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A GENERALIZED CHISQUARE KINEMATIC
FITTING PROCEDURE APPLICABLE TO MULTIWIRE
PROPORTIONAL CHAMBERS

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ABSTRACT

A chisquare fitting procedure is presented which allows a hypothesis to be fit directly to measured parameters. The hypothesis is exhibited through a series of constraint equations which may contain variables with no measured counterpart. Thus this method is a generalization of previous procedures.

1. INTRODUCTION

This report presents a general method of analyzing data by performing a chisquare (χ^2) fit to the data using kinematic parameters. This method is more applicable to counter experiments than older versions.¹ The data are analyzed through a series of constraint equations. Thus, for example, if one were measuring πp elastic scattering, one would "fit" a set of kinematic variables which obey the πp kinematics to a set of measured quantities. The degree of agreement between the measured and fitted sets of variables is given by a number "chisquare" defined in the next section. One needs only a single parameter, χ^2 , to determine whether or not an event is acceptable, where hopefully the maximum acceptable chisquare is determined unambiguously.

The previous methods suffer by the requirement that a set of variables appearing in all the constraint equations must have "measured" counterparts.^{2,3} For bubble chamber experiments, this is a perfectly adequate method, but for counter or chamber experiments, where one measures points in

space rather than kinematic variables, certain adjustments must be made. One frequently derives kinematic variables from the measured points (a sometimes difficult procedure), then does the kinematic fitting. The assumption is also generally made that the derived "measured" values are not correlated, i.e., the error matrix is diagonal (see the next section). This approximation is not accurate for many experiments.

In this report a generalization of the previous fitting procedure is introduced which allows a hypothesis to be fit directly to measured parameters (points in space, etc.) These direct measurements are the least likely to have correlations so that it is more appropriate to use a diagonal error matrix. Also, the necessity of doing intermediate calculations is eliminated. So if one were measuring elastic πp scattering with a two-arm spectrometer, one could fit kinematic variables obeying the πp kinematics directly to the measured points in space. The fit is performed using the constraint equations defining the relationship between the measured set of values and the kinematic variables, the constraint equations defining the kinematics,

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and a constraint equation indicating a point vertex for the interaction. The chi-square determined from the fit is then the single parameter needed to tell whether an event is acceptable.

We have used this method while taking data on line using a Digital PDP 11/20⁺ with good results. Below is a description of the technique.

11. METHOD

We wish to minimize "chi-square" defined as

$$\chi^2 = \sum_{ab} (X_a - X_a^M) G_{ab} (X_b - X_b^M),$$

where

X_a^M = a th measured variable (M superscript denotes "measured"),

X_a = a th fitted variable corresponding to X_a^M ,

and

G_{ab} = symmetric error matrix.

The idea is to use measured quantities such as position in a multiwire proportional chamber which, in the absence of multiple scattering, are independent noncorrelated measurements. In this case G_{ab} is diagonal.

The X variables are connected through various constraint equations or straight line equations, which may contain other variables which are defined as

Y , a set of variables with no measured counterpart which appear at least once in a constraint equation with an X variable, and

Z , a set of variables with no measured counterpart which never appear in the same constraint equation as an X variable.

The Y variables could be slopes and intercepts of rays which are fitted to measured points in space (the X variables). The Z variables allow for further transformations.

Two sets of constraint equations are defined.

$F_k = F_k(X, Y) = k$ th constraint equation which is a function of X and Y only with at least one X variable in each equation.

$S_k = S_k(Y, Z) = k$ th constraint equation which is a function of Y and Z only.

These constraint equations are equal to zero when satisfied. They may, for instance, represent momentum and energy conservation.

The following conventions are adopted for the calculations given below.

a, b, c denote X variable subscripts.
 j, k, l denote Y variable subscripts.
 r, s, t denote Z variable subscripts.
 u, v, w label the F constraint equations.
 i, j, k label the S constraint equations.
 α_k = Lagrange multiplier for F_k .
 β_k = Lagrange multiplier for S_k , and
 $n, n-1$ superscripts denote the n th and $(n-1)$ th iteration.

