THEORY OF CREEP BENDING - A STATE VARIABLE APPROACH

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ABSTRACT

In this brief note direct constitutive relations between moment, curvature rate and "moment hardness" are derived from Hart's [5] state variable equations for creep of materials under uniaxial tension. This leads naturally to theories of creep of beams and plates. An approximate theory is deduced by assuming that hardness and thus moment hardness remains constant while a structure is undergoing creep deformations. Both formulations can be applied directly to problems of creep bending of rectangular beams and thin circular plates. The resulting analyses are much simpler than the complete transient analyses reported elsewhere [2,3].

INTRODUCTION

The relatively high design temperatures for nuclear reactor and aircraft components are such that creep behavior often occurs. Thus, this aspect of the inelastic behavior of structural metals is a significant consideration in the design and analysis of such components.

Interesting cases are the creep bending of beams and plates. A survey of past investigations in that area can be found in Rabotnov's book [1] and in the two reports by the present authors [2,3]. It must be emphasized that all classical analyses were based on steady-state, time-hardening and strain-hardening creep laws. These classical laws are, however, incapable of accounting for the effect of past history upon creep deformation [4]. To predict creep behavior more accurately, a state-variable theory due to Hart [5] has received much attention, and tensile experiments to determine material constants for a wide class of metals has been performed [6,7]. The theory was recently chosen for analysis of creep bending of beams by the present authors [2], and was then extended to the creep bending of circular plates [3]. All the results showed the strong influence upon creep deformation of the state-variable called "hardness".

In most engineering designs, the determination of creep deformation is the only target whereas the calculation of stress redistribution is secondary or even unnecessary. As pointed out by the present authors [2,3], the complete analysis of creep bending (including stress redistribution) in terms of stress, strain rate and hardness (the state-variable) is quite involved. Our objective here is to deduce direct constitutive relations between generalized parameters, namely, moment, curvature rate and related state-variables and consequently to simplify the analysis and computation.

By introducing a rate potential, Kachanov [8] formulated and then solved the steady-state creep bending of beams and plates in terms of generalized parameters with a variational principle. Its extension to transient creep analysis using a time-hardening law was also demonstrated [8]. Shesterikov [9] followed the same approach with a strain-hardening law. However, in applying the variational equation, the introduction of strain-hardening parameters raised serious difficulties and it was practically impossible to obtain effective results [1]. Using a mixed variational principal due to Sanders, McComb and Schlechte [10], and the assumption that stress distribution remained linear during the course of creep, Rabotnov [1] succeeded in formulating a creep-bending theory of beams and plates in terms of generalized parameters based on a strain-hardening law. The structure of the constitutive equation was exactly the same as that of the initial strain-hardening law. Due to the assumption of a constant linear stress distribution, an error which increased in magnitude with time was inevitable. To compensate for this drawback, Rabotnov examined the corresponding steady-state creep case and introduced a correction factor. However that remedy led to an accurate prediction of the long-time creep only, which was not the main concern in a transient creep analysis.

In this brief note, Hart's kinetic laws are modified and then transformed into constitutive relations between moment, curvature rate and a state variable called "moment hardness" (derived from "hardness", the original state-variable). No definite restriction is imposed upon the stress distribution, in contrast to Rabotnov's work [1], and hence both transient and stationary creep predictions are feasible. The approach is generalized to bending of plates in the framework of classical plasticity. As hardness usually changes by an insignificant amount [2,3], an approximate formulation

is deduced by assuming that moment hardness also remains constant.

Following the theoretical formulation, we construct algorithms to find solutions of problems of bending of rectangular beams and circular plates.

Our previous reports [2,3] showed that Hart's kinetic laws for uniaxial tension could be generalized to solve problems of transient creep bending of beams and plates. This study indicates that an alternate approach to analyze the same problems based on constitutive relations between generalized parameters results in much simpler computation. It might be appropriate to conduct experiments on beams to establish directly the generalized constitutive relations.

RELATIONS IN TERMS OF GENERALIZED PARAMETERS

The aim here is to formulate constitutive relations relating moment, curvature rate and related state-variables, and then to extend these equations to multiaxial case through a flow rule.

