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CONDUCTION HEAT LOSS SCALING
IN OPEN FIELD LINE
GEOMETRIES

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ABSTRACT

A scaling law governing the cooling of a finite-length column due to electron thermal conduction loss at the column ends, where the magnetic field lines are intercepted by a cold material wall, is derived. In addition to the electron temperature, both the electron number density and the magnetic field strength are allowed to vary along the field lines. It is shown that the cooling of the column is not very sensitive to moderate spatial variations of number density and magnetic field strength and is well represented by a characteristic cooling time given by $\tau_0 = \frac{5}{2} n L^2 / K_0$, where L is the column half-length and n and K_0 are the number density and coefficient of thermal conductivity at the midplane.

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I. INTRODUCTION

Recent works¹⁻⁵ on end stoppering in linear plasma columns by solid and gaseous plugs to eliminate or reduce end loss of plasma energy and particles have generated a resurgence of interest in plasma confinement by linear magnetic field geometries for controlled thermonuclear reaction purposes. In particular, analyses^{2,4} have shown that when plasma flow is eliminated, electron thermal conduction along field lines emerges as the dominant loss in the determination of reactor criteria for a linear system.

In open field line geometries for plasma confinement, plasma cooling by thermal conduction along the field lines has received in the past relatively little attention when compared to the extensive literature⁶ on particle loss due to plasma flow out of the ends. In interpreting electron heating experiments in Q-machines, Levine et al⁷ obtained a steady-state temperature profile based on the heat-flow condition at the ends. In θ -pinches, effects of electron thermal conduction along field lines were analyzed by Green et al⁸ for interpreting the observed electron temperature limitations. They assumed separable spatial and time dependences of electron temperature. Although a solution satisfying the heat-flow boundary conditions was not given, the electron temperature scaling in θ -pinches was shown to depend on column length. Morse⁹ also constructed the simple heat-loss model which satisfies the heat flow condition at the end. This landmark work is remarkably useful in

predicting electron temperature scaling in θ -pinches when an empirical end-zone column length is used, in spite of the fact that the interior heat diffusion process was neglected. More recently Dreicer¹⁰ also obtained, independently, a separable solution to the electron thermal conduction equation; the spatial dependence of the electron temperature is given by a quadrature.

The present work analyzes the effect of electron thermal conduction along the magnetic field lines for a finite length column. To be of further practical interest, two parameters in addition to the electron temperature are allowed to vary along the field lines: the electron number density and the magnetic field strength; the latter in the form of a plasma tube of varying cross sectional area. The electron temperature so defined is therefore a temperature averaged over the cross-sectional area, and such a representation renders the important effect of magnetic field geometry on thermal conduction loss amenable to a simple analysis.

Our objective is primarily to arrive at a scaling law predicting the cooling of the electron temperature due to thermal conduction loss at the ends of a finite-length plasma column so that reactor requirements based on the scaling can be readily obtained. Detailed electron temperature profiles for cases of different electron number density and magnetic field geometries, therefore, will not be emphasized, although solutions for some of these cases will be presented for the purpose of illustrating the scaling law. Parts of the present work have been reported previously^{1,2,4,5} in abbreviated form.

II. THE SCALING LAW

The equation governing the variation of electron temperature T due to electron thermal conduction along a plasma tube of cross sectional area A is

$$\frac{3}{2} n \frac{\partial T}{\partial t} = \frac{1}{A} \frac{\partial}{\partial z} \left(AK \frac{\partial T}{\partial z} \right), \quad (1)$$

where n is the electron number density, K is the coefficient of thermal conductivity, and z is the coordinate along the field lines. At the ends of the plasma column where field lines run through the material wall, electrons contact the wall and assume the wall temperature $T=0$. In the interior of the plasma, where conductivity is high, spatial inhomogeneities in electron temperature will be quickly smoothed, and the temperature should assume a quasi-steady spatial profile. We therefore seek separable solutions of Eq. (1), subject to the boundary conditions for the time $t>0$

$$\frac{\partial T}{\partial z} = 0 \quad \text{at } z = 0, \quad (2a)$$

$$\text{and} \quad T = 0 \quad \text{at } z = L, \quad (2b)$$

where L is the column half-length.