A comma (,) denotes differentiation ($\partial S_k / \partial X_{i,j,n-1}$).

A (-1) superscript denotes matrix inversion (square matrices).

By adopting the method of Lagrange multipliers, we may treat all variables as being independent and introduce a "generalized" chi-square function χ as

$$\chi = \sum_{ab} (X_a - X_a^M) G_{ab} (X_b - X_b^M) + 2 \sum_k \alpha_k F_k + \sum_k \beta_k S_k. \quad (1)$$

The factor of two placed in front of the constraint equation is for convenience. In this calculation the error matrix is not assumed to be diagonal, but in using this procedure on our data, the equations were specialized for a diagonal error matrix. The problem is solved when χ is minimized. A relative minimum is found by satisfying the following equations.

$$1/2 \partial \chi / \partial X_a = \sum_b G_{ab} (X_b - X_b^M) + \sum_j \alpha_j F_{j,a} = 0. \quad (2)$$

$$1/2 \partial \chi / \partial Y_k = \sum_j \alpha_j F_{j,k} + \sum_l \beta_l S_{l,k} = 0. \quad (3)$$

$$1/2 \partial \chi / \partial Z_r = \sum_l \beta_l S_{l,r} = 0. \quad (4)$$

$$1/2 \partial \chi / \partial \alpha_k = F_k = 0, \text{ and} \quad (5)$$

$$1/2 \partial \chi / \partial \beta_k = S_k = 0. \quad (6)$$

Since these equations are generally difficult or impossible to solve exactly, an iterative procedure is used. Given x^n, y^n, z^n, a^n, b^n after the n^{th} iteration, determine $x^{n+1}, y^{n+1}, z^{n+1}, a^{n+1}, b^{n+1}$ for the $(n+1)^{\text{th}}$ iteration. This is done by expanding the F_{λ} and S_{δ} constraint equations to first order in a Taylor series, giving the following set of equations.

$$\sum_b G_{ab} (x_b^{n+1} - x_b^n) + \sum_{\lambda} a_{\lambda}^{n+1} F_{\lambda,a}^n = 0. \quad (2a)$$

$$\sum_{\lambda} a_{\lambda}^{n+1} F_{\lambda,k}^n + \sum_{\delta} \beta_{\delta}^{n+1} S_{\delta,k}^n = 0. \quad (3a)$$

$$\sum_{\delta} \beta_{\delta}^{n+1} S_{\delta,r}^n = 0. \quad (4a)$$

$$F_{\lambda}^n = \sum_a F_{\lambda,a}^n (x_a^{n+1} - x_a^n) + \sum_k F_{\lambda,k}^n (y_k^{n+1} - y_k^n) = 0, \text{ and} \quad (5a)$$

$$S_{\delta}^n = \sum_k S_{\delta,k}^n (y_k^{n+1} - y_k^n) + \sum_r S_{\delta,r}^n (z_r^{n+1} - z_r^n) = 0. \quad (6a)$$

Note that

$$F_{\lambda,k}^n = F_{\lambda,k} (x^n, y^n)$$

and

$$S_{\delta,r}^n = S_{\delta,r} (y^n, z^n).$$

From Eq. (2a) we obtain

$$x_b^{n+1} = x_b^n + \sum_{\lambda} a_{\lambda}^{n+1} G_{ba}^{-1} F_{\lambda,a}^n. \quad (7)$$

Substituting Eq. (7) into Eq. (5a) gives

$$B_{\lambda} = \sum_b H_{\lambda b} a_{\lambda}^{n+1} + \sum_k F_{\lambda,k}^n (y_k^{n+1} - y_k^n) = 0, \quad (8)$$

where

$$B_{\lambda} = F_{\lambda}^n + \sum_a F_{\lambda,a}^n (x_a^n - x_a^n) \quad (9)$$

$$H_{\lambda b} = \sum_a F_{\lambda,a}^n G_{ab}^{-1} F_{b,b}. \quad (10)$$

Solve for a_{λ}^{n+1} from Eq. (8).

$$a_{\lambda}^{n+1} = \sum_{\lambda} H_{\lambda\lambda}^{-1} [B_{\lambda} + \sum_k F_{\lambda,k}^n (y_k^{n+1} - y_k^n)]. \quad (11)$$

If there are no Y variables, the problem is solved since

$$a_{\lambda}^{n+1} = \sum_{\lambda} H_{\lambda\lambda}^{-1} B_{\lambda}.$$

This is the solution used by previous fitting procedures.