As creep is a time-dependent process, rate equations are constructed. First we assume that in the uniaxial case the total strain rate is the sum of an elastic strain rate $\dot{\epsilon}$ and a non-elastic strain rate $\dot{\epsilon}^n$,

$$\dot{\epsilon}^t = \dot{\epsilon} + \dot{\epsilon}^n \quad (\dot{}) = \partial()/\partial t \quad (1)$$

Two separate creep strain rate components have been experimentally identified [11], i.e.,

$$\dot{\epsilon}^n = \dot{\epsilon}^a + \dot{p} \quad (2)$$

where $\dot{\epsilon}^a$ is the recoverable creep strain rate and $\dot{p}$ is the irrecoverable creep strain rate. If the loading is steady or slowly varying ϵ^a will readily approach a saturation value ϵ^{as} and consequently its rate $\dot{\epsilon}^a \approx 0$. Henceforth

$$\dot{\epsilon}^t = \dot{\epsilon} + \dot{p} \quad (3)$$

The constitutive relations include Hooke's law and Hart's kinetic equations of creep in terms of a state variable σ^* , called "hardness" [5].

$$\dot{\epsilon} = \dot{\sigma}/E \quad (4)$$

$$\dot{p} = f(\sigma, \sigma^*, T) \quad (5)$$

$$\dot{\sigma}^* = g(\sigma, \sigma^*, T)$$

where f and g are experimentally determined functions, σ is the stress, $\dot{p}$, the irrecoverable strain rate, T , the temperature and E , Young's modulus. Equations (5) are sometimes referred to as evolution equations [4]. Under isothermal conditions, we assume that f and g are power functions of σ and σ^* , and hence

$$\begin{aligned}\dot{p} &= A \sigma^{*\ell} \sigma^n \\ \dot{\sigma}^* &= B \sigma^{*L} \sigma^N\end{aligned}\tag{6}$$

where A , B , ℓ , L , n and N are material constants. For simplicity, all subsequent derivations will be formulated in terms of dimensionless quantities. A specific choice which depends on the nature of the problem will be elaborated in the next section. The simplified Hart's equations are

$$\begin{aligned}p' &= \sigma^{*\ell} \sigma^n \\ \sigma^{*'} &= \lambda \sigma^{*L} \sigma^N\end{aligned}\tag{7}$$

Let the stress distribution be in the following form:

$$\sigma(t, y) = y^{\beta(t)}\tag{8}$$

where y is a one-dimensional spatial coordinate, and $\beta(t)$ is a function of time, t . It must be emphasized that equation (8) incorporates the effect of stress redistribution in contrast to Rabotnov's assumption of constant stress distribution. In fact $\beta(t)$ must be in a form such that

$$\beta(0) = 1, \text{ the elastic state,}\tag{9}$$

$$\text{and approximately } \beta(\infty) = k, \text{ the stationary state,}\tag{10}$$

where k is a constant governed by the initial state of the material.

Based on the results of the previous reports [2, 3], we assume that there is no spatial variation of hardness during the course of creep, i.e.,

$$\sigma^*(t, y) = \alpha(t) \quad (11)$$

To preserve dimensional uniformity of y in the second equation of (7) the following restriction is necessary

$$N\beta(t) = 0 \quad (12)$$

The generalized parameters, namely, "moment hardness" and moment are defined as follows:

$$\begin{aligned} M^* &= 2 \int_0^c \sigma^* y \, dy \\ M &= 2 \int_0^c \sigma y \, dy \end{aligned} \quad (13)$$

where c is one half of the thickness of the component. By substituting (8) and (11) in (13), we deduce

$$\begin{aligned} M^* &= c^2 \alpha(t) = \gamma_1(t) \\ M &= 2(c)^{(2+\beta(t))} / (2+\beta(t)) = \gamma_2(t) \end{aligned} \quad (14)$$

