} - \bar{R}_{jcm}) . \quad (32)$$

Differentiating Eq. (32) and substituting it into Eq. (31) yields

$$\sum_p m_{jp} \ddot{\bar{R}}_{jp} + \sum_p \dot{m}_{jp} \dot{\bar{R}}_{jp} - \sum_p \dot{m}_{jp} \dot{\bar{p}}_{jp} \Delta m = m_j \ddot{\bar{R}}_{jcg} + 2\dot{m}_j \dot{\bar{R}}_{jcg} + \ddot{m}_j (\bar{R}_{jcg} - \bar{R}_{jcm}) , \quad (33)$$

which can be reduced to the required particle mass moment summation second derivatives.

Substituting Eqs. (1), (15), (18), and their derivatives along with Eqs. (12), (13), (22), and (23) as needed into Eqs. (29) and (33), the particle mass moment summation derivatives of interest can be expressed as

$$\sum_p \dot{m}_{jp} \bar{p}_{jp} = \dot{m}_j \bar{p}_{je} . \quad (34)$$

$$\sum_p \dot{m}_{jp} \ddot{\bar{\rho}}_{jp} + \sum_p \dot{m}_{jp} \dot{\bar{\rho}}_{jp} - \sum_p \dot{m}_{jp} \dot{\bar{\rho}}_{jp} \Delta m = \dot{m}_j \ddot{\bar{\rho}}_{jcg} + 2\dot{m}_j \dot{\bar{\rho}}_{jcg} + \ddot{m}_j (\bar{\rho}_{jcg} - \bar{\rho}_{je}) + \dot{m}_j (\ddot{\bar{R}}_o + \ddot{\bar{r}}_j). \quad (35)$$

Using Eq. (13), the summation involving $\bar{u}_j$ in the applied force term of Eq. (24) can be written as

$$\left(\sum_p \dot{m}_{jp} \right) \bar{u}_j = \dot{m}_j \bar{u}_j = \bar{T}_j \quad (36)$$

where $\bar{T}_j$ is the total thrust acting on the j th body. Using Eqs. (13), (34), and (36), the summations involving $\bar{u}_j$ in the applied moment term of Eq. (25) become

$$\bar{r}_j \times \left(\sum_p \dot{m}_{jp} \right) \bar{u}_j = \bar{r}_j \times \dot{m}_j \bar{u}_j = \bar{r}_j \times \bar{T}_j \quad (37)$$

$$\left(\sum_p \dot{m}_{jp} \bar{\rho}_{jp} \right) \times \bar{u}_j = \bar{\rho}_{je} \times \dot{m}_j \bar{u}_j = \bar{\rho}_{je} \times \bar{T}_j \quad (38)$$

Substituting Eq. (36) into Eq. (24), the applied force acting on the j th body can be described by

$$\bar{F}_j = \sum_p \bar{F}_{jp} + \bar{T}_j \quad (39)$$

Substituting Eqs. (37) and (38) into Eq. (25), the applied moment becomes

$$\bar{M}_{j_o} = \sum_p \bar{M}_{jp_o} + (\bar{r}_j + \bar{\rho}_{je}) \times \bar{T}_j + \sum_p \dot{m}_{jp} \bar{\rho}_{jp} \times \dot{\bar{\rho}}_{jp} \Delta e \quad (40)$$

Because thrust is an external force, hereafter, as indicated by Eqs. (39) and (40), its contribution to applied force and moment will be included in $\bar{F}_j$ and $\bar{M}_{j_o}$, respectively, without being indicated separately.

Equation (38) describes the thrust misalignment moment which acts about the origin of the $X_j Y_j Z_j$ body-fixed axis system. The sum of Eqs. (37) and (38), the second term on the right-hand side of Eq. (40), represents the total moment of the thrust force acting about "o" (Figure 1).

The first terms on the right-hand sides of Eqs. (39) and (40), the particle force and moment summations, include only contributions due to external force and gravitational force. The particle interaction forces do not contribute anything because they occur in pairs of equal but opposite forces. Since the external forces (aerodynamic force, interaction force from other bodies, etc.) and the gravitational force acting on the j th body can be directed along lines which do not include the origin of the $X_j Y_j Z_j$ body-fixed axes, their resulting moments in Eq. (40), like the thrust

induced moments, can consist of a portion acting about the origin of the $X_j Y_j Z_j$ axes and a portion consisting of the vector product of $\bar{\mathbf{r}}_j$ and the external and gravitational forces.

The last term on the right-hand side of Eq. (40) is a mass variation induced couple. This moment results from particle motion within the incoming or outgoing mass slug; therefore, knowledge concerning these particle motions is required for its evaluation. The presence of this moment indicates that the motion of the j th body can be affected by motions of the mass slug other than those responsible for the thrust force; therefore the often used treatment of the mass slug as a point mass is incorrect. Because it can have compensating effects, this moment should not be neglected under conditions where the moment resulting from mass variation induced rates of change of moments and products of inertia is considered to be important. Like the particle moment summation and thrust induced moments, as indicated by Eq. (40), the mass variation induced couple will be included in $\bar{\mathbf{M}}_{j_o}$.

Substituting Eqs. (12), (13), (16), (34), and (35) into Eqs. (24) and (25), the force acting on the j th body and its moment about "o" become

$$\bar{\mathbf{F}}_j = \frac{d}{dt} \left[m_j (\dot{\bar{\mathbf{R}}}_o + \dot{\bar{\mathbf{r}}}_j + \dot{\bar{\rho}}_{jcg}) \right] + \dot{m}_j (\dot{\bar{\rho}}_{jcg} - \dot{\bar{\rho}}_{je}) + \ddot{m}_j (\bar{\rho}_{jcg} - \bar{\rho}_{je}) \quad (41)$$

$$\begin{aligned} \bar{\mathbf{M}}_{j_o} = & \dot{\bar{\mathbf{H}}}_j + m_j (\bar{\mathbf{r}}_j + \bar{\rho}_{jcg}) \times (\ddot{\bar{\mathbf{R}}}_o + \ddot{\bar{\mathbf{r}}}_j) + m_j \bar{\mathbf{r}}_j \times \ddot{\bar{\rho}}_{jcg} \\ & + \dot{m}_j \bar{\mathbf{r}}_j \times (\dot{\bar{\mathbf{R}}}_o + \dot{\bar{\mathbf{r}}}_j + 2\dot{\bar{\rho}}_{jcg} - \dot{\bar{\rho}}_{je}) - \dot{m}_j \bar{\rho}_{je} \times \dot{\bar{\rho}}_{je} + \ddot{m}_j \bar{\mathbf{r}}_j \times (\bar{\rho}_{jcg} - \bar{\rho}_{je}) \end{aligned} \quad (42)$$

where

$$\bar{\mathbf{H}}_j = \left(\sum_p m_{jp} \bar{\rho}_{jp} \times \dot{\bar{\rho}}_{jp} \right) \quad (43)$$

is the apparent moment of momentum of the j th body to an observer fixed in the $X_j Y_j Z_j$ coordinate system. As shown later, the vector $\bar{\mathbf{H}}_j$ is partially defined in terms of the j th body's moments and products of inertia.

The mass variation induced jet damping forces and moments, those terms involving $\dot{m}_j$, $\bar{\rho}_{je}$, and their first derivatives, are apparent in Eqs. (41) and (42). Because the cg is not necessarily located at the origin of the $X_j Y_j Z_j$ body-fixed axes, corrections to these forces and moments exist in these equations. One moment correction is contained in $\dot{\bar{\mathbf{H}}}_j$, while the remaining corrections are provided by some of the terms involving $\dot{m}_j$, $\bar{\rho}_{jcg}$ and their first derivatives.