For classical electron heat conduction, the coefficient of thermal conductivity is

$$K = \frac{1.94 \times 10^{21} T^{5/2}}{Z \ln \Lambda} \text{ sec}^{-1} \text{ cm}^{-1}, \quad (3)$$

where T is in ev. The effective ion charge and the Coulomb logarithm will be regarded as constant, so that Eq. (3) becomes

$$K = \kappa T^{5/2}, \quad (4)$$

and κ is evaluated for plasma conditions at $z = 0$. Substitution of Eq. (4) into Eq. (1) gives

$$\frac{3}{2} n \frac{\partial T}{\partial t} = \frac{\kappa}{A} \frac{\partial}{\partial z} (A T^{5/2} \frac{\partial T}{\partial z}) \quad (5)$$

Separable solutions of Eq. (5) may be written in the form

$$T(z, t) = T_t(t) \theta(z), \quad (6)$$

where T_t is the value of electron temperature at the midplane at time t , and θ is the spatial temperature profile, with the normalization

$$\theta(0) = 1. \quad (7)$$

For convenience, we shall use a normalized spatial coordinate

$$\xi = z/L, \quad (8)$$

and the derivative notations

$$\dot{T}_t = \frac{dT_t}{dt} \quad (9)$$

and

$$A' = \frac{dA}{d\xi} \quad (10)$$

Then if n and A are functions of position only, Eq. (5) becomes

$$\begin{aligned} T_t^{-7/2} \dot{T}_t &= \frac{2\kappa}{3nL^2} \left[(\theta^{3/2} \frac{A'}{A} + \frac{5}{2} \theta^{1/2} \theta') \theta' + \theta^{3/2} \theta'' \right] \\ &= \frac{2\kappa \theta''(0)}{3n(0)L^2}, \end{aligned} \quad (11)$$

where the separation constant $2\kappa \theta''(0)/[3n(0)L^2]$ follows by evaluating the spatial part of Eq. (11) at the midplane.

The temporal part of Eq. (11) may now be integrated directly. The electron temperature at the midplane is then given by

$$T_t = T_0 \left(1 + \frac{5}{2} \frac{t}{\tau_0} \right)^{-2/5} \quad (12)$$

where T_0 is the temperature at the time $t = 0$, when the plasma assumes a quasi-static profile. The initial cooling time τ_0 is

$$\tau_0 = \frac{3 n(0) L^2}{2K_0(-\theta''(0))} , \quad (13)$$

where K_0 is evaluated from Eq. (3) with $T = T_0$.

Equation (12) is the desired scaling law predicting the cooling of the electron temperature in a finite length column. The evaluation of the characteristic cooling time τ_0 is therefore the focal point of our remaining effort.

Equations (12) and (13) specify the electron temperature at the center of the plasma in terms of initial plasma parameters at the midplane. In order to complete the evaluation of the initial cooling time, we must examine the spatial part of Eq. (11).

With the substitution

$$\theta = u^{2/7} , \quad (14)$$

the spatial part of Eq. (11) becomes

$$\frac{n}{n(0)} = u^{-2/7} \frac{u'}{u''(0)} \left(\frac{u''}{u'} + \frac{A'}{A} \right) , \quad (15)$$

where

$$u''(0) = \frac{7}{2} \theta''(0) \quad (16)$$

follows from Eq. (14).

We shall not attempt an exact solution of Eq. (15) for specific n and A profiles. Rather we shall obtain an approximation for $u''(0)$ in terms of the simplest characterization of n and A , namely the curvatures $n''(0)$ and $A''(0)$ at the midplane.

The Taylor series expansion of u is

$$u = 1 + \frac{1}{2} u''(0) \xi^2 + \frac{1}{24} u'''(0) \xi^4 + \dots \quad (17)$$

By differentiating Eq. (15) twice and evaluating at the midplane, we obtain the relation

$$\frac{u'''(0)}{u''(0)} - \frac{2}{7} u''(0) = \frac{n''(0)}{n(0)} - 2 \frac{A''(0)}{A(0)} = \chi \quad (18)$$

If we ignore higher order terms in Eq. (17), then the boundary condition at $\xi = 1$ gives

$$1 + \frac{1}{2} u''(0) + \frac{1}{24} u'''(0) = 0 \quad (19)$$

Combining Eqs. (18) and (19) yields the desired result,

$$u''(0) = -21 \left(1 + \frac{\chi}{12}\right) \left[1 - \left(1 - \frac{4}{21 \left(1 + \frac{\chi}{12}\right)^2}\right)^{1/2}\right], \quad (20)$$

and

$$u'''(0) = -24 \left[1 + \frac{1}{2} u''(0)\right] \quad (21)$$

The accuracy of Eq. (20) may be estimated by comparing with results for particular choices of n and A which permit the solution of Eq. (15) to be reduced to quadratures. For example, if we choose