In accordance with the assumption that plane sections remain plane and from equation (3), the remaining generalized parameter, curvature rate κ' , is

$$\kappa' y = \sigma' + p' \quad (15)$$

We integrate (15) to deduce, according to (13),

$$\kappa' = \kappa'^e + \kappa'^p, \quad (16)$$

where $\kappa'^e = 3M'/(2c^3)$, the elastic curvature rate, and (17)

$$\kappa'^p = (3/c^3) \int_0^c p'y \, dy, \quad \text{the creep curvature rate.} \quad (18)$$

Substituting (7) together with (8) and (11) in (18), and using (14) we deduce

$$\kappa'^p = \delta_1(t) M^{*\ell} M^n \quad (19)$$

where $\delta_1(t) = 3(\gamma_2(t))^{-n}(c)^{(n\beta(t)-2\ell-1)}/(n\beta(t)+2)$. Note that $\delta_1(t)$ does not depend on $\alpha(t)$. Likewise the rate version of the first equation in (13) can be derived from the second equation of (13), (8), (11) and (14)

$$M^{*'} = \delta_2(t) M^{*L} M^N \quad (20)$$

where $\delta_2(t) = 2\lambda(c)^{(N\beta(t)+2-2L)}(\gamma_2(t))^{-N}/(N\beta(t)+2)$ and, like $\delta_1(t)$, $\delta_2(t)$ does not depend on $\alpha(t)$. It is interesting to compare (19) and (20) with the creep law used by Venkatraman and Hodge [12], namely

$$\kappa' = M^n \delta(t) \quad (21)$$

Obviously (21) is the limiting case of the present formulation with both $M^{*'}$ and κ'^e equal to zero.

The extension of (19) and (20) into the corresponding multiaxial case is carried out within the framework of classical plasticity. For simplicity, we examine the special case of a circular plate under axisymmetric loading.

The stress state is fully specified by the principal bending moments M_r and M_θ . With the assumption of a creep potential, a Mises type criterion can readily be derived [1] in the following form:

$$M_o = (M_r^2 + M_\theta^2 - M_r M_\theta)^{1/2} \quad (22)$$

$$\kappa_o^{p'} = (4/3)(\kappa_r^{p'2} + \kappa_\theta^{p'2} + \kappa_r^{p'} \kappa_\theta^{p'})^{1/2} \quad (23)$$

$$\kappa_r^{p'} = (\kappa_o^{p'} / M_o)(M_r - M_\theta/2) \quad (24)$$

$$\kappa_\theta^{p'} = (\kappa_o^{p'} / M_o)(M_\theta - M_r/2) \quad (25)$$

Finally equations (19) and (20) are generalized, namely,

$$\kappa_o^{p'} = \delta_1(t) M^{*L} M_o^N \quad (26)$$

$$M^{*I} = \delta_2(t) M^{*L} M_o^N \quad (27)$$

Equations (22) to (27) form the complete constitutive relations of creep-bending of circular plates.

If, instead of (11), we assume that hardness does not change with time [2,3], equations (20) and (27) become redundant and the corresponding analyses will be straightforward. This will be demonstrated in the next section.

RECTANGULAR BEAMS AND CIRCULAR PLATES

Following the formulation of constitutive relations between generalized parameters, we construct algorithms and deduce solutions to problems of bending of rectangular beams and circular plates.

A. Rectangular Beams

As reported by Marriot and Leckie [13] and also confirmed by the present authors [2], there exists a "skeletal point" in the creep-bending of a rectangular beam, where stress almost remains constant. That point thus becomes the natural choice of reference stress in the non-dimensionalization of equation (8), i.e.