Continuing by substituting Eq. (11) into Eq. (3a), defining

$$C_{k\lambda} = \sum_{\lambda} F_{\lambda,k}^n H_{\lambda\lambda}^{-1} \quad (12)$$

and

$$D_{k\delta} = \sum_{\lambda} C_{k\lambda} F_{\lambda,k}^n, \quad (13)$$

and solving for y_j^{n+1} we obtain

$$y_j^{n+1} = y_j^n + \sum_{\lambda} D_{jk}^{-1} C_{k\lambda} B_{\lambda} + \sum_{\delta} \beta_{\delta}^{n+1} D_{jk}^{-1} S_{\delta,k}^n. \quad (14)$$

(If there are no S -type constraint equations, the problem is solved.) Defining

$$N_j = - \sum_{\lambda} D_{jk}^{-1} C_{k\lambda} B_{\lambda}, \quad (15)$$

$$R_{j\delta} = \sum_{\lambda} D_{jk}^{-1} S_{\delta,k}^n, \quad (16)$$

$$U_{\delta\sigma} = \sum_{\delta} S_{\delta\delta} R_{\delta\sigma}, \text{ and} \quad (17)$$

$$T_{\delta} = S_{\delta}^n + \sum_{\delta} S_{\delta,\delta} N_{\delta}. \quad (18)$$

substituting Eq. (14) into Eq. (6a), and solving for β_{σ}^{n+1} gives

$$\beta_{\sigma}^{n+1} = \sum_{\delta} U_{\sigma\delta}^{-1} [T_{\delta} + \sum_r S_{\delta,r}^n (z_r^{n+1} - z_r^n)]. \quad (19)$$

(The problem is solved if there are no z variables.) Define

$$O_{\sigma} = \sum_{\delta} U_{\sigma\delta}^{-1} T_{\delta}, \quad (20)$$

$$P_{\sigma r} = \sum_{\delta} U_{\sigma\delta}^{-1} S_{\delta,r}^n, \quad (21)$$

$$Q_{sr} = \sum_{\delta} S_{\delta,s}^n P_{\sigma r}, \text{ and} \quad (22)$$

$$V_s = \sum_{\sigma} S_{\sigma,s}^n O_{\sigma}. \quad (23)$$

Substitute Eq. (19) into Eq. (4a) and solve for Z_t^{n+1} .

$$Z_t^{n+1} = Z_t^n - \sum_s Q_{ts}^{-1} V_s. \quad (24)$$

For the first iteration ($n=1$), the X variables may be initialized with their measured values, the Y and Z variables with values corresponding to the "expected" result. The fit is complete when χ^2 reaches a minimum and the constraint equations are satisfied to within an acceptable tolerance.

III. EXAMPLE

As an example to illustrate the method, consider the two body elastic scattering process $\pi p \rightarrow \pi p$. The apparatus is illustrated in Fig. 1. The scattered pion is detected in a spectrometer consisting of two sets of MWPCs (multiwire proportional chambers) before the magnet and two sets after. The recoil proton is detected by two sets of MWPCs. We also assume that the incident pion trajectory and momentum are known to within errors determined by the phase space.

We have the following set of measured variables: the points in space ($X_1, Y_1, \dots, X_6, Y_6$) determined from the MWPCs and the incident pion variables (P_0, θ_0, ϕ_0). The errors in the position measurements are determined by the MWPC resolution and multiple scattering. The errors in the incident pion parameters are determined by the beam channel phase space acceptance. These measured variables constitute the X -type variables defined in the previous section.