At this point, equations have been developed which describe the motion of the j th body using vector quantities resolved relative to the $X_1 Y_1 Z_1$ inertially fixed directions. To accomplish the objectives of this derivation, the vector quantities must be resolved relative to the XYZ reference coordinate directions. Since the XYZ reference axes rotate with angular velocity $\vec{\Omega}$ (Figure 1) relative to inertial space, the following replacements for the derivatives in Eqs. (41) and (42) are required.

$$\dot{\vec{H}}_j = \dot{\vec{H}}_j + \vec{\Omega} \times \vec{H}_j \quad (44)$$

$$\dot{\vec{r}}_j = \dot{\vec{r}}_j + \vec{\Omega} \times \vec{r}_j \quad (45)$$

$$\ddot{\vec{r}}_j = \ddot{\vec{r}}_j + \dot{\vec{\Omega}} \times \vec{r}_j + 2\vec{\Omega} \times \dot{\vec{r}}_j + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_j) \quad (46)$$

$$\dot{\vec{p}}_{jcg} = \dot{\vec{p}}_{jcg} + \vec{\Omega} \times \vec{p}_{jcg} \quad (47)$$

$$\ddot{\vec{p}}_{jcg} = \ddot{\vec{p}}_{jcg} + \dot{\vec{\Omega}} \times \vec{p}_{jcg} + 2\vec{\Omega} \times \dot{\vec{p}}_{jcg} + \vec{\Omega} \times (\vec{\Omega} \times \vec{p}_{jcg}) \quad (48)$$

$$\dot{\vec{p}}_{je} = \dot{\vec{p}}_{je} + \vec{\Omega} \times \vec{p}_{je} \quad (49)$$

$$\dot{\vec{R}}_o = \vec{V}$$

Resolving the velocity vector $\vec{V}$ (velocity of "o") relative to the XYZ reference directions and taking its derivative gives

$$\ddot{\vec{R}}_o = \dot{\vec{V}} + \vec{\Omega} \times \vec{V} \quad (50)$$

Equations (44) through (50) indicate the equivalence of magnitude and direction shared by the vector quantities on either side of the equal signs; however, note that the vector components on opposing sides of these equations are resolved relative to different coordinates.

Substituting, as needed, Eqs. (44) through (50) into Eqs. (41) and (42) and then transforming the remaining vectors, the equations of motion for the j th body can be written relative to the XYZ reference coordinates as

$$\vec{F}_j = m_j \left\{ \dot{\vec{V}} + \vec{\Omega} \times \vec{V} + \left(\ddot{\vec{r}}_j + \ddot{\vec{p}}_{jcg} \right) + \vec{\Omega} \times \left(\dot{\vec{r}}_j + \dot{\vec{p}}_{jcg} \right) \right\}$$

$$\begin{aligned}
& + 2\vec{\Omega} \times (\vec{r}_j + \vec{\rho}_{jcg}) + \vec{\Omega} \times [\vec{\Omega} \times (\vec{r}_j + \vec{\rho}_{jcg})] \Big\} \\
& + \dot{m}_j \left\{ \vec{v} + \left(\vec{r}_j + 2\vec{\rho}_{jcg} - \vec{\rho}_{je} \right) + \vec{\Omega} \times \left(\vec{r}_j + 2\vec{\rho}_{jcg} - \vec{\rho}_{je} \right) \right\} \\
& + \ddot{m}_j (\vec{\rho}_{jcg} - \vec{\rho}_{je}) .
\end{aligned} \tag{51}$$

$$\begin{aligned}
\vec{M}_{j_o} = & \vec{H}_j + \vec{\Omega} \times \vec{H}_j + m_j (\vec{r}_j + \vec{\rho}_{jcg}) \times \left[\vec{v} + \vec{\Omega} \times \vec{v} + \ddot{\vec{r}}_j \right. \\
& + \vec{\Omega} \times \vec{r}_j + 2\vec{\Omega} \times \vec{r}_j + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_j) \Big] + m_j \vec{r}_j \times [\ddot{\vec{\rho}}_{jcg} \\
& + \vec{\Omega} \times \vec{\rho}_{jcg} + 2\vec{\Omega} \times \vec{\rho}_{jcg} + \vec{\Omega} \times (\vec{\Omega} \times \vec{\rho}_{jcg}) \Big] + \dot{m}_j \vec{r}_j \times \left[\vec{v} \right. \\
& + \vec{r}_j + 2\vec{\rho}_{jcg} - \vec{\rho}_{je} + \vec{\Omega} \times (\vec{r}_j + 2\vec{\rho}_{jcg} - \vec{\rho}_{je}) \Big] - \dot{m}_j \vec{\rho}_{je} \times \ddot{\vec{\rho}}_{je} \\
& - \dot{m}_j \vec{\rho}_{je} \times (\vec{\Omega} \times \vec{\rho}_{je}) + \ddot{m}_j \vec{r}_j \times (\vec{\rho}_{jcg} - \vec{\rho}_{je}) .
\end{aligned} \tag{52}$$

It is assumed that the forces and moments acting on the jth body and the vector quantities which depend on the physical properties of the jth body will be initially defined relative to the $X_j Y_j Z_j$ body fixed coordinate system. Therefore, the vectors $\vec{F}_j$, $\vec{M}_{j_o}$, $\vec{H}_j$, $\vec{\rho}_{jcg}$, and $\vec{\rho}_{je}$ that appear in the preceding equations are obtained by transforming $\hat{F}_j$, $\hat{M}_{j_o}$, $\hat{H}_j$, $\hat{\rho}_{jcg}$, and $\hat{\rho}_{je}$. A generalized form of the required angular transformation is given as

$$\left\{ \hat{A}_j \right\} = \left[\ell_j \right] \left\{ \hat{A}_j \right\} ,$$

which when expanded becomes

$$\begin{pmatrix} A_{X_j} \\ A_{Y_j} \\ A_{Z_j} \end{pmatrix} = \begin{bmatrix} \ell_{j11} & \ell_{j12} & \ell_{j13} \\ \ell_{j21} & \ell_{j22} & \ell_{j23} \\ \ell_{j31} & \ell_{j32} & \ell_{j33} \end{bmatrix} \begin{pmatrix} A_{X_{jj}} \\ A_{Y_{jj}} \\ A_{Z_{jj}} \end{pmatrix} . \tag{53}$$

Here, components of a vector quantity resolved relative to the body-fixed coordinates

$(\hat{A}_j = A_{X_{jj}} i_j + A_{Y_{jj}} j_j + A_{Z_{jj}} k_j)$ are transformed into components of the same vector quantity

relative to the XYZ reference coordinates ($\vec{A}_j = A_{Xj}i + A_{Yj}j + A_{Zj}k$). The direction cosine elements ℓ_{j11} , ℓ_{j12} , etc., of the transformation matrix [Eq. (53)]^j are defined using Euler rotation sequences,² quaternions,⁶ etc. An abbreviated vector notation indicating the transformation is given as

$$\vec{A}_j = \begin{bmatrix} \vec{\ell}_j \end{bmatrix} \left\{ \hat{A}_j \right\} \quad (54)$$

where

$$\begin{bmatrix} \vec{\ell}_j \end{bmatrix} = \begin{bmatrix} i & o & o \\ o & j & o \\ o & o & k \end{bmatrix} \begin{bmatrix} \ell_{j11} & \ell_{j12} & \ell_{j13} \\ \ell_{j21} & \ell_{j22} & \ell_{j23} \\ \ell_{j31} & \ell_{j32} & \ell_{j33} \end{bmatrix} \cdot$$

