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EFFECTS OF PRESSURE ANISOTROPY ON THE STABILITY
OF THE GUIDING CENTER PLASMA

Progress Report

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Abstract

The stability analysis of nearly cylindrical magnetically confined plasma is generalized to include effects of pressure anisotropy in the guiding center model of Grad. The equilibrium and normal mode equations for helically symmetric plasmas are found and it is shown that these equations reduce to all previous diffuse profile bumpy θ pinch and helical calculations (in the appropriate limits). The equations presented are valid for arbitrary helical wave number k , arbitrary degree of helicity L and $O(1)$ pressure anisotropy. It is shown that the "new" and "old" Scyllac orderings can be unified in a single calculation.

Drift orbit diffusion induced by turbulence acting on trapped electrons is shown to reduce and broaden the magnetic drift resonance and produce the dominant nonlinear saturation mechanism for the dissipative trapped electron instability. The fluctuation level obtained is consistent with present experimental observations.

Analytic FCT equilibria have been found in cylindrical geometry.

Ideal 3D incompressible MHD turbulence in cylindrical geometry shows that meansquare turbulent velocity fields $\langle \vec{v}^2 \rangle \neq 0$ for virtually all initial conditions (including quiescent ones), except for a state of extremal helicity. A conjecture, based on the inverse cascade of magnetic helicity, is put forward to explain J. B. Taylor's hypothesis for reversed field pinches.

Preface

Although the original proposal was written just for

1. Effects of Pressure Anisotropy on the Stability of the Guiding Center Plasma
the scope of this ERDA (40-1)-5260 contract was extended to include the following topics

2. Perturbed Magnetic Field Phase Slip for Tokamaks

3. Nonlinear Saturation of the Drift Dissipative Trapped Electron Instability

4. Analytic Flux Conserving Tokamak Equilibria

5. Three Dimensional Magnetohydrodynamic Turbulence in Cylindrical Geometry

due to interactions with colleagues at ORNL during the summers of 1976, 1977.

Topic No. 5 was done in collaboration with David Montgomery (now at William and Mary) and Leaf Turner (LASL).

1. Effects of Pressure Anisotropy on the Stability of the Guiding Center Plasma

In studying the magnetic confinement of plasma one has to resort to fairly simple models so as to be able to incorporate into the theory the geometry. One of the major models that has currently received extensive analysis is that of (one fluid) ideal magnetohydrodynamics - a collision-dominated model. In particular, for high β plasmas, the stability of (straight) helically symmetric $\ell = 1$ ¹⁻⁶ and $\ell = 2$ ⁷ systems have been considered as well as the bumpy θ pinch^{8,9} ($\ell = 0$) and the bumpy screw pinch¹⁰ (the case of adding small shear to the bumpy θ pinch fields).

The results from ideal magnetohydrodynamics for the growth rates for the $m = 1$ mode are in good agreement with current high β experiments (Scyllac and Isar I) even though in these experiments the ions are collisionless and have finite Larmor radius - a regime where ideal magnetohydrodynamics does not apply. However, for the $m \geq 2$ modes there is strong disagreement between theory and experiment: in most cases ideal magnetohydrodynamics predicts larger growth rates for the $m \geq 2$ modes over the $m = 1$ mode, while experimentally the $m \geq 2$ modes have not been observed. Freidberg¹¹ has given a possible explanation for the absence of the $m = 2$ mode experimentally by considering a Vlasov-fluid model to estimate finite Larmor radius stabilization effects on these modes ($m \geq 2$). The electrons were assumed tied to the field lines and were treated macroscopically. The ions satisfied the Vlasov equation.

However, as one scales the experiments to a reactor parameter regime the plasma radius becomes large compared to the ion Larmor radius so that we must look for other effects to stabilize these systems. (The plasma is collisionless.)

In order to study such collisionless plasma effects on plasma stability, Weitzner¹² has considered the guiding center plasma model first introduced by Grad.¹³ In this model the ions and the electrons behave microscopically (obeying a 1D Vlasov-like equation) along field lines but macroscopically (MHD-like) across field lines.

