Conf-760922--1

LA-UR-76-1549

TITLE: HYPERSCALING IN THE ISING MODEL

AUTHOR(S): GEORGE A. BAKER, JR.

SUBMITTED TO:

To be presented at the International Conference
on Magnetism, September 6-10, 1976, Amsterdam,
The Netherlands, and published with the
Conference Proceedings in Physica B

By acceptance of this article for publication, the
publisher recognizes the Government's (license) rights
in any copyright and the Government and its authorized
representatives have unrestricted right to reproduce in
whole or in part said article under any copyright
secured by the publisher.

The Los Alamos Scientific Laboratory requests that the
publisher identify this article as work performed under
the auspices of the USERDA.

los alamos
scientific laboratory
of the University of California
LOS ALAMOS, NEW MEXICO 87544

An Affirmative Action/Equal Opportunity Employer

NOTICE

This report was prepared as an account of work
sponsored by the United States Government. Neither
the United States nor the United States Energy
Research and Development Administration, nor any of
their employees, nor any of their contractors,
subcontractors, or their employees, makes any
warranty, express or implied, or assumes any legal
liability or responsibility for the accuracy, completeness
or usefulness of any information, apparatus, product or
process disclosed, or represents that its use would not
infringe privately owned rights.

MASTER

HYPERSCALING IN THE ISING MODEL*

George A. Baker, Jr.**

Theoretical Division, University of California
Los Alamos Scientific Laboratory, Los Alamos, New Mexico, 87545

The high temperature series expansions relevant to the hyperscaling relation $\Delta = \frac{1}{2}(d\nu + \gamma)$ are extended and re-examined. Hyperscaling is found to hold in 2 dimensions as expected, but fails in 3 and 4 dimensions. The triviality of hyperstrong-coupling, Euclidean, Boson, ϕ^4 field theory follows.

*Work performed under the auspices of the Energy Research and Development Administration.

**On leave from Brookhaven National Laboratory.

The modern renormalization group theory of critical phenomena¹, as currently practiced, implicitly assumes that the hyperscaling relationship holds between the critical indices. Specifically, we consider the usual spin- $\frac{1}{2}$, Ising model defined by the partition function

$$Z = \sum_{\{\sigma_i = \pm 1\}} \exp \left[K \sum_{\substack{\text{nearest} \\ \text{neighbors}}} \sigma_i \sigma_j + H \sum_i \sigma_i \right] \quad (1)$$

where the sum in the exponential is over all the nearest neighbor pairs (i, j) of a regular space-lattice in d -dimensions of N sites. In space dimensions d higher than 1, this model has a critical temperature, T_c at which the correlation length ξ diverges to infinity. Letting $\tau = (1 - T_c/T)$ we have, just above the critical temperature, the behavior

$$\chi = N^{-1} \frac{\partial^2 \ln Z}{\partial H^2} \approx A_+ \tau^{-\gamma}, \quad \frac{\partial^2 \chi}{\partial H^2} \approx -B_+ \tau^{-\gamma-2\Delta}, \quad (2)$$

$$\xi^2 = \sum_{\mathbf{r}} r^2 \langle \sigma_0 \sigma_{\mathbf{r}} \rangle / (2d\chi) \approx D_+^2 \tau^{-2\nu}$$

The hyperscaling relation is then

$$\Delta = \frac{1}{2}(d\nu + \gamma) \quad (3)$$

This relation has been proven for $d = 2$ by Kadanoff² and in general it has been shown that $\Delta \leq \frac{1}{2}(d\nu + \gamma)$ by Baker, Krinsky, and Schrader³. The best previously available high temperature series estimates^{4,5} yield $\frac{1}{2}(d\nu + \gamma) - \Delta \approx 0.019 \pm 0.006$ in 3 dimensions which is suggestive of a failure of hyperscaling. The consequences of such a possible failure are significant. For example, the renormalization group theory of critical phenomena would need to be substantially revised, and the quantum field theory $\lambda: \phi^4:_{\mathbf{d}}$ would be trivial (i.e., free of real scattering).

We can check on this possibility, and incidently compute a number of interest to field theory by considering the dimensionless, renormalized coupling constant

$$w = g m^{d-4} \propto -v \frac{\partial^2 \chi}{\partial H^2} / \chi^2 \xi^d \approx [v B_+ / A_+^2 (a D_+)^d] \tau^{d\nu + \gamma - 2\Delta} \quad (4)$$

which is finite, if hyperscaling holds, and vanishes if it does not. The quantity, v , is the specific hyper-volume per site on the lattice, and a , the lattice spacing.