Using Eq. (54), the transformed vector quantities appearing in Eqs. (51) and (52) are defined as

$$\vec{F}_j = \begin{bmatrix} \vec{\ell}_j \end{bmatrix} \left\{ \hat{F}_j \right\} \quad (55)$$

$$\vec{M}_{j_o} = \begin{bmatrix} \vec{\ell}_j \end{bmatrix} \left\{ \hat{M}_{j_o} \right\} \quad (56)$$

$$\vec{H}_j = \begin{bmatrix} \vec{\ell}_j \end{bmatrix} \left\{ \hat{H}_j \right\} \quad (57)$$

$$\vec{\rho}_{jcg} = \begin{bmatrix} \vec{\ell}_j \end{bmatrix} \left\{ \hat{\rho}_{jcg} \right\} \quad (58)$$

$$\vec{\rho}_{je} = \begin{bmatrix} \vec{\ell}_j \end{bmatrix} \left\{ \hat{\rho}_{je} \right\} \quad (59)$$

To develop an expression for the apparent moment of momentum $\hat{H}_j$ in terms of physical properties of the jth body, Eq. (43) is written as

$$\hat{H}_j = \sum_p m_{jp} \hat{\rho}_{jp} \times \dot{\hat{\rho}}_{jp} + \sum_p m_{jp} \hat{\rho}_{jp} \times (\hat{\omega}_j \times \hat{\rho}_{jp}) \quad (60)$$

where $\hat{\omega}_j$ (Figure 1) is the angular velocity relative to inertial space of the $X_j Y_j Z_j$ body fixed coordinates. As demonstrated in Reference 7, the last summation term in Eq. (60) can be represented in terms of products of the rotational inertia properties of the jth body (moments and products of inertia) and components of $\hat{\omega}_j$. The first summation term of Eq. (60) involves terms that have similarities to those portions of the rates of change of product of inertia which result from changes in configuration. Equation (60) can be rewritten as

$$\hat{H}_j = \hat{\mathcal{J}}_j + \hat{\mathcal{J}}_j \cdot \hat{\omega}_j \quad (61)$$

where the vector

$$\hat{\mathcal{J}}_j = \sum_p m_{jp} \hat{\rho}_{jp} \times \dot{\hat{\rho}}_{jp} ,$$

and

$$\hat{\mathcal{J}}_j \cdot \hat{\omega}_j = \sum_p m_{jp} \hat{\rho}_{jp} \times (\hat{\omega}_j \times \hat{\rho}_{jp}) = \int_{j\text{th body}} \hat{\rho}_{jp} \times (\hat{\omega}_j \times \hat{\rho}_{jp}) dm .$$

The inertia dyadic for the j th body is given as

$$\hat{\mathcal{J}}_j = \begin{bmatrix} I_{X_j i j i} & -I_{X_j Y_j i j j} & -I_{X_j Z_j i j k} \\ -I_{X_j Y_j j j i} & I_{Y_j j j j} & -I_{Y_j Z_j j j k} \\ -I_{X_j Z_j k j i} & -I_{Y_j Z_j k j j} & I_{Z_j k j k} \end{bmatrix} . \quad (62)$$

Its elements, the moments and products of inertia I_{X_j} , $I_{X_j Y_j}$, etc., vary with time due to changes in mass and configuration.

The $\hat{\mathcal{J}}_j$ term in Eq. (61) can become nonzero when there is particle motion relative to the $X_j Y_j Z_j$ body-fixed axes. This term is difficult to evaluate because individual particle trajectory information is required. Sometimes it is possible to place the body-fixed coordinates within the body so that $\hat{\mathcal{J}}_j$ is eliminated. For example, this term becomes approximately zero when there exists an even distribution of particles that have particular symmetrical motion patterns, like radial motion, parallel motion, etc., relative to a body-fixed axis system. However, certain symmetrical motions of evenly distributed particles, like swirling motion, can cause the term to be nonzero. Fortunately, for many problems of interest, detailed evaluation of $\hat{\mathcal{J}}_j$ is not required because it is negligibly small in comparison to the remaining term in Eq. (61).

Note that the following derivatives of Eqs. (57) through (59) and Eq. (61) are required in Eqs. (51) and (52).

$$\dot{\hat{H}} = \begin{bmatrix} \dot{\gamma}_j \end{bmatrix} \left\{ \hat{H}_j \right\} + \begin{bmatrix} \dot{\gamma}_j \end{bmatrix} \left\{ \hat{H}_j \right\} \quad (63)$$

$$\dot{\hat{H}} = \dot{\hat{\mathcal{J}}}_j + \dot{\hat{\mathcal{J}}}_j \cdot \hat{\omega}_j + \dot{\hat{\mathcal{J}}}_j \cdot \hat{\omega}_j \quad (64)$$

$$\dot{\hat{\rho}}_{jcg} = \begin{bmatrix} \dot{\ell}_j \end{bmatrix} \left\{ \dot{\hat{\rho}}_{jcg} \right\} + \begin{bmatrix} \dot{\ell}_j \end{bmatrix} \left\{ \hat{\rho}_{jcg} \right\} \quad (65)$$

$$\ddot{\hat{\rho}}_{jcg} = \begin{bmatrix} \ddot{\ell}_j \end{bmatrix} \left\{ \dot{\hat{\rho}}_{jcg} \right\} + 2 \begin{bmatrix} \dot{\ell}_j \end{bmatrix} \left\{ \dot{\hat{\rho}}_{jcg} \right\} + \begin{bmatrix} \ddot{\ell}_j \end{bmatrix} \left\{ \hat{\rho}_{jcg} \right\} \quad (66)$$

$$\dot{\hat{\rho}}_{je} = \begin{bmatrix} \dot{\ell}_j \end{bmatrix} \left\{ \dot{\hat{\rho}}_{je} \right\} + \begin{bmatrix} \dot{\ell}_j \end{bmatrix} \left\{ \hat{\rho}_{je} \right\} \quad (67)$$

Inspecting Eqs. (51) and (52), together with Eqs. (55) through (67), it becomes obvious that for variable mass bodies many time varying parameters must be specified before Eqs. (51) and (52) can be used. These parameters are listed below:

$$m_j, \dot{m}_j, \ddot{m}_j$$

$$\hat{\rho}_{je}, \dot{\hat{\rho}}_{je}$$

$$\hat{\rho}_{jcg}, \dot{\hat{\rho}}_{jcg}, \ddot{\hat{\rho}}_{jcg}$$

$$I_{X_j}, \dot{I}_{X_j}$$

$$I_{Y_j}, \dot{I}_{Y_j}$$

$$I_{Z_j}, \dot{I}_{Z_j}$$

$$I_{X_j Y_j}, \dot{I}_{X_j Y_j}$$

$$I_{Y_j Z_j}, \dot{I}_{Y_j Z_j}$$

$$I_{X_j Z_j}, \dot{I}_{X_j Z_j}$$

$$\hat{\mathcal{J}}_j, \dot{\hat{\mathcal{J}}}_j$$

$$\hat{F}_j, \hat{M}_{jo}$$

To use Eqs. (51) and (52) for variable configuration bodies, the same list of parameters must be specified with the exception of m_j , $\hat{\rho}_{je}$, and their time derivatives. For rigid bodies, only $\hat{r}_j$ and $\hat{M}_{j_o}$ must be specified as functions of time since all of the remaining parameters in the list are either constant or zero.

Estimates of the above listed parameters will most likely be used in place of exact evaluations. The exact time variations of many of these parameters would be practically impossible to obtain since knowledge concerning the trajectories of individual particles within the moving body would be required.

