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CALCULATION OF TRANSPORT COEFFICIENTS IN AN AXISYMMETRIC PLASMA

Dana E. Shumaker
(Ph.D. thesis)

January, 1977

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assumption eliminates the electrostatic potential from the transport equations for total particle density and total energy density.

A variational principle is used to compute the transport coefficients. This variational principle differs from others in that it not only yields an approximation of the distribution functions, but it also yields an approximation of the inductive component of the electric field parallel to the magnetic field.

The trial functions used in the variational principle are orthogonal polynomials in velocity space. A computer code was written to perform the calculation. Good agreement with previous results are obtained in the Pfirsch Schluter, high collision frequency regime.

In this work the full Landau collision operator is used. No mass ratio expansion is made. This makes it easy to extend the calculation to include more than one ion species.

A computer code was written to calculate the transport coefficients for a three species plasma, electrons and two species of ions. This is useful for calculating the transport coefficients of a plasma which contains impurities. It was found that the particle and energy transport coefficients are increased by a large amount, and the transport coefficients for the toroidal magnetic field

are reduced by a small amount. For example, a deuterium plasma with 1.3% oxygen, one of the particle transport coefficients is increased by a factor of about four. The transport coefficients for the toroidal magnetic flux are reduced by about 20%. The increase in the particle transport coefficient is due to the collisional scattering of the deuterons by the heavy oxygen ions which is larger than the deuteron electron scattering, the normal process for particle transport in a two species plasma. The reduction in the toroidal magnetic flux transport coefficients are left unexplained.

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I INTRODUCTION

1.1 TOKAMAKS

One of the most active areas of research on controlled thermonuclear fusion involves an experimental device designed to confine a hot toroidal or doughnut-shaped plasma. These devices are named Tokamaks after the first Russian devices of this type. Tokamaks have shown promise for development into a useful controlled thermonuclear energy source. The toroidal plasma is confined by a poloidal magnetic field (in the direction of the short way around the torus, see Figure 1). The poloidal magnetic field is produced by a toroidal current flowing in the plasma. Also a large toroidal magnetic field is needed to suppress dangerous magnetohydrodynamic instabilities. This field is produced by external poloidal coils. In the traditional style Tokamak apparatus shown in Figure 2, the toroidal electric field which drives the plasma current is induced by an iron core transformer which links the torus. The plasma is positioned in r and z by eddy currents in a copper shell. These eddy currents are induced by the plasma current. The r , θ , z coordinate system is shown in Figure 3. Recent improvements in the design of Tokamaks include the use of air core transformers which avoid flux saturation of an iron core and also fit into a smaller space. Also the toroidal copper shell has been replaced in several devices by poloidal field coils with electronically controlled

Figure 1 Tokamak Field Structure

Figure 2 Tokamak Apparatus

Figure 3 r, θ, z Coordinate system

currents which control the r, z position of the plasma. Table 1 gives some parameters for the more predominant Tokamak experiments, [1], [2]. In the table the name and location of the device is given. Also given are the major radii, the length from the z axis to the center of the minor circle, the minor radii, the radii of the circle in the $\theta = \text{constant}$ plane, the toroidal plasma current, and the toroidal magnetic field strength.

In order to obtain any significant amount of thermonuclear fusion, the temperature of the plasma must be high; about 10 keV for a deuterium tritium plasma. In the early Tokamak experiments the only heating source for the plasma was Ohmic heating due to the toroidal plasma current. However; this method has severe limitations. It is good for a maximum temperature of about 3 keV, [2]. This is due to the fact that the conductivity increases with increasing temperature, T , as $T^{3/2}$. At higher temperature the electric field is less for a given current, thus the heating rate, which is the product of the electric field and current, is less.

Another method of heating a Tokamak which is now being used is by neutral beams. High energy neutrals, (20 to 100 keV) are injected into the torus where they are ionized by collisions and contained in the magnetic field, [2], [19], [20], [21], [22].

TABLE I PARAMETERS OF SOME TOKAMAK DEVICES

MODEL	LOCATION	MAJOR RADIUS	MINOR RADIUS	ASPECT RATIO	TOROIDAL MAGNETIC FIELD	PLASMA CURRENT	PERFECT
		(cm)	(cm)		(kG)	(kA)	
T-4	USSR	100	17	5.9	50	200	3
TFR	France	98	20	4.9	60	200	4
ORMAK	Oak Ridge	80	23	3.5	25	120	5
ATC	Princeton	90-36(1)	17-11	5.3-3.3	20-50	70-180	6
DOUBLET II	San Diego	59	20/50 (2)		10	100	7
ST	Princeton	109	14	7.8	50	70	8
ALCATOR	MIT	54	10	5.4	100	150	9
JFT-2	Japan	90	28	3.2	10	120	10
CLEO	England	90	18	5.0	20	80	11
PULSATOR	Germany	70	12	5.8	30	80	12
TM-3	USSR	40	8	5.0	40	30	13
TO-1	USSR	60	12	5.0	20	20	14
T-6	USSR	70	25	2.8	15	50	15
T-9	USSR	36	7/28(?)		15	10	16
T-10	USSR	150	40	3.8	50	1070	17
PLT	Princeton	130	45	2.9	50	1600	18

(1). Uses compression in major radius to heat plasma.

(2). Noncircular flux surface shape.

To obtain a net energy production in a fusion reactor, not only must the plasma be heated to ignition temperature, but, as pointed out by Lawson, [23], the product of the particle density, N , and confinement time, τ_c , must be above a certain value. For a deuterium tritium plasma this condition can be written as:

$$N \tau_c > 10^{14} \text{ sec cm}^{-3} \quad (1-1)$$

This is to be interpreted as a lower limit on $N \tau_c$. In an actual reactor, the presence of losses not included in the Lawson condition (1-1); such as line radiation from impurities, charge