$$A = \text{const.} , \quad (22)$$

$$\frac{n}{n(0)} = \theta^\gamma , \quad (23)$$

then the solution of Eq. (15) gives

$$u''_\gamma(0) = -\frac{7}{2} \frac{\pi}{(9+2\gamma)} \left[\frac{\Gamma(\frac{7}{9+2\gamma})}{\Gamma(\frac{7}{9+2\gamma} + \frac{1}{2})} \right]^2 , \quad (24)$$

where Γ is the Gamma function. Differentiation of Eq. (23) then yields

$$\chi_\gamma = \frac{2}{7} \gamma u''_\gamma(0) . \quad (25)$$

From a comparison of values obtained from Eq. (24) with those given by Eq. (20), using values of χ from Eq. (25), it appears that Eq. (20) will yield results accurate to within a few percent for

$$|\chi| \gtrsim 2 . \quad (26)$$

In particular, the results are exact for the important special case of pressure balance ($nT = \text{const.}$) in straight geometry.

In Fig. 1 the characteristic cooling time τ_0 , normalized by $n(0)L^2/K_0$, is plotted as a function of the midplane curvature parameter χ , in which the case of constant area and constant density corresponds to $\chi=0$. ($\tau_0/[n(0)L^2/K_0]$ for $\chi=0$ evaluated from Eq. (24) is 2.463.) From Fig. 1 we observe that converging area and increasing density toward the end of the tube ($\chi>0$) increases the cooling time while diverging tube area and decreasing density ($\chi<0$) shortens the cooling time. In Fig. 2, temperature profiles for several cases of χ are plotted. Fuller temperature profiles correspond to longer cooling times.

III. DISCUSSION

The separable solution given in Section II will apply after any initial spatial inhomogeneities in electron temperature have been smoothed. In order to investigate the early transient conditions immediately after a hot plasma column contacts a cold boundary, we construct a solution to the thermal conduction problem for a semi-infinite column, but with a similarity variable to transform Eq. (1) into an ordinary differential equation.¹

We consider the simple case of constant density n and constant area. The similarity variable is

$$\eta = \frac{z}{\left(\frac{2K_0 t}{3n}\right)^{1/2}} \quad (27)$$

Equation (1) is then transformed to

$$\frac{\eta}{2} \frac{dq}{d\eta} + \frac{d}{d\eta} (q^{5/2} \frac{dq}{d\eta}) = 0 \quad (28)$$

where $q \equiv T/T_\infty$ and T_∞ is the unperturbed temperature at infinity.

The boundary conditions are

$$q = 0 \quad \text{at} \quad \eta = 0, \quad (29a)$$

$$\text{and} \quad q = 1 \quad \text{at} \quad \eta \rightarrow \infty, \quad (29b)$$

where we have chosen $z = 0$ corresponding to the wall location.

The similarity variable is defined only for time $t > 0$.

The numerical solution of Eq. (1), satisfying the boundary conditions Eqs. (29a) and (29b) is shown in Fig. 3. Also shown in Fig. 3 is the parameter S corresponding to the heat flux,

$$S \equiv \frac{dq^{7/2}}{d\eta} \quad (30)$$

The temperature profiles in z , according to the similarity solution should only begin to become nearly indistinguishable from that based on the separable solution at a certain time following $t = 0$. The equivalent heat flux parameter S shows that, indeed, the heat flux becomes infinitesimal at large η .

To bridge the time gap between the similarity solution, which governs the initial period immediately after $t = 0$, and the separable solution, which becomes valid after a certain elapsed time, we extrapolate the linear portion of S to intercept the η -axis at η_c ,

$$\eta_c = \frac{z_c}{\left(\frac{2K_0 t_c}{3n}\right)^{1/2}} = 2.35 \quad (31)$$

The time t_c which separates the two solutions is therefore,

$$t_c = \frac{3nL^2}{2K_0 \eta_c^2} \quad (32)$$

Temperature profiles for three times, $t/t_c = 0.01, 0.1$, and 1.0 are plotted and compared with that of the separable solution in Fig. 4. For the time period $t \ll t_c$ the temperature profiles from the similarity solution are significantly different from that of the separable solution.

Although the particular method used to choose η_c is somewhat arbitrary, the value of z_c defined by Eq. (31) can be regarded as a characteristic penetration distance of a cooling layer toward the interior in the time period immediately after $t = 0$. We note that the form of t_c Eq. (32), is identical to that of the cooling time τ_0 of Eq. (13) with the midplane curvature parameter $-\theta''(0)$ replaced by η_c^2 . Furthermore, since

$$\frac{t_c}{\tau_0} = - \frac{\theta''(0)}{\eta_c^2} \approx 0.11, \quad (33)$$

for the practical purpose of calculating the electron temperature cooling, Eq. (12) can be used for all $t > 0$.