Weitzner¹² considered a helically symmetric equilibrium with helical wave number ϵ , $\epsilon \ll 1$ and a fractional distortion, δ , of the magnetic flux surfaces from cylinder, $\epsilon \ll \delta \leq 1$. The ordering used was the "new Scyllac" ordering.⁴ Weitzner considered only very weak, of $O(\epsilon^2)$, pressure anisotropies. The stability of the system was considered by linearizing the equations of motion and performing successive expansions in the parameters ϵ and δ . The class of stable distribution functions is limited so as to exclude the fire hose, mirror, and velocity gradient instabilities. Under these conditions Weitzner¹² has shown that the unstable modes and growth rates are the same as ideal magnetohydrodynamics. Thus, the range of applicability of ideal magnetohydrodynamics is extended into this collisionless regime and the $m \geq 2$ modes are not stabilized under Weitzner's assumptions.

We proposed to explore the question of what is the effect of varying the pressure anisotropy on the stability of axial and helically symmetric systems (in both ideal magnetohydrodynamics and in Freidberg's model¹¹ the equilibrium pressure was isotropic) and in particular on the growth rates and possible stabilization of the $m \geq 2$ modes.

The project was attacked in the following manner:

1. It was thought that the double perturbation expansion technique^{8,10,12} could be relaxed by allowing ϵ to be finite. This program was carried out for MHD for the axially symmetric case by Cayton and Vahala.¹⁴ A finite wavelength theory is also necessary because of violation of the double perturbation expansion

ordering (e.g., in Scyllac $\delta = 0.15$, $\epsilon = 0.2$; in EBT $\delta = 0.33$, $\epsilon = 2.4$).

One of the interesting results was the study of the evolution of two unstable classes of modes from a single unstable class as β is increased.

2. Concurrently to the above, Schmidt and Vahala applied the double perturbation expansion ($\epsilon \ll \delta \leq 1$) technique to axially symmetric GCP plasma - first¹⁵ for the Weitzner case of very weak pressure anisotropies (of $O(\epsilon^2)$) and then for $O(1)$ anisotropies.¹⁶ We found that the anisotropy must be $O(1)$ for the growth rates for GCP plasmas to be different from MHD plasmas. In particular, for $p_{||} > p_{\perp}$ the GCP growth rates can be substantially decreased relative to MHD growth rates for the $m = 1$ mode.

3. Vahala and Vahala¹⁷ then applied a finite wavelength GCP theory for axially symmetric systems, studying wall stabilization effects and the effects of finite wavelength on the growth rates. It was shown that (global) modes with zero nodes in the radial eigenfunction can be wall-stabilized, but the local modes with non-zero nodes are not wall-stabilized. The experimental implications are not known. It was also shown that $m \geq 2$ modes have higher GCP growth rates than $m = 1$ mode, but these growth rates can be substantially lower than their MHD counterparts. Thus, the suppression of $m \geq 2$ modes cannot be found in the zero Larmor radius GCP theory.

More importantly, it was found that the long wavelength theory was a special case of our finite wavelength theory. In particular, Refs. 8, 14 and 15 are special limiting cases.

4. Finally, we are investigating¹⁸ the effects of large pressure anisotropies on helically symmetric GCP. In particular, because of the unified theory in Ref. 17, we felt that there might be a connection between the "new" and "old" Scyllac orderings.

The equations we derive in this paper¹⁸ include as limiting cases all previous diffuse profile bumpy theta pinch and helical calculations. For example, the helical stability theories known as the "new" scyllac ordering and the "old" scyllac ordering turn out to be limiting cases of a unified diffuse profile theory - a result that has not been fully appreciated or utilized by other authors. (An exception is the sharp boundary $L = 1$ hybrid calculation of Freidberg.)¹⁹

The equilibrium considered possesses a magnetic flux function of the form:

$$\psi(r, \phi) = \psi_0(r) + \delta \psi_1(r) \cos L\phi ; \quad \phi = \theta + kz$$

The one expansion parameter δ measures the percent deviation of the radius of the surface of the plasma to the mean radius r_0 of that surface. k is the wave number of the helical period of the plasma (referred to as the helical wavenumber) and L is the degree of helicity of the plasma.