$$\sigma_0 = 0.65 \sigma_{\max} \quad \text{and} \quad \sigma_{\max} = 3M(0)/2, \quad (28)$$

where $\sigma_{\max}$ is the maximum elastic stress and correspondingly in (13)

$$c = 1.54. \quad (29)$$

$$\text{At } t = 0, \quad M^*(0, y) = M^*(0),$$

$$\kappa(0) = 1, \quad \kappa^p(0) = 0,$$

$$w_{z=1/2}(0) = 1. \quad (30)$$

where $w_{z=1/2}$ is the deflection at the mid-point of the beam. For $t \geq 0$, use (16), (19) and (20),

$$\kappa'(t) = 3M'(t)/(2c^3) + \kappa^{p'}(t)$$

$$\kappa^{p'}(t) = \delta_1(t) M^* M^n$$

$$M^{*'}(t) = \delta_2(t) M^{*L} M^N \quad (31)$$

Forward integration in time is achieved with any standard routine. Finally the deflection is given by

$$w_{z=1/2}(t) = \kappa(t) \quad (32)$$

If we restrict ourselves to the case $M' = 0$ (i.e. constant moment) and neglect the change in moment hardness, then the normalized deflection rate is given by

$$\bar{w}'_{z=1/2} = M'^n \quad (33)$$

where $\bar{w}'_{z=1/2} = w'_{z=1/2}(t)/(\delta_1(t)M'^{*l})$. Note that equation (33) is independent of time.

B. Circular Plates

The discovery of the existence of a skeletal point in creep-bending of a rectangular beam [13] led to the concept of reference stress. Recently Sim [14] employed that technique to solve the problem of creep-bending of plates. Therefore equations (28) and (29) are equally applicable to circular plates.

The governing equation of circular plates under uniform lateral loading rate q' ,

$$\partial((1/r)(\partial(rv')/\partial r))/\partial r = rq'/2 - (\partial\Delta M'_r/\partial r + (\Delta M'_r - \Delta M'_\theta)/r) \quad (34)$$

where v' is the deflection slope rate, ν is Poisson's ratio

$$\Delta M'_r = \kappa_r p'_r + \nu \kappa_\theta p'_\theta$$

$$\text{and} \quad \Delta M'_\theta = \kappa_\theta p'_\theta + \nu \kappa_r p'_r \quad (35)$$

The solution to equation (34) is

$$v' = -Ar + B/r + q'r^3/16 - I'/r \quad (36)$$

$$\text{where } I' = \left(\int_0^r (\Delta M'_r - \Delta M'_\theta) d\xi / \xi \right) (r^2/2) + \left(\int_0^r (\Delta M'_r + \Delta M'_\theta) \xi d\xi \right) / 2 \quad (37)$$

and A and B are integration constants to be determined by boundary conditions of a specific problem. The deflection rate w' is given by

$$\partial w' / \partial r = v' \quad (38)$$

and another boundary condition. For the sake of definiteness, consider the simply supported case. The boundary conditions are,

$$\text{at } t = 0, \quad v(0,0) = M_r(0,1) = 0 \quad M_r = M_\theta, \quad r = 0$$

$$\text{for } t \geq 0, \quad v'(t,0) = M_r(t,1) = 0 \quad \text{and } t \geq 0$$

$$w(t,1) = 0. \quad (39)$$

The complete routine is as follows:

$$\text{at } t = 0, \quad M_r(0,r) = (1-r^2)/6$$

$$M_\theta(0,r) = (1-(1+3\nu)r^2/(3+\nu))/6$$

$$M^*(0,r) = M^*$$

$$v(0,r) = -r(1/(1+\nu)-r^2/(3+\nu))/6$$

$$\kappa_r^p(0,r) = \kappa_\theta^p(0,r) = 0. \quad (40)$$

$$\text{For } t \geq 0, \quad M_0(t, r) = (M_r^2 + M_\theta^2 - M_r M_\theta)^{1/2}$$

$$\kappa_0^{p'}(t, r) = \delta_1(t) M^{*L} M_0^n$$

$$M^{*'}(t, r) = \delta_2(t) M^{*L} M_0^N$$

$$\kappa_r^{p'}(t, r) = (\kappa_0^{p'}/M_0)(M_r - M_\theta/2)$$

$$\kappa_\theta^{p'}(t, r) = (\kappa_0^{p'}/M_0)(M_\theta - M_r/2)$$

$$\Delta M_r'(t, r) = \kappa_r^{p'} + v \kappa_\theta^{p'}$$

$$\Delta M_\theta'(t, r) = \kappa_\theta^{p'} + v \kappa_r^{p'} \quad (41)$$

The integrals deduced from (39) and (36) are defined,

$$I_1'(r) = \int_0^r (\Delta M_r' - \Delta M_\theta') d\xi / \xi$$

$$I_{11}' = I_1'(1)$$

$$I_2'(r) = I_1'(r) + \Delta M_r'$$

$$I_3'(r) = r^2 I_1'/2 + (\int_0^r (\Delta M_r' + \Delta M_\theta') \xi d\xi) / 2$$