Fig. 1 Geometry for χ^2 fitting example.

An acceptable event must satisfy the kinematic constraint equations for the πp scattering. These equations require the scattered pion variables ($\theta_\pi, \phi_\pi, P_\pi$) and the recoil proton variables (θ_p, ϕ_p, P_p). The variables (X_v, Y_v, Z_v) which define the position of the vertex are also needed. These derived variables appear below in the same constraint equations as the measured parameters and thus meet the criterion for Y -type variables.

The constraint equations are listed below. We define the functions A to illustrate the variable dependence rather than explicitly writing the equations. The constants R_1 to R_6 are the known positions of the chambers. Constraint equations F_1 to F_{12} relate the measured points in space to the kinematic parameters and vertex position. Equations F_{13} , F_{14} , and F_{15} require momentum conservation and F_{16} requires energy conservation.

$$\begin{aligned} F_1 &= X_1 - A_1(\theta_\pi, \phi_\pi, X_v, Y_v, Z_v, R_1), \\ F_2 &= Y_1 - A_2(\theta_\pi, \phi_\pi, X_v, Y_v, Z_v, R_1), \\ F_3 &= X_2 - A_1(\theta_\pi, \phi_\pi, X_v, Y_v, Z_v, R_2), \\ F_4 &= Y_2 - A_2(\theta_\pi, \phi_\pi, X_v, Y_v, Z_v, R_2), \\ F_5 &= X_3 - A_3(\theta_\pi, \phi_\pi, P_\pi, X_v, Y_v, Z_v, R_3), \\ F_6 &= Y_3 - A_4(\theta_\pi, \phi_\pi, P_\pi, X_v, Y_v, Z_v, R_3), \\ F_7 &= X_4 - A_3(\theta_\pi, \phi_\pi, P_\pi, X_v, Y_v, Z_v, R_4), \\ F_8 &= Y_4 - A_4(\theta_\pi, \phi_\pi, P_\pi, X_v, Y_v, Z_v, R_4), \\ F_9 &= X_5 - A_5(\theta_p, \phi_p, X_v, Y_v, Z_v, R_5), \\ F_{10} &= Y_5 - A_6(\theta_p, \phi_p, X_v, Y_v, Z_v, R_5), \\ F_{11} &= X_6 - A_5(\theta_p, \phi_p, X_v, Y_v, Z_v, R_6), \\ F_{12} &= Y_6 - A_6(\theta_p, \phi_p, X_v, Y_v, Z_v, R_6), \\ F_{13} &= F_{13}(P_0, \theta_0, \phi_0, P_\pi, \theta_\pi, \phi_\pi, P_p, \theta_p, \phi_p), \\ F_{14} &= F_{14}(P_0, \theta_0, \phi_0, P_\pi, \theta_\pi, \phi_\pi, P_p, \theta_p, \phi_p), \\ F_{15} &= F_{15}(P_0, \theta_0, \phi_0, P_\pi, \theta_\pi, \phi_\pi, P_p, \theta_p, \phi_p), \text{ and} \\ F_{16} &= F_{16}(E_0, E_\pi, E_p). \end{aligned}$$

All of the above equations are F type. If the beam parameters were nonmeasured, F_{13} to F_{16} would contain no measured X -type variables and would then be S -type constraint equations. (P_0, θ_0, ϕ_0) appear only in these equations and would be Z type variables.

IV. CONCLUSION

This iterative procedure has been used and shown to converge. We have successfully used this technique to analyze two different experiments; one, a two-arm experiment to measure elastic $\pi^2\text{H}$ scattering⁴ and the other, a single arm experiment to look at elastic $\pi^3\text{H}$ and $\pi^3\text{He}$ scattering.⁵ The S-type constraint equations and Z-type variables were not necessary for this analysis and were therefore eliminated in the computer program. This resulted in a gain in speed and efficiency in the computer routines. We found that three iterations were generally sufficient to give an adequate fit. FORTRAN computer programs for the above fitting procedure exist and listings may be obtained from the author.

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