The various limits of δ , kr_0 , and L represent all the previous nearly cylindrical plasma calculations. The "old" scyllac ordering has the limits $L = 1$, $\delta \ll 1$ and $kr_0 \sim \text{finite}$. Since one always uses δ as an expansion parameter to obtain analytic results, the true "new" scyllac ordering is $L = 1$, $rk_0 < \delta \ll 1$. It is assumed that these results still hold in some sense for $\delta \sim \text{finite}$.

The other limit of interest is the axisymmetric bumpy theta pinch limit. In this limit $L \rightarrow 0$ and $k \rightarrow \infty$ such that the product $Lk = k'$ where k' is the bumpy theta pinch wave number. In this limit, the two major cases that have been studied are the finite wavelength case¹⁷ ($r_0 k' \sim \text{finite}$, $\delta \ll 1$), and the long wavelength case¹⁶ ($rk_0' < \delta \ll 1$). Thus, the "new" and "old" scyllac ordering in the helical

calculation are equivalent to the long wavelength and finite wavelength scalings in the axisymmetric bumpy theta pinch calculation.

We have numerically studied the equilibrium using the rigid rotor profile of Berge and Freidberg. We calculate the effect of pressure anisotropy and helical wave number on the equilibrium helical shift for the $L = 1$ system. This shift is a qualitative indication of the magnitude of the growth rates and is of experimental interest in stellerator designs.

The growth rates for helically systems are currently being investigated. This is the only part of the proposal not yet completed.

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2. Perturbed Magnetic Field Phase Slip for Tokamaks¹

Coherent helical structures have been observed in Tokamaks, and these structures appear to rotate within the plasma. The typical experimental diagnostics are (i) magnetic field pickup loops located near the vacuum wall and (ii) arrays of collimated soft X-ray detectors scanning along chords passing through the plasma. For a give mode (in this case an $m = 2$ mode) both the magnetic loops and the X-ray detectors observe persistent coherent oscillations --the Mirnov oscillations--in the frequency range 1-30 kHz at exactly the same frequency.

It is believed that the Mirnov oscillations are due to helically perturbed current structures corresponding to magnetic islands within the plasma and the soft X-ray emission is due primarily to electron temperature perturbations coincident with the same magnetic islands. The oscillation in the observed signal is apparently associated with the rotation of the island structure. Since the electron temperature and the current perturbations are evidently associated with the same island structure, the oscillation frequency of the observed X-ray and magnetic field perturbations should be (and is) identical.

The idea behind this calculation is the following: if the highly conducting plasma between the $m = 2$ island and the magnetic loops is not rotating with the same angular velocity as the island structure then there should be an experimentally measurable phase difference between the observed X-ray and the magnetic field oscillations. That is, there should be a relationship between these phase differences and the relationship between the mode and plasma fluid rotations. The development of the mode-fluid rotation relationship during the Tokamak discharge may significantly effect plasma confinement and stability.

In this paper¹ we set up a very simple model giving this phase difference Δ as a function of σS , where σ is the electrical conductivity of the plasma and S is the slip between the frequency of the signal, ω , and the poloidal and/or toroidal rotational frequency of the plasma

$$S = \omega - \frac{mV_\theta}{r} - kV_z$$

S may be non-zero because the magnetic islands may have a different apparent rotation velocity from the surrounding plasma because of diamagnetic and other effects in the dispersion relation for resistive tearing modes. Alternatively, the angular velocity of the intervening plasma may not be uniform, since there is not a prior reason to expect the plasma to rotate as a rigid body.

In this model we wish to determine the effects of fluid rotation on the magnetic fields produced by a $m = 2$ magnetic island assuming the magnetic fields in the absence of rotation are known. In practice, we will use the (unrotated) saturated $m = 2$ magnetic island current as found from the nonlinear computer codes of White, Monticello, Rosenbluth and Waddell.²

We assume helical symmetry in cylindrical geometry and ignore the vacuum region between the plasma and the wall. Rosenbluth's reduced MHD equations^{2,3} are generalized so as to include equilibrium flow velocities, using the standard large aspect ratio expansion in the low β Tokamak ordering. To avoid having to solve the resulting nonlinear partial differential equations by nontrivially extending the nonlinear computer codes of White, et al., (which are very complex and heavy users of machine time) we ignore all coupling effects of the rotation on the island saturation mechanism. This is a reasonable assumption for low β , and allows us to dispense with the force balance equation. The resulting set of equations are just ordinary differential equations which are quite simple and speedily solved on the computer.