A second point related to the immediate cooling of the column interior is the validity regime in time of the differential equation, Eq. (1). In this equation, activities of free particles are not included, and they are important in the time period immediately following the imposing of the cold boundary. A full account of the effect of these free particles can be obtained from a kinetic description. For the present purpose, it is sufficient to note that the solution according to Eq. (12) becomes valid after a time of the order of $1/\nu$, where ν is the collision frequency.

Finally, we note that the boundary condition Eq. (2b), implies that the wall can be brought to zero temperature instantaneously. This assumption is justified in the case that the wall is an ideal heat sink. In reality, the wall may have only finite heat capacity and finite thermal conductivity when compared to those of the plasma column as might be the case if high mass density neutral gas is used for end stoppering.⁵ For those cases, the transient problem of the wall temperature beginning from the initial temperature must be solved in terms of the thermal properties of the plug material. Discussions concerning these transient solutions will be reported elsewhere.¹¹

IV. SUMMARY AND CONCLUSION

A scaling law governing the cooling of the electron temperature in a finite plasma column due to electron thermal conduction along the magnetic field lines has been obtained. In addition to the electron temperature, both the electron number density and the magnetic field strength are allowed to vary along the field lines. The effect of these variations on the characteristic cooling time τ_0 in the scaling law is represented by a single parameter χ which characterizes the curvatures of the density profiles and the magnetic field lines at the midplane. For the case of constant magnetic field strength and constant density along the field lines, the normalized τ_0 is

$$\frac{\tau_0}{\frac{nL^2}{K_0}} \approx 2.5$$

For variable density and field strength in a range of practical interest, the normalized characteristic cooling time varies between 2 to 3; increasing density and converging field lines toward the column end result in longer cooling time. It is further shown that the time required for a cooling layer to penetrate the column and reach the midplane is relatively short when compared to the cooling time. Therefore, the derived scaling law, Eq. (12), is applicable for all times $t > 0$.

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REFERENCES

- ¹T. K. Chu and L. C. Johnson, Bull. Am. Phys. Soc. 21, 832 (1976).
- ²T. K. Chu and L. C. Johnson, Fourth International Symposium of Plasma Heating in Toroidal Devices, Varenna, Italy (September 1976).
- ³R. J. Comisso, C. A. Ekdahl, K. F. McKenna, W. E. Quinn, Bull. Am. Phys. Soc. 21, 1035 (1976).
- ⁴T. K. Chu and L. C. Johnson, Bull. Am. Phys. Soc. 21, 1035 (1976).
- ⁵L. C. Johnson and T. K. Chu, Bull. Am. Phys. Soc. 21, 1036 (1976).
- ⁶A review of θ -pinch end loss has been recently given by J. P. Friedberg and H. Weitzner, Nuclear Fusion 15, 217 (1975).
- ⁷A. M. Levine, A. F. Kuckes and R. W. Motley, J. of Appl. Phys. 38, 4435 (1967).
- ⁸T. s. Green, D. L. Fisher, A. H. Gabriel, F. J. Morgan and A. A. Newton, Phys. Fluids 10, 1663 (1967).
- ⁹R. L. Morse, Phys. Fluids 16, 545 (1973).
- ¹⁰H. Dreicer, Bull. Am. Phys. Soc. 21, 107 (1976); Fourth International Symposium of Plasma Heating in Toroidal Devices, Varenna, Italy, (September, 1976)
- ¹¹L. C. Johnson and T. K. Chu (to be published).

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Fig. 1. Characteristic cooling time τ_0 , normalized by $\bar{n}(0)L^2K_0$, vs. the midplane density and area curvature parameter χ . Constant cross sectional area and constant density correspond to $\chi=0$. Increasing density and converging tube area toward the column end correspond to $\chi>0$; decreasing density and diverging tube area toward the column end correspond to $\chi<0$.

Fig. 2. Temperature profiles for three values of the midplane curvature parameter χ .

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Fig. 3. Temperature and equivalent heat flux vs. the similarity variable η , in a semi-infinite plasma column of constant density and area. η_c is obtained by extrapolating the linear portion of the equivalent heat flux parameter S to the η -axis. For a given time, η_c determines a column length z_c which characterizes the penetration distance of a cooling layer originating from the wall.

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Fig. 4. Temperature profiles based on the similarity solution for three values of t/t_c : 0.01, 0.1, and 1.0. The solid curve is the temperature profile from the separable solution. All profiles are for constant density and tube area.