System Dynamics

The total mass of the system of j bodies and the cg location of that system relative to the inertially fixed $X_1Y_1Z_1$ coordinates (Figure 1) are found by summing the masses of the j bodies and their mass moments as follows:

$$\sum_j m_j = m \quad (68)$$

$$\sum_j m_j \bar{R}_{jcg} = m \bar{R}_{cg} \quad (69)$$

where

$$\bar{R}_{cg} = \bar{R}_o + \bar{r}_{cg} \quad (70)$$

Differentiating Eq. (68)

$$\sum_j \dot{m}_j = \dot{m} \quad (71)$$

Substituting Eqs. (15) and (70) into Eq. (69) and then resolving vectors relative to the XYZ reference coordinate frame, the cg location of the system of bodies relative to the origin "o" of the reference XYZ coordinates (Figure 1) is given as

$$\sum_j m_j (\vec{r}_j + \vec{\rho}_{jcg}) = m \vec{r}_{cg} \quad (72)$$

The total force acting on the system of bodies and the moment of that force about "o" are obtained by summing Eqs. (51) and (52) for all j bodies and using Eqs. (68), (71), and (72)

$$\begin{aligned}
\vec{F} = \sum_j \vec{F}_j &= m(\dot{\vec{v}} + \vec{\Omega} \times \vec{v}) + \dot{m}\vec{v} + \dot{\vec{\Omega}} \times m\vec{r}_{cg} \\
&+ \vec{\Omega} \times (\vec{\Omega} \times m\vec{r}_{cg}) + \left(\sum_j m_j (\ddot{\vec{r}}_j + \ddot{\vec{\rho}}_{jcg}) \right) + 2\vec{\Omega} \times \left(\sum_j m_j (\dot{\vec{r}}_j + \dot{\vec{\rho}}_{jcg}) \right) \\
&+ \left(\sum_j \dot{m}_j (\dot{\vec{r}}_j + 2\dot{\vec{\rho}}_{jcg} - \dot{\vec{\rho}}_{je}) \right) + \vec{\Omega} \times \left(\sum_j \dot{m}_j (\vec{r}_j + 2\vec{\rho}_{jcg} - \vec{\rho}_{je}) \right) \\
&+ \left(\sum_j \ddot{m}_j (\vec{\rho}_{jcg} - \vec{\rho}_{je}) \right). \tag{73}
\end{aligned}$$

$$\begin{aligned}
\vec{M}_o = \sum_j \vec{M}_{j_o} &= \left(\sum_j \dot{\vec{H}}_j \right) + \vec{\Omega} \times \left(\sum_j \vec{H}_j \right) + m\vec{r}_{cg} \times \left[\dot{\vec{v}} + \vec{\Omega} \times \vec{v} \right] \\
&+ \left(\sum_j m_j (\vec{r}_j + \vec{\rho}_{jcg}) \times \ddot{\vec{r}}_j \right) + \left(\sum_j m_j (\vec{r}_j + \vec{\rho}_{jcg}) \times (\vec{\Omega} \times \vec{r}_j) \right) \\
&+ 2 \left(\sum_j m_j (\vec{r}_j + \vec{\rho}_{jcg}) \times (\vec{\Omega} \times \dot{\vec{r}}_j) \right) + \left(\sum_j m_j (\vec{r}_j + \vec{\rho}_{jcg}) \times [\vec{\Omega} \times (\vec{\Omega} \times \vec{r}_j)] \right) \\
&+ \left(\sum_j m_j \vec{r}_j \times \ddot{\vec{\rho}}_{jcg} \right) + \left(\sum_j m_j \vec{r}_j \times (\vec{\Omega} \times \vec{\rho}_{jcg}) \right) + 2 \left(\sum_j m_j \vec{r}_j \times (\vec{\Omega} \times \dot{\vec{\rho}}_{jcg}) \right) \tag{74} \\
&+ \left(\sum_j m_j \vec{r}_j \times [\vec{\Omega} \times (\vec{\Omega} \times \vec{\rho}_{jcg})] \right) + \left(\sum_j \dot{m}_j \vec{r}_j \times (\dot{\vec{r}}_j + 2\dot{\vec{\rho}}_{jcg} - \dot{\vec{\rho}}_{je}) \right) \\
&+ \left(\sum_j \dot{m}_j \vec{r}_j \times [\vec{\Omega} \times (\vec{r}_j + 2\vec{\rho}_{jcg} - \vec{\rho}_{je})] \right) + \left(\sum_j \dot{m}_j \vec{r}_j \times \vec{v} \right) \\
&- \left(\sum_j \dot{m}_j \vec{\rho}_{je} \times \dot{\vec{\rho}}_{je} \right) - \left(\sum_j \dot{m}_j \vec{\rho}_{je} \times (\vec{\Omega} \times \vec{\rho}_{je}) \right) + \left(\sum_j \ddot{m}_j \vec{r}_j \times (\vec{\rho}_{jcg} - \vec{\rho}_{je}) \right).
\end{aligned}$$

Motions of the j individual bodies are described by Eqs. (51) and (52), together with their auxiliary equations, while Eqs. (73) and (74) describe the motion of the system of j bodies. Unless the relative motions of the bodies have been specified a priori, equations from both sets are required to solve a multiple body problem.

The forces and moments acting on the j individual bodies include those resulting from gravitational forces, body interaction forces, and other external forces such as thrust and aerodynamic force. Forces and moments resulting from body interactions occur in pairs whose contributions are equal in magnitude but opposite in sign. Therefore, in the summation to obtain Eqs. (73) and (74), the contributions to these pairs cancel, leaving only the summations of the remaining external forces and moments and gravitational forces and moments, which are represented by $\vec{F}$ and $\vec{M}_O$.

Example Cases

In order to demonstrate the usefulness of these general equations of motion, two examples will be given. For the first example, equations of motion will be developed relative to a nonrolling coordinate system (aeroballistic axes) for a spinning-thrusting vehicle containing an internal rotating mass. The equations of motion for a spinning-thrusting combination of bodies having a flexible joint will be developed as a second example.

Rotating Internal Mass

The two-body system considered here (Figure 3) consists of an outer spinning-variable mass (thrusting) body which contains an inner rigid body that spins at a different rate. For this example, the $X_1 Y_1 Z_1$ coordinate system is fixed in Body 1 with its origin located away from the cg of the body ($\hat{p}_{1cg} = x_{1cg}, 0, 0$), while the $X_2 Y_2 Z_2$ coordinate frame is fixed in Body 2 with its origin at the body's cg ($\hat{p}_{2cg} = 0$). Note that the cg of Body 1 and Body 2 lie on the common axis of symmetry of these bodies. As shown in Figure 3, the colinear X_1 and X_2 axes are directed toward the nose of Body 1 along the axis of symmetry. The position of Y_2 relative to Y_1 and Z_2 relative to Z_1 is described by the angle γ .

The XYZ nonrolling reference coordinate system (Figure 3) is placed in this system of bodies so that its origin "o" is coincident with the origin of the $X_1 Y_1 Z_1$ body fixed axes ($\hat{r}_1 = 0, \hat{r}_2 = x_2, 0, 0$). The X_1 and X axes, which have identical direction, remain colinear while the position of Y_1 relative to Y and Z_1 relative to Z is described by the roll angle ϕ . Therefore, like Body 2, the reference coordinate system pitches and yaws with Body 1 but does not roll with it.