The chief results are (using typical ORMAK parameters and conductivity profiles): (i) as $S \rightarrow 0$ a jump of π occurs in the phase difference Δ as we cross the zero of the island current with Δ becoming insensitive to the location of the X-ray diode (i.e., all diodes scanning positions to the right of the island zero have $\Delta = \pi/2$, while all those to the left have $\Delta = -\pi/2$). However, for even small slips ($S \geq 20$) Δ starts to exhibit a strong dependence on the radial scanning location of the diode. (ii) for peaked conductivity profiles, Δ is observed to saturate and then decrease at large values of S ($S \geq 2000$).

At the present time Julian Dunlap of ORNL is examining the experimentally observed Δ 's on some ORMAK discharges and we shall apply the above model to his experimental results.

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3. Nonlinear Saturation of the Drive Dissipative Trapped Electron Instability¹

The existence of drift waves in tokamak-type toroidal confinement devices has been demonstrated by microwave scattering experiments in the Adiabatic Toroidal Compressor and Tokamak Fontenay-aux-Roses devices.² Because the trapped electron instability is a type of drift wave which is destabilized by trapped electrons, it is expected that it will be operative in larger and hotter tokamaks and will persist even in a reactor regime. The anomalous transport due to this instability is generally believed to be a dominant factor affecting the confinement time of a high temperature plasma. While there is already a fairly complete picture (except for the two-dimensional mode structure) for the linear theory, the nonlinear theory is still rather primitive. The most serious attempt so far has been a one-dimensional, resonant mode-coupling process,³ which has difficulty in treating the short wavelength part of the spectrum because of the strong dispersion of the real part of the frequency. In addition, this theory gives a saturation level which is too high to be consistent with the experimental value. Other nonlinear saturation mechanisms, such as quasi-linear plateauing of the density or temperature profiles, have been observed in numerical simulations.⁴ However, this is highly improbable in real experiments because in a tokamak, energy is continuously pumped into the system through ohmic heating or neutral beam injection.

The nonlinear model we investigate in this paper¹ is based on the renormalized turbulence theory of Dupree.⁵ This model is chosen because it apparently gave satisfactory agreement between theory and drift wave experiments in Q-machines.⁶ The major difference between drift waves in Q-machines and those in tokamaks lies in the complexity of the geometry involved. Hence, generalizing Dupree's theory⁵ from slab to tokamak geometry should provide an appropriate nonlinear description for drift waves in a tokamak.

In the original theory on trapped electron modes,⁷ the instability arises purely due to electron collisions. Energy is exchanged between the electrons and the waves through the stochastic process of trapping and detrapping of electrons by pitch angle scattering. Linear stability is attained if the rate of energy transfer from the electrons to the wave is balanced by the radial convection of the wave energy from the rotational surface to regions where the waves are ion Landau damped. In this theory, the instability does not exist if there are no collisions.

The picture is significantly changed if the magnetic drift resonance is included in the theory. It is now known that with the inclusion of magnetic drift resonance, the instability persists even without electron collisions.⁸ In other words, the trapped electron instability exists in very collisionless plasmas such as the plasma in a reactor regime. The energy exchange between the wave and the electrons is no longer through the random trapping and detrapping of the electrons, but through a Landau type resonance between the trapped electrons and the wave. In the single particle picture, trapped electrons have banana orbits bouncing in poloidal and toroidal directions. At the same time, their average position (or the center of the banana) drifts along the toroidal direction due to the magnetic curvature and gradient. This drift velocity is proportional to the energy of the electron and is in the same direction as the phase velocity of the wave. Therefore, there is always a small group of trapped electrons having the correct energy and experiencing a dc electric field on the average. These trapped electrons can exchange energy with the wave just as occurs in the wave-particle resonance in the well-known theory of Landau resonance. The collisions no longer serve as the catalyst of energy exchange, but rather serve to reduce the strength and broaden the resonance. In fact, as the effective

collision frequency of the trapped electrons increases to the same order of magnitude as the trapped electron bounce frequency, the trapped electron destabilizing term in most cases is reduced to a level that can be stabilized by shear in present experimental conditions.⁹ In other words, present experiments are not far from the stability boundary of trapped electron instability.