With one exception, the series necessary to analyze (4) are available in the literature. (We will treat the simple quadratic, triangular, simple cubic, body-centered cubic, face-centered cubic and hyper-simple-cubic lattices.) The exception is $\frac{\partial^2 \chi}{\partial H^2}$ for

$d > 3$. We have computed this series on the hyper-simple-cubic lattice by the method of Rushbrooke and Scoins⁷ through order $(\tanh K)^9$, where $u = \tanh K$.

The analysis of these series will next be discussed. The general methods employed are of those of Padé approximation⁸. In particular, heavy use is made of two ideas. The first is "critical point renormalization"^{5,8}. If we have

$$\begin{aligned} f(x) &= \sum f_j x^j \approx A(1 - yx)^{-\psi} + B \\ g(x) &= \sum g_j x^j \approx C(1 - yx)^{-\phi} + D \end{aligned} \quad (5)$$

for the same value of y then

$$h(x) = \sum f_j x^j / g_j \approx (1 - x)^{-(\psi - \phi + 1)} \quad (6)$$

for x near 1. The second idea is that one can obtain an estimate of the difference $(\chi - \phi)$ in (6) by forming Padé approximations to

$$(1 - x)(d/dx) \log h(x) \quad (7)$$

and evaluating them at $x = 1$.

One of the problems which can arise in the analysis of series such as these is the occurrence of confluent singularities. By this remark is meant that near the singular point

$$k(x) \approx (1 - yx)^{-\phi} \left(1 + A(1-yx)^{\tau} + o[(1 - yx)^{\tau}] \right) \quad (8)$$

where $\tau < 1$. For the 3-dimensional, spin- $\frac{1}{2}$ Ising model case we are treating, Camp, et al⁹ have found, both for the susceptibility χ , and the second moment series used to define ξ in eq. (2) that such confluent singularities are absent. (These singularities do occur, however, for $s > 1/2$.) We have examined the structure of $\frac{\partial^2 \chi}{\partial H^2}$. A Padé analysis of $d \log \frac{\partial^2 \chi}{\partial H^2} / dK$ reveals a clearly-defined, isolated pole corresponding to the critical point. The confluent singularity analysis of Baker and Hunter¹⁰ when applied here is strongly interfered with by the presence of non-confluent singularities. Only the close-packed lattices, triangular and F.C.C. permit a conclusion by this method. The appearance is that confluent singularities do not appear there either.

In order to reduce the number of separately estimated quantities in (4) from three to two we have analysed the dimensionless quantity (also independent of spin-length scale) $\left[\frac{\partial^2 \chi}{\partial H^2} / \chi^2 \right]$. This quantity leads to stable Padé estimates. By the procedures of eq. (5)-(7) we conclude that $2\Delta - \gamma$ has the values

$$\begin{aligned} \text{S.Q.}, & 1.998 \pm 0.002; \quad \text{T.}, 2.00 \pm 0.01; \quad \text{S.C.}, 1.885 \pm 0.006; \\ \text{B.C.C.}, & 1.886 \pm 0.003; \quad \text{F.C.C.}, 1.8868 \pm 0.003; \\ \text{H.S.C.}, & 1.918 \pm 0.006 \end{aligned} \quad (9)$$

It is to be noted that the results for the 2-dimensional lattices are consistent with the exact value of 2.

We have reanalysed carefully the series for ξ^2 . By the methods of eq. (5)-(7) we conclude for ν

$$\begin{aligned} \text{S.Q.}, & 0.97 \pm 0.1; \quad \text{T.}, 1.0 \pm 0.05; \quad \text{S.C.}, 0.642 \pm 0.002; \\ \text{B.C.C.}, & 0.6384 \pm 0.0007; \quad \text{F.C.C.}, 0.6384 \pm 0.0006; \quad (10) \\ \text{H.S.C.}, & 0.555 \pm 0.008 \end{aligned}$$

We remark that $\nu = 1$ is the exact answer for $d = 2$. The results of (9) and (10) agree with hyperscaling for $d = 2$. For $d = 3$, the F.C.C. results are $2\Delta - \gamma - d\nu = -0.028 \pm 0.002$ which is a sufficiently strong result to reject the renormalization group hypothesis of zero, in my opinion, and implies that w of (4) is zero. One concludes that the present renormalization group is unfaithful and does not include the Ising fixed point.