For the arrangement of coordinate systems and constraints pictured in Figure 3, the equations of motion for the system of bodies, Eq. (52) together with Eqs. (68) and (71) through (74), reduce to the following vector set

$$m\vec{r}_{cg} = m_1\vec{r}_{1cg} + m_2\vec{r}_2 \quad (75)$$

Figure 3. Spinning-Thrusting Body With Rotating Internal Component (Nonrolling Coordinates)

$$\vec{M}_O = \vec{H}_2 + \vec{\Omega} \times \vec{H}_2 + m_2 \vec{r}_2 \times \left[\vec{v} + \vec{\Omega} \times \vec{v} + \dot{\vec{\Omega}} \times \vec{r}_2 + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_2) \right] \quad (76)$$

$$\begin{aligned} \vec{F} = & m \left[\dot{\vec{v}} + \vec{\Omega} \times \vec{v} \right] + \dot{m}_1 \vec{v} + m \dot{\vec{\Omega}} \times \vec{r}_{cg} + m \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_{cg}) \\ & + m_1 \ddot{\vec{\rho}}_{1cg} + 2m_1 \vec{\Omega} \times \dot{\vec{\rho}}_{1cg} + 2\dot{m}_1 \dot{\vec{\rho}}_{1cg} + \dot{m}_1 \vec{\Omega} \times (2\vec{\rho}_{1cg} - \vec{\rho}_{1e}) \\ & + \ddot{m}_1 (\vec{\rho}_{1cg} - \vec{\rho}_{1e}) \end{aligned} \quad (77)$$

$$\begin{aligned} \vec{M}_O = & \vec{H}_1 + \vec{H}_2 + \vec{\Omega} \times (\vec{H}_1 + \vec{H}_2) + \vec{r}_{cg} \times \vec{F} - m \vec{r}_{cg} \times \left[\dot{\vec{\Omega}} \times \vec{r}_{cg} \right. \\ & \left. + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_{cg}) \right] - m_1 \vec{r}_{cg} \times \left[\ddot{\vec{\rho}}_{1cg} + 2\vec{\Omega} \times \dot{\vec{\rho}}_{1cg} \right] \\ & + m_2 \vec{r}_2 \times \left[\dot{\vec{\Omega}} \times \vec{r}_2 + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_2) \right] - \dot{m}_1 \vec{r}_{cg} \times \left[\vec{v} + 2\dot{\vec{\rho}}_{1cg} \right. \\ & \left. + \vec{\Omega} \times (2\vec{\rho}_{1cg} - \vec{\rho}_{1e}) \right] - \dot{m}_1 \vec{\rho}_{1e} \times (\vec{\Omega} \times \vec{\rho}_{1e}) - \ddot{m}_1 \vec{r}_{cg} \times (\vec{\rho}_{1cg} - \vec{\rho}_{1e}) \end{aligned} \quad (78)$$

where $m = m_1 + m_2$, $\dot{m} = \dot{m}_1$, and $\ddot{m} = \ddot{m}_1$. The force equation for Body 2 has been omitted since that body has only a single angular degree of freedom.

With the exception of $\vec{H}_1$ and $\vec{H}_2$, the vector quantities which appear in Eqs. (75) through (78) are defined in terms of their components relative to the XYZ nonrolling reference axis system as

$$\vec{\rho}_{1cg} = (x_{1cg}, 0, 0) \quad (79)$$

$$\vec{\rho}_{1e} = (x_{1e}, 0, 0) \quad (\text{constant}) \quad (80)$$

$$\vec{r}_2 = (x_2, 0, 0) \quad (\text{constant}) \quad (81)$$

$$\vec{M}_O = (M_{X_2}, M_{Y_2} - m_2 x_2 g_Z, M_{Z_2} + m_2 x_2 g_Y) \quad (82)$$

$$\vec{F} = (F_X + mg_X, F_Y + mg_Y, F_Z + mg_Z) \quad (83)$$

$$\vec{M}_O = \left[M_X, M_Y - (m_1 x_{1cg} + m_2 x_2) g_Z, M_Z + (m_1 x_{1cg} + m_2 x_2) g_Y \right] \quad (84)$$

$$\vec{V} = (\tilde{u}, \tilde{v}, \tilde{w}) \quad (85)$$

$$\vec{\Omega} = (\tilde{p}, \tilde{q}, \tilde{r}) \quad (86)$$

For this example, with the exceptions of F_X and M_{X_2} , the nongravitational forces and moments (F_Y , F_Z , M_X , etc.) appearing in Eqs. (82) through (84) result entirely from aerodynamic effects. The axial force F_X contains thrust in addition to aerodynamic force. Since Body 2 is internal, the moment M_{X_2} is composed entirely of moments input to Body 2 by some action relative to Body 1; e. g., a motor attached to Body 1 and applying a torque to Body 2. These interaction moments do not appear in M_X because of the moments of equivalent magnitude but opposite sign that they induce in the two-body system.

The inertially referenced angular velocities of the bodies resolved into components along their respective body-fixed coordinate directions are given as

$$\vec{\omega}_j = (p_j, q_j, r_j) \quad (j = 1, 2) \quad (87)$$

These components and those of $\vec{\Omega}$ [Eq. (86)] are related as follows:

$$\tilde{p} = p_1 - \dot{\phi} = p_2 - (\dot{\phi} + \dot{\gamma})$$

$$\tilde{q} = q_1 \cos \phi - r_1 \sin \phi = q_2 \cos (\phi + \gamma) - r_2 \sin (\phi + \gamma)$$

$$\tilde{r} = q_1 \sin \phi + r_1 \cos \phi = q_2 \sin (\phi + \gamma) + r_2 \cos (\phi + \gamma)$$

Because the rotational inertial properties of these bodies will be measured relative to the X_j, Y_j, Z_j ($j = 1, 2$) body-fixed axes, the apparent moment of momentum vectors for the bodies will initially be defined relative to these same axes; i. e., as $\hat{H}_j$ ($j = 1, 2$). Using Eqs. (61), (62), and (87), assuming a symmetric distribution relative to the X_1 and X_2 axes of the rigid and/or moving mass, and neglecting $\hat{J}_1$ ($\hat{J}_2 = 0$, Body 2 is rigid)

$$\hat{H}_j = (I_{X_j} p_j, I_{Y_j} q_j, I_{Z_j} r_j) \quad (j = 1, 2) \quad (88)$$

Substituting Eq. (88) together with the respective vector transformation matrices [see Eq. (54) for matrix form] into Eq. (57) and writing the lateral rates in terms of those of the XYZ reference system, the apparent moment of momentum vectors become

$$\vec{H}_j = (I_{X_j} p_j, I_{Y_j} \tilde{q}, I_{Z_j} \tilde{r}) \quad (j = 1, 2) \quad (89)$$

Combining Eqs. (89) together with Eqs. (75) through (86), the vector equations of motion can be expanded into the following scalar forms:

$$\ddot{x}_{cg} - \left(\frac{m_1}{m}\right) \ddot{x}_{1cg} + \left(\frac{m_2}{m}\right) \ddot{x}_2 \quad (90)$$

$$M_{X_2} = I_{X_2} (\ddot{p}_1 + \ddot{\gamma}) = I_{X_2} \ddot{p}_2 \quad (91)$$

$$\begin{aligned} F_X + mg_X &= m(\ddot{u} + \ddot{q}\ddot{w} - \ddot{r}\ddot{v}) - m\ddot{x}_{cg}(\ddot{q}^2 + \ddot{r}^2) + m_1\ddot{x}_{1cg} \\ &+ \dot{m}_1(\ddot{u} + 2\dot{x}_{1cg}) + \dot{m}_1(x_{1cg} - x_{1e}) \end{aligned} \quad (92)$$

$$\begin{aligned} F_Y + mg_Y &= m\left[\ddot{\dot{v}} + \ddot{r}\left(\ddot{u} + 2\left(\frac{m_1}{m}\right)\dot{x}_{1cg}\right) - \ddot{p}\ddot{w}\right] + m\ddot{x}_{cg}(\ddot{r} + \ddot{p}\ddot{q}) \\ &+ \dot{m}_1(\ddot{v} + x_{1cg}\ddot{r}) + \dot{m}_1(x_{1cg} - x_{1e})\ddot{r} \end{aligned} \quad (93)$$