Because collisions play such an important role in reducing the trapped electron destabilizing effect, it is natural to expect in a nonlinear theory that the effective electron collisions caused by the electrostatic turbulence are the dominant mechanism of saturation. This is the nonlinear effect we investigate in this paper.¹ The turbulence effects on the ions are ignored for the time being because the orbit diffusion effect on the ions will change the ion response in such a way that the radial eigenfunction equation for the mode is modified. Such a modification will increase the effective shear and ion viscous damping. If the effective trapped electron Coulomb collision frequency is much smaller than the average trapped electron bounce frequency, the trapped electron destabilizing term in the dispersion relation is much larger than the shear and ion collision terms. In this case, the effect of reducing the trapped electron destabilizing term is more important than enhancing the stabilizing shear and ion collision terms.

In Dupree's turbulence theory of drift waves, a turbulent collision frequency $k_{\perp}^2 D$ was found, where $k_{\perp}$ is the perpendicular wave number and D is the diffusion coefficient. The meaning of $k_{\perp}$ is unambiguous in a homogeneous plasma. In an inhomogeneous plasma with gradients perpendicular to the magnetic field, it is not easy to decide which $k_{\perp}$ we should use. In particular, in a tokamak with gradients in the radial direction, it is not clear what combination of k_r and k_e should appear in $k_{\perp}^2 D$. The problem is more complicated if we realize that the

eigenmode in the radial direction is not a Fourier series but is a sum of parabolic cylinder functions. The radial wave number can only be defined in an average sense. In this paper,¹ we systematically deduce that the turbulent collision is $k_r^2 D$ and find an averaging procedure to obtain this k_r . One of the defects of Dupree's theory is the lack of closure: the fluctuating spectrum cannot be determined within the framework of this theory. If we assume the instability to be stabilized by the turbulent collisions of the trapped electrons, we can then determine the test particle diffusion coefficient such that the most unstable mode has a zero nonlinear growth rate. With the knowledge of this diffusion coefficient we can then find the fluctuation level for an assumed shape of spectrum, and then find the particle and heat fluxes due to trapped electron instability.

In the calculation, we first derive the turbulent response of the trapped electrons from a bounce averaged drift kinetic equation. The turbulence affects the trapped electron nonadiabatic response mainly through the turbulent collisions. The nonlinear dispersion relation is then derived and the turbulent response of the trapped electrons is treated perturbatively.

We have shown that turbulence-induced collisions acting on trapped electrons are very significant in the nonlinear saturation of the dissipative trapped electron instability, particularly when the plasma is deep in the banana regime. When the shape of the fluctuation spectrum is assumed to be similar to those observed in experiments, i.e., peaked around $k_\theta \rho_i \approx 1$, the fluctuation level and diffusion coefficients obtained from the theory are consistent with the experimental values.

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4. Analytic Flux Conserving Tokamak Equilibria¹⁰

Recently, Clarke¹ observed that if one starts with an Ohmically heated low β tokamak plasma into which intense neutral beams are injected, the plasma β can be increased in a flux conserving manner. That is, if the energy deposition time of the beams are shorter than the resistive skin time, then the magnetic flux and the safety factor profile (which is a function of poloidal flux) will be conserved during the energy deposition.