$$I_{33}' = I_3'(1) \quad (42)$$

Then v' , M'_r and M'_θ are determined,

$$v'(t,r) = -q'r(1/(1+\nu) - r^2/(3+\nu))/6 + r((1-\nu)I'_{33}-I'_{11})/(1+\nu) - I'_3/r$$

$$M'_r(t,r) = q'(1-r^2)/6 + ((1-\nu)I'_{33}-I'_{11}-(1-\nu)I'_3/r^2+\nu I'_2)-\Delta M'_r$$

$$M'_\theta(t,r) = q'(1-(1+3\nu)r^2/(3+\nu)) + ((1-\nu)I'_{33}-I'_{11}+(1-\nu)I'_3/r^2+\nu I'_2)-\Delta M'_\theta \quad (43)$$

Once again forward integration in time is achieved with any standard routine. Finally deflections w are obtained by integrating v with the last boundary condition. Note that except for the new constitutive relations between generalized parameters and the associated flow rule, most of the expressions given in equations (39) to (43) are identical to that reported previously [3]. The same approach can be carried out for the clamped case.

If we restrict ourselves to the case $q' = 0$ (i.e. constant lateral load) and neglect the change in moment hardness, then the deflection slope rate v' can be normalized with respect to $\delta_1(t)M^{*\ell}$, i.e.

$$\bar{v}' = v' / (\delta_1(t)M^{*\ell}) \quad (44)$$

Like $\bar{w}'_{z=1/2}$ in (33), $\bar{v}'$ is also time-independent.

DISCUSSION

A complete transient creep analysis using Hart's kinetic equations in terms of a state-variable called "hardness", has recently been employed by the present authors [2] to solve the problem of creep-bending of beams. The same approach was successfully extended to a multiaxial stress state problem of bending of circular plates [3]. All the results compared qualitatively with those obtained from classical theories, and in addition they showed the strong influence of "hardness" upon creep deformation. Those investigations serve to establish a rational basis for incorporating the new model in analysis and design. The approach in its entirety is, however, quite formidable as noted in the previous reports.

The purpose of this paper is twofold: to formulate direct constitutive relations between generalized parameters, and then to investigate their feasibility as a tool to simplify analysis by constructing solutions to creep-bending of rectangular beams and circular plates.

Following the development in the previous sections, it was observed that (i) All the analyses incorporating the direct constitutive relations were much simpler than the corresponding ones reported previously. This is due to the fact that a complete determination of stress components (which depend on one more spatial dimension than moment components) was avoided.

(ii) The direct constitutive relations were derived without the assumption of constant linear stress in contrast to Rabotnov's approach, and hence were capable of predicting both transient and long-term creep behavior. In fact Rabotnov's formulation, namely,

$$\kappa^{p'} = (\kappa^p)^{-\alpha} (R)^{1+\alpha} M^n / (1+\alpha)$$

is a special case of (19) and (20).

- (iii) The reference stress technique [15], which has recently become one of the major research interests in creep design, was readily adopted for explicit solutions to problems of creep-bending.
- (iv) Further simplification in computation was achieved with the assumption of constant "moment hardness" as applied to the case of steady loading. The normalized creep deformation rates in equations (33) and (44) were time-independent
- (v) The constitutive equations defined by (19) and (20) were independent of the time variation of "hardness" given by (11).

In view of the analyses in the previous sections as well as the above observations, we feel that the aim of this study has been achieved.

An alternative approach to derive constitutive equations in the form of (19) and (20) is to establish relations between generalized parameters directly from experiments on bending of beams.

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