$$\begin{aligned} F_Z + mg_Z &= m\left[\ddot{\dot{w}} + \ddot{p}\ddot{v} - \ddot{q}\left(\ddot{u} + 2\left(\frac{m_1}{m}\right)\dot{x}_{1cg}\right)\right] + m\ddot{x}_{cg}(\ddot{p}\ddot{r} - \ddot{q}\ddot{\dot{q}}) \\ &+ \dot{m}_1(\ddot{w} - x_{1cg}\ddot{q}) - \dot{m}_1(x_{1cg} - x_{1e})\ddot{q} \end{aligned} \quad (94)$$

$$M_X - M_{X_2} = I_{X_1} \ddot{p}_1 + I_{X_1} \ddot{p}_1 \quad (95)$$

$$\begin{aligned} M_Y &= \left[(I_1 + I_2 + m_2x_2^2) - m\ddot{x}_{cg}^2\right](\ddot{q} - \ddot{p}\ddot{r}) + \left[(I_{X_1} + I_{X_2})\ddot{p}_1 + I_{X_2}\ddot{\dot{r}}\right]\ddot{r} \\ &+ \left\{\left[\ddot{I}_1 - 2m_1\ddot{x}_{cg}\dot{x}_{1cg} - \dot{m}_1\ddot{x}_{cg}x_{1cg}\right] - \dot{m}_1\left[\ddot{x}_{cg}(x_{1cg} - x_{1e}) + x_{1e}^2\right]\right\}\ddot{q} \\ &- \ddot{x}_{cg}(F_Z - \dot{m}_1\ddot{w}) \end{aligned} \quad (96)$$

$$\begin{aligned}
M_Z = & \left[(I_1 + I_2 + m_2 x_{cg}^2) - m x_{cg}^2 \right] (\ddot{r} + \ddot{pq}) - \left[(I_{X_1} + I_{X_2}) p_1 + I_{X_2} \dot{\gamma} \right] \ddot{q} \\
& + \left\{ \left[\dot{I}_1 - 2m_1 x_{cg} \dot{x}_{1cg} - \dot{m}_1 x_{cg} x_{1cg} \right] - \dot{m}_1 \left[x_{cg} (x_{1cg} - x_{1e}) + x_{1e}^2 \right] \right\} \ddot{r} \\
& + x_{cg} (F_Y - \dot{m}_1 \ddot{v}) \quad (97)
\end{aligned}$$

The mass variation induced effects of thrust, rate of change of moment of inertia, variation of cg location, and jet damping are all included in Eqs. (90) through (97). These equations can be used to investigate the motion of spinning vehicles having rotating internal flywheels, stable platforms, etc., with or without the effects of mass variation being present.

Flexible Joint

The multiple body system used for this example (Figure 4) consists of two spinning bodies connected by a flexible joint (spring). Motion of the aft variable mass (thrusting) body (Body 2) relative to the rigid forebody (Body 1) is constrained to small rotations about a pivot point located at the base of Body 1 on its axis of symmetry. The $X_1 Y_1 Z_1$ body-fixed coordinates (Body 1, Figure 4) and the XYZ reference system are coincident with their colinear X and X_1 axes directed forward along the axis of symmetry of Body 1 and their origins are fixed at the cg of the body ($\hat{r}_1 = \hat{\rho}_{1cg} = 0$). The $X_2 Y_2 Z_2$ axis system is fixed in Body 2 with its origin at the pivot point ($\hat{r}_2 = x_2, 0, 0$) and its X_2 axis directed forward along the axis of symmetry of Body 2. Note that the cg of Body 2, which is located away from the origin of the $X_2 Y_2 Z_2$ axes, is also assumed to lie on the Body 2 axis of symmetry ($\hat{\rho}_{2cg} = X_{2cg}, 0, 0$). As shown in Figure 4, the angular position of $X_2 Y_2 Z_2$ relative to $X_1 Y_1 Z_1$ ($X_{1p} Y_{1p} Z_{1p}$ is mutually parallel with $X_1 Y_1 Z_1$), i.e., the angular position of Body 2 relative to Body 1, is specified by the Euler rotation angles Ψ, θ in a yaw-pitch sequence.

A set of vector equations which describe the motion of this system of bodies can be obtained from Eqs. (52), (68), and (71) through (74) as

$$m \vec{r}_{cg} = m_2 (\vec{r}_2 + \vec{\rho}_{2cg}) \quad (98)$$

$$\begin{aligned}
\vec{F} = m & \left[\dot{\vec{v}} + \vec{\Omega} \times \vec{v} + \dot{\vec{\Omega}} \times \vec{r}_{cg} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_{cg}) \right] + m_2 (\ddot{\vec{\rho}}_{2cg} + 2\vec{\Omega} \times \dot{\vec{\rho}}_{2cg}) \\
& + \dot{m}_2 \left[\vec{v} + 2\dot{\vec{\rho}}_{2cg} - \dot{\vec{\rho}}_{2e} + \vec{\Omega} \times (\vec{r}_2 + 2\vec{\rho}_{2cg} - \vec{\rho}_{2e}) \right] + \ddot{m}_2 (\vec{\rho}_{2cg} - \vec{\rho}_{2e}) \quad (99)
\end{aligned}$$

Figure 4. Spinning-Thrusting Body with Flexible Joint

$$\begin{aligned}
\vec{M}_O = & \dot{\vec{H}}_1 + \dot{\vec{H}}_2 + \vec{\Omega} \times (\vec{H}_1 + \vec{H}_2) + (\vec{r}_2 + \vec{\rho}_{2cg}) \times \left[\vec{F} - m_1(\dot{\vec{v}} + \vec{\Omega} \times \vec{v}) \right] \\
& - m_2 \vec{\rho}_{2cg} \times \left[\ddot{\vec{\rho}}_{2cg} + \vec{\Omega} \times \vec{\rho}_{2cg} + 2\vec{\Omega} \times \dot{\vec{\rho}}_{2cg} + \vec{\Omega} \times (\vec{\Omega} \times \vec{\rho}_{2cg}) \right] \\
& - \dot{m}_2 \vec{\rho}_{2cg} \times \left[\dot{\vec{v}} + 2\dot{\vec{\rho}}_{2cg} - \dot{\vec{\rho}}_{2e} + \vec{\Omega} \times (\vec{r}_2 + 2\vec{\rho}_{2cg} - \vec{\rho}_{2e}) \right] \\
& - \dot{m}_2 \vec{\rho}_{2e} \times \left[\dot{\vec{\rho}}_{2e} + (\vec{\Omega} \times \vec{\rho}_{2e}) \right] + \dot{m}_2 \vec{\rho}_{2cg} \times \vec{\rho}_{2e}
\end{aligned} \tag{100}$$

$$\vec{M}_O - \vec{M}_2 = \dot{\vec{H}}_1 + \vec{\Omega} \times \vec{H}_1 \tag{101}$$

where $m = m_1 + m_2$, $\dot{m} = \dot{m}_2$, and $\ddot{m} = \ddot{m}_2$. To minimize the number of terms in the above vector equations, the moment equation for Body 1 [Eq. (101)] has been used in place of the Body 2 moment equation. Both moment equations contain the spring interaction moment.

Certain vector quantities used to characterize this problem (those that do not require derivation or transformation) can be written directly in terms of their components relative to the various coordinate systems. These are given as

$$\hat{\rho}_{2e} = (x_{2e}, 0, 0) \quad (\text{constant}) \tag{102}$$

$$\hat{\rho}_{2cg} = (x_{2cg}, 0, 0) \quad (\text{constant}) \tag{103}$$

$$\hat{r}_2 = (x_2, 0, 0) \quad (\text{constant}) \tag{104}$$

$$\vec{\Omega} = \hat{\omega}_1 = (p, q, r) \tag{105}$$

$$\hat{\omega}_2 = (p_2, q_2, r_2) \tag{106}$$

$$\vec{v} = (u, v, w) \tag{107}$$

$$\vec{F} = (F_X + mg_X, F_Y + mg_Y, F_Z + mg_Z) \tag{108}$$

$$\begin{aligned}
\vec{M}_O = & \left\{ M_X + m_2(y_{2cg}g_Z - z_{2cg}g_Y), M_Y + m_2[z_{2cg}g_X - (x_2 + x_{2cg})g_Z], M_Z \right. \\
& \left. + m_2[(x_2 + x_{2cg})g_Y - y_{2cg}g_X] \right\}
\end{aligned} \tag{109}$$

$$\begin{aligned} \vec{M}_{20} = & \left\{ M_{X_2} + m_2(y_{2cg}g_Z - z_{2cg}g_Y), M_{Y_2} - (K\dot{\theta} + C\ddot{\theta}) + m_2[z_{2cg}g_X \right. \\ & \left. - (x_2 + x_{2cg})g_Z], M_{Z_2} - (K\dot{\psi} + C\ddot{\psi}) + m_2[(x_2 + x_{2cg})g_Y - y_{2cg}g_X] \right\} \end{aligned} \quad (110)$$

The terms x_{2cg} , y_{2cg} , and z_{2cg} used in Eqs. (109) and (110) are components of $\vec{p}_{2cg}$ which will be established after transforming $\hat{p}_{2cg}$ [Eq. (103)]. Note the presence of the interaction moments $-(K\dot{\theta} + C\ddot{\theta})$ and $-(K\dot{\psi} + C\ddot{\psi})$ in Eq. (110). The forces and moments F_X , F_Y , F_Z , M_Y , and M_Z in Eqs. (108) and (109) contain contributions from aerodynamic reaction and thrust, while M_X , M_{X_2} , M_{Y_2} , and M_{Z_2} of Eqs. (109) and (110) result only from aerodynamic reaction.

Using Eqs. (61), (105), and (108), assuming a symmetrical distribution of the rigid and/or moving mass relative to the body-fixed axes, and neglecting $\hat{J}_2$ ($\hat{J}_1 = 0$, Body 1 is rigid), the apparent moment of momentum vectors for Bodies 1 and 2 can be written as

$$\vec{H}_1 = \hat{H}_1 = I_{X_1} p, I_1 q, I_1 r \quad (111)$$

$$\hat{H}_2 = I_{X_2} p_2, I_2 q_2, I_2 r_2 \quad (112)$$

where I_{X_1} , I_1 are constants, I_{X_2} , I_2 are variables, and

$$p_2 = p \cos \theta \cos \Psi + q \cos \theta \sin \Psi - (r + \dot{\psi}) \sin \theta$$

$$q_2 = (q \cos \Psi + \dot{\theta}) - p \sin \Psi$$

$$r_2 = (r + \dot{\psi}) \cos \theta + p \sin \theta \cos \Psi + q \sin \theta \sin \Psi$$

To complete the definitions of the vector quantities that appear in Eqs. (98) through (101) in terms of their components relative to the XYZ reference system, it is necessary to transform the components of $\hat{p}_{2e}$, $\hat{p}_{2cg}$, and $\hat{H}_2$. Substituting Eqs. (102), (103), and (112) together with the vector transformation matrix [see Eq. (54) for matrix form] into Eqs. (59), (58), and (57), respectively,

$$\vec{p}_{2e} = (X_{2e} \cos \theta \cos \Psi, X_{2e} \cos \theta \sin \Psi, -X_{2e} \sin \theta) \quad (113)$$

$$\vec{p}_{2cg} = (x_{2cg}, y_{2cg}, z_{2cg}) = (X_{2cg} \cos \theta \cos \Psi, X_{2cg} \cos \theta \sin \Psi, -X_{2cg} \sin \theta) \quad (114)$$

$$\vec{H}_2 = (H_{X_2}, H_{Y_2}, H_{Z_2}) \quad (115)$$

where

$$\begin{aligned}
H_{X_2} = & \left(I_{X_2} \cos^2 \theta + I_2 \sin^2 \theta \right) \left(p \cos \Psi + q \sin \Psi \right) \cos \theta \\
& + I_2 \left[p \sin \Psi - \left(q \cos \Psi + \dot{\theta} \right) \right] \sin \Psi \\
& + \left(I_2 - I_{X_2} \right) \left(r + \dot{\Psi} \right) \sin \theta \cos \theta \cos \Psi
\end{aligned}$$

$$\begin{aligned}
H_{Y_2} = & I_2 \left[\left(q \cos \Psi + \dot{\theta} \right) - p \sin \Psi \right] \cos \Psi + \left(I_2 - I_{X_2} \right) \left(r + \dot{\Psi} \right) \sin \theta \cos \theta \sin \Psi \\
& + \left(I_{X_2} \cos^2 \theta + I_2 \sin^2 \theta \right) \left(q \sin \Psi + p \cos \Psi \right) \sin \Psi
\end{aligned}$$

$$H_{Z_2} = \left(I_{X_2} \sin^2 \theta + I_2 \cos^2 \theta \right) \left(r + \dot{\Psi} \right) + \left(I_2 - I_{X_2} \right) \left(p \cos \Psi + q \sin \Psi \right) \sin \theta \cos \theta.$$

Note that since X_{2e} and X_{2cg} were assumed constant [Eqs. (102) and (103)], $\vec{p}_{2e}$ and $\vec{p}_{2cg}$ [Eqs. (113) and (114)] are variables only due to the presence of Ψ and θ . A constant X_{2e} realistically fits this problem. Constant X_{2cg} was introduced principally to simplify the final form of the equations; however, there are motors, like radial burning rockets, that have characteristics which fit this assumption. The characteristics of these radial burning rockets could also satisfy the previous assumption concerning symmetric distribution of the rigid and moving mass relative to axes of symmetry.

A completely general expansion of Eqs. (98) through (101) to scalar form would result in a set of equations with an extremely large number of terms involving products of trigonometric functions. Because the angular displacements Ψ and θ are small, $\cos \Psi \approx \cos \theta \approx 1$, $\sin \Psi \approx \Psi$, and $\sin \theta \approx \theta$. In order to further simplify the final form of these equations, which would otherwise include many terms having negligible effect on the motion, it was assumed that terms involving products of Ψ and/or θ could be ignored; however, all indicated differentiation in Eqs. (99) through (101) was completed prior to applying this assumption.

Using the approximations for the trigonometric functions and neglecting small terms in the combination and expansion of Eqs. (98) through (115), the scalar differential equations of motion for the two-body system shown in Figure 4 can be written as

$$\begin{aligned}
F_X + mg_X &= m(\dot{u} + qw - rv) + \dot{m}_2 u - m_2 X_{2cg} \left\{ \left[(\dot{r} + \ddot{\Psi}) - qp \right] \Psi \right. \\
&\quad \left. + \left[(\dot{q} + \ddot{\theta}) + rp \right] \theta + (q + \dot{\theta})^2 + (r + \dot{\Psi})^2 \right\} - m_2 x_2 (q^2 + r^2) \\
&\quad - \dot{m}_2 (2X_{2cg} - X_{2e}) \left[(q + \dot{\theta}) \theta + (r + \dot{\Psi}) \Psi \right] + \ddot{m}_2 (X_{2cg} - X_{2e})
\end{aligned} \tag{116}$$