The determination of such a sequence of flux conserving tokamak equilibria (and, more generally, the adiabatic problem) yields a non-standard mathematical problem whose well-posedness has been investigated by Grad^{2,3} in his theory of generalized differential equations. Numerical toroidal flux conserving equilibria have been found by Dory and Peng.⁴ Clarke and Sigmar,⁵ using a virial theory approach, have investigated global properties of flux conserving equilibria assuming the safety factor profile $q(\psi) = \text{const.}$ for a circular shell in fully toroidal geometry. Mizoguchi and Kamash⁶ have recently generalized the work of Ref. 5 to an elliptical shell. Dory and Peng⁴ also solved the most elementary problem of a flux conserving cylinder with uniform $q(\psi)$ profile analytically, and a peaked current profile case (similar to ours) purely numerically. Related unpublished calculations have been performed by Bateman.⁷ These authors prescribed a fixed $q(r)$ profile and generated a series of somewhat arbitrary pressure or current profiles such that flux was conserved.

In this paper,¹⁰ an analytic solution is presented for the physically more meaningful case where neighboring pressure profiles are connected via an adiabatic equation of state, driven by some heat source profile (due to auxiliary heating). (Such a problem was recently solved purely numerically by Nelson.)⁸

The flux conserving equilibria for circular flux surfaces in cylindrical geometry with non-constant q and $p'(\psi)$ - profiles are found by performing perturbation expansions in $\epsilon \ll 1$, where ϵ is the ratio of poloidal to toroidal magnetic fields. The sequence of equilibria connects an initially low $\beta \sim 0(\epsilon)$ equilibrium to high $\beta \sim 0(\epsilon)$ equilibria conserving total flux and conserving the $q(\psi)$ profile. It should be noted that while no upper limit on β has been found in the numerical codes of Dory and Peng,⁴ there does appear to be a critical high β of 5-10% from the linear stability codes of Bateman and Peng.⁹ This motivates our perturbation technique in the smallness of ϵ , suitable even for so called "high" beta Tokamaks. The $q(\psi)$ profile was so chosen to give zero toroidal current density at the wall.

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5. 3D MHD Turbulence in Cylindrical Geometry¹

The idea of approaching fluid turbulence through the statistical mechanics of the velocity field received an early and simple presentation in the paper of Lee.² The technique was enlarged by Kraichnan² to infer equilibrium probability distributions for the Fourier coefficients in a truncated representation of an ideal Navier-Stokes fluid in two dimensions. Kraichnan's predictions were to a considerable extent verified by numerical solutions of the Navier-Stokes equations by Basdevant and Sadourny³ and by Seyler, et al.⁴

The analogous problem for an ideal incompressible magnetohydrodynamic plasma has been treated in three dimensions by Frisch, et al.,⁵ and Pouquet, et al.,⁶ and in two dimensions by Fyfe, et al.^{7,8,9} A review of the situation for all four cases (Navier-Stokes and magnetohydrodynamics, two dimensions and three dimensions) has been given by Montgomery.¹⁰

A truncated Fourier representation of ideal fluid of magnetohydrodynamic equations admittedly leaves out important effects: all those associated with non-zero viscosity or resistivity. Particularly in the case of high Reynolds number turbulence, these omissions can be serious, even though the numerical values of the dissipation coefficients can be very small. For instance, for turbulent cascade processes, an entirely different steady state is expected for the dissipative and non-dissipative systems. In the dissipative case the spectral functions are approximately constant, while for the non-dissipative case they are identically zero, and different power laws result.

There is, nevertheless, considerable utility in studying the equilibrium statistics of truncated Fourier representations in ideal fluids or magnetohydrodynamics. The statistical mechanical equilibria represent time-dependent steady states toward which the turbulent field continually tries to relax, and to which it would relax if the relaxation were not thwarted at the upper end by dissipation.

Three situations of the previously mentioned four (two and three dimensions in magnetohydrodynamics, and two dimensions in Navier-Stokes hydrodynamics) also show a tendency for coherent macroscopic structures to form,²⁻¹⁰ with or without dissipation. These are easiest to see in the truncated Fourier representation without dissipation. We shall, finally, argue later in this article that considerations derived from ideal, truncated discretizations provide a novel way of formulating the problem of magnetohydrodynamic stability, leading to some new conclusions. In particular, an awareness of the crucial role of magnetic helicity for the stability problem will be seen to emerge.