For the case of four dimensions we compute $2\Delta - \gamma - d\nu = -0.302 \pm 0.014$. We conclude that w of (4) is zero in four dimensions. Renormalization group theory¹ has predicted that w vanishes like $1/(\log \tau)$ instead of $\tau^{0.3}$, but we have not been successful in attempting to verify this hypothesis. We would have expected $\left[\frac{\partial^2 \chi}{\partial H^2} / \chi \right]$ to be free of log terms, but it has a more erratic Padé analysis than $\left[\frac{\partial^2 \chi}{\partial H^2} / \chi^2 \right]$. Also our attempts to treat $\chi \xi^4$ as though it was proportional to $\log \tau$ were not successful either.

Finally, we present in Table 1 the amplitude factors of eq. (4). The values of (B_+/A_+^2) were estimated by the method of Essam and Hunter⁴ and D_+ by the method of Fisher and Burford.⁶ These amplitudes assume (except for $d = 4$) that hyperscaling holds

Table 1 - Amplitude Factors

Lattice	(B_+/A_+^2)	D_+	w
S.Q.	4.72	.5672	14.68
T.	4.683	.526	14.66
S.C.	3.173	.505	24.64
B.C.C.	3.231	.469	24.11
F.C.C.	3.266	.457	24.20
H.S.C.	1.221	.400	0

and compute v from $2\Delta - \gamma$. The relation to the Callan-Symanzik equation approach of Baker, et al¹¹ is given by how v^* compares to $9w/(48\pi)$, or 1.42 compares to $1.44 \pm .02$. The values of D_+ in 3 dimensions are biased high by about 5 percent. If the best estimate for v is used the values are closer to those of Fisher and Burford.⁶

The author is pleased to acknowledge helpful discussions with B. Nickel, and R. Schrader as well as assistance by D. Meiron with some of the computations reported herein.

References

1. K. G. Wilson and J. B. Kogut, Phys. Rep. 12C, 75 (1974);
E. Brézin, J. C. LeGuillou and J. Zinn-Justin, Phase Transitions and Critical Phenomena, C. Domb and M.S. Green, eds., Vol. 6, (Academic Press, New York, to be published).
2. L. P. Kadanoff, Phys. Rev. 188, 859-863 (1969). This proof depends on a hypothesis shown by J. Stephenson, J. Math. Phys. 5, 1009 (1964), as explained by B. McCoy and T. T. Wu, The Two-Dimensional Ising Model, pp. 186-199, (Harvard Univ. Press, Cambridge, 1973). I am grateful to C. Tracy for this reference.
3. G. A. Baker, Jr., and S. Krinsky, J. Math. Phys. (to be published);
R. Schrader, preprint.
4. J. W. Essam and D. L. Hunter, J. Phys. C, 1, 392 (1968) and private communication (1976).
5. D. S. Gaunt and G. A. Baker, Jr., Phys. Rev. B, 1, 1184 (1970);
M. A. Moore, D. Jasnow, and M. Wortis, Phys. Rev. Letters, 22, 940 (1969).
6. M. E. Fisher and D. S. Gaunt, Phys. Rev. 133, A224 (1964); M. E. Fisher and R. J. Burford, Phys. Rev. 156, 583 (1967); M. A. Moore, Phys. Rev. B, 1, 2238 (1970); G. A. Baker, Jr., Phys. Rev. B, 9, 4908 (1974); C. Domb, Phase Transitions and Critical Phenomena, C. Domb and M. S. Green, eds. Vol. 3, pg. 357 et seq. (Academic Press, New York, 1974).
7. G. S. Rushbrooke and H. I. Scoins, J. Math. Phys. 3, 176 (1962).
8. See, for example, G. A. Baker, Jr., Essentials of Padé Approximants (Academic Press, New York, 1975).

9. W. J. Camp and J. P. Van Dyke, Phys. Rev. B 11, 2579 (1975);
W. J. Camp, D. M. Saul, J. P. Van Dyke and M. Wortis, preprint.
10. G. A. Baker, Jr. and D. L. Hunter, Phys. Rev. B 7, 3377 (1973).
11. G. A. Baker, Jr., B. G. Nickel, M. S. Green, and D. I. Meiron,
Phys. Rev. Letters 36, 1351 (1976).