$$\begin{aligned}
F_Y + mg_Y &= m(\dot{v} + ru - pw) + \dot{m}_2 (v + x_2 r) \\
&\quad + m_2 X_{2cg} \left\{ (\dot{r} + \ddot{\Psi}) - \left[p^2 + (r + \dot{\Psi})^2 + (\dot{\theta})^2 \right] \Psi - \left[(rq - \dot{p}) \right. \right. \\
&\quad \left. \left. + 2\dot{\theta} (r + \dot{\Psi}) \right] \theta + (q + 2\dot{\theta}) p \right\} + m_2 x_2 (\dot{r} + p q) \\
&\quad + \dot{m}_2 (2X_{2cg} - X_{2e}) \left[(r + \dot{\Psi}) + p \theta \right] + \ddot{m}_2 (X_{2cg} - X_{2e}) \Psi
\end{aligned} \tag{117}$$

$$\begin{aligned}
F_Z + mg_Z &= m(\dot{w} + pv - qu) + \dot{m}_2 (w - x_2 q) \\
&\quad - m_2 X_{2cg} \left\{ (\dot{q} + \ddot{\theta}) - \left[p^2 + (q + \dot{\theta})^2 \right] \theta - \left[(qr + \dot{p}) + 2q\dot{\Psi} \right] \Psi \right. \\
&\quad \left. - (r + 2\dot{\Psi}) p \right\} + m_2 x_2 (pr - \dot{q}) + \dot{m}_2 (2X_{2cg} - X_{2e}) \left[p \Psi - (q + \dot{\theta}) \right] \\
&\quad - \ddot{m}_2 (X_{2cg} - X_{2e}) \theta
\end{aligned} \tag{118}$$

$$\begin{aligned}
M_X &= \left(I_{X_1} + I_{X_2} \right) \dot{p} + I_{X_2} \left\{ \left[q\dot{\Psi} - (r + \dot{\Psi}) \dot{\theta} + \dot{q}\Psi - (\dot{r} + \ddot{\Psi}) \theta \right] + \left[(q + 2\dot{\theta}) \theta \right. \right. \\
&\quad \left. \left. + (r + 2\dot{\Psi}) \Psi \right] p \right\} + \left(I_2 - m_2 X_{2cg}^2 \right) \left\{ \left[(\dot{r} + \ddot{\Psi}) \theta - (\dot{q} + \ddot{\theta}) \Psi \right] + \left[(q + 2\dot{\theta}) \theta \right. \right. \\
&\quad \left. \left. + (r + 2\dot{\Psi}) \Psi \right] p \right\} - \left[I_2 - \dot{m}_2 \left(X_{2cg}^2 + X_{2cg} (X_{2cg} - X_{2e}) + X_{2e}^2 \right) \right] \left[(q + \dot{\theta}) \Psi \right.
\end{aligned} \tag{119}$$

$$\begin{aligned}
& - (r + \dot{\Psi}) \theta \Big] + \dot{I}_{X_2} \left[q\dot{\Psi} - (r + \dot{\Psi})\dot{\theta} \right] - \dot{m}_2 X_{2cg} \left[(w + x_2 q) \dot{\Psi} + (v - x_2 r) \dot{\theta} \right] \\
& + X_{2cg} \left\{ \left[F_Z - m_1 (\dot{w} + pv - qu - g_Z) \right] \dot{\Psi} + \left[F_Y - m_1 (\dot{v} + ru - pw - g_Y) \right] \dot{\theta} \right\}
\end{aligned} \tag{119}$$

$$\begin{aligned}
M_Y = & I_1 \dot{q} + \left[I_2 - m_2 X_{2cg}^2 \right] \left\{ (\dot{q} + \ddot{\theta}) - 2\dot{\Psi}(p + q\dot{\Psi}) - \left[p^2 - r(r + \dot{\Psi}) \right] \dot{\theta} \right. \\
& - (\dot{p} + qr)\dot{\Psi} \Big\} + \left[\left(I_{X_1} + I_{X_2} \right) - (I_1 + I_2) + m_2 X_{2cg}^2 \right] pr + I_{X_2} \left\{ p\dot{\Psi} \right. \\
& + \left[(\dot{p} + qr) - \dot{\theta}(r + \dot{\Psi}) \right] \dot{\Psi} + \left[p^2 - r(r + \dot{\Psi}) \right] \dot{\theta} \Big\} - m_2 X_{2cg}^2 \dot{\Psi} (r + \dot{\Psi}) \dot{\theta} \\
& + \dot{I}_{X_2} p\dot{\Psi} + \left[\dot{I}_2 - \dot{m}_2 \left(X_{2cg}^2 + X_{2cg} (X_{2cg} - X_{2e}) + X_{2e}^2 \right) \right] \left[(q + \dot{\theta}) - p\dot{\Psi} \right] \\
& + \dot{m}_2 X_{2cg} (w + u\dot{\theta} - x_2 q) - (x_2 + X_{2cg}) \left[F_Z - m_1 (\dot{w} + pv - qu - g_Z) \right] \\
& - X_{2cg} \left[F_X - m_1 (\dot{u} + qw - rv - g_X) \right] \dot{\theta}
\end{aligned} \tag{120}$$

$$\begin{aligned}
M_Z = & I_{Z_1} \dot{r} + \left[I_2 - m_2 X_{2cg}^2 \right] \left\{ (\dot{r} + \ddot{\Psi}) + 2\dot{\theta} \left[(p + q\dot{\Psi}) - (r + \dot{\Psi}) \dot{\theta} \right] \right. \\
& + (\dot{p} - rq)\dot{\theta} - (p^2 - q^2)\dot{\Psi} \Big\} + \left[\left(I_1 + I_2 \right) - \left(I_{X_1} + I_{X_2} \right) - m_2 X_{2cg}^2 \right] pq \\
& + I_{X_2} \left\{ \left[p^2 - q(q + \dot{\theta}) \right] \dot{\Psi} - p\dot{\theta} + \left[(qr - \dot{p}) + 2\dot{\theta}(r + \dot{\Psi}) \right] \dot{\theta} \right\} \\
& - \dot{I}_{X_2} p\dot{\theta} + \left[\dot{I}_2 - \dot{m}_2 \left(X_{2cg}^2 + X_{2cg} (X_{2cg} - X_{2e}) + X_{2e}^2 \right) \right] \left[(r + \dot{\Psi}) + p\dot{\theta} \right] \\
& - \dot{m}_2 X_{2cg} (v - u\dot{\theta} + x_2 r) - X_{2cg} \left[F_X - m_1 (\dot{u} + qw - rv - g_X) \right] \dot{\Psi} \\
& + (x_2 + X_{2cg}) \left[F_Y - m_1 (\dot{v} + ru - pw - g_Y) \right]
\end{aligned} \tag{121}$$

$$M_Y - M_{Y_2} + (K\dot{\theta} + C\dot{\theta}) = I_1 \dot{q} + (I_{X_1} - I_1) pr \tag{122}$$

$$M_Z - M_{Z_2} + (K\dot{\Psi} + C\dot{\Psi}) = I_1 \dot{r} + (I_1 - I_{X_1}) pq \tag{123}$$

Equations (116) through (123) are useful for studying the effects of joint stiffness and damping on both the relative motion and overall motion of a spinning two-body system with or without the effects of mass variation being present.

Conclusion

A general vector differential equation set has been developed to describe the motion of a system of j ($j = 1, 2, \dots, n$) rotating-translating variable-mass variable configuration bodies relative to a rotating reference frame. These equations include: (1) a complete treatment of the effects of varying mass; (2) complete flexibility in the placement and motion of the reference coordinate frame within the system of bodies (nonrolling reference coordinates are one of many types that can be implemented); (3) complete flexibility in the placement of the body-fixed axes within each body of the system (mass asymmetry can be introduced), and (4), the provision for relative translation and rotation to exist between the reference coordinates and each body of the system as well as between the respective bodies. Through the use of simple restraints, these general equations are easily reduced to the governing equations for the motions of many dynamic systems of interest. Two examples are given to illustrate their flexibility and usefulness.

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