All the theoretical treatments referenced above rely on an assumed rectangular symmetry and periodic boundary conditions. The mathematics may seem remote from the concerns of those who work on fusion plasmas contained in a cylinder or torus. For the sake of greater experimental utility, we consider the case of a long straight cylinder with a rigid, perfectly-conducting wall at radius $r = a$. A torus can be simulated to some extent by requiring periodicity in the z -direction, along the axis of the cylinder. One pays a price for this realism, however, in the loss of the simple spectral-method numerical capability of Orszag and Patterson. These techniques have been invaluable in substantiating theoretical predictions in rectangular geometry.⁴⁻¹⁰

This article,¹ specifically, is concerned with the implications of the canonical distribution for the expansion coefficients of an ideal magnetohydrodynamic fluid inside cylindrical boundaries. The work in effect is a direct generalization of that of Frisch, et al.,⁵ to a cylindrical region, but modifications to include the effects of externally-imposed fluxes are required. First, the basic dynamics of the fields are set out. Then, a representation of the fields are given in terms of a particularly convenient set of eigenfunctions replacing the more usual expansion in Fourier series. The basic statistical mechanical considerations can then be derived.

This calculation offers a new way of considering the problem of non-linear magnetohydrodynamic stability and enunciates a principle of extremal helicity, which may be of some use in designing nearly quiescent confinement devices. Typically, magnetohydrodynamic stability theory has been perturbation theoretic, proceeding from linearized equations of motion around quiescent ($\bar{v} \approx 0$) equilibria. One either follows the evolution of small perturbations in detail, or attempts to establish the existence or non-existence of exponentially growing normal modes by variational methods (the "energy principle"). Some advantages of the linear stability point of view are: (1) explicitness, particularly as regards the growth rate of the instability; and (2) insight into the character of whatever growing modes may exist. Some disadvantages are: (1) the difficulty of treating realistic spatial profiles, and the wide variety of such profiles there is to treat (one can literally go on forever, changing the profiles slightly and recalculating their stability); and (2) the impossibility of determining the effects of instabilities once they have grown out of the linear regime. Clearly, the impact of conventional MHD stability theory has not been so overwhelming as to preclude alternative viewpoints on the stability problem. Our development suggests one that is essentially thermodynamic in character.

Our calculation predicts in general a finite mean square velocity for every mode in the system. This is true for initially quiescent systems as well ($\bar{v} \approx 0$). Four scalar numbers, the energy $\langle \epsilon \rangle$, the magnetic helicity $\langle H_M \rangle$, the toroidal flux $\langle \psi_T \rangle$, and the "poloidal" flux $\langle \psi_P \rangle$ suffice to determine the level of eventual excitation of every mode in the system for an initially quiescent system. In this sense, almost all initial profiles are surely unstable. This conclusion is independent of linear analysis and its inevitable severe approximations.

The quiescent limit is one in which all of the excitation is locked into the extremal helicity state by the simultaneous constancy of $\int d^3 x \underline{A} \cdot \underline{B}$ and $\int d^3 x (\underline{B}^2 + \underline{v}^2)$. It is the state of minimum energy for given magnetic helicity, or equivalently, the state of extremal helicity for given energy. The Euler equation for the two variational problems is the same. We shall call this the "maximal helicity state." It is a force-free state.

There is a well-defined value of $\langle \Psi_T \rangle$ for the maximal helicity state, and any attempt to maintain $\langle \Psi_T \rangle$ at some other value will lead to the inaccessibility of the maximal helicity state and to turbulent velocity fields $\langle \underline{v}^2 \rangle \neq 0$. Experimentally, this translates roughly into there being a single value of axial current which will permit the maximal helicity state to dominate.

The lesson for stability appears to be, in most general terms, that control of the turbulent velocity field is effected by control of the turbulent energy-to-helicity ratio. Low values of this ratio are desirable. Our experimental insights into helicity, how to manipulate it and how to maximize it, have a way to go before we are in a position to make effective use of this idea in the laboratory.

We also suggest a basis for a possible relaxation process which J. B. Taylor has conjectured for cylindrical turbulent z-pinch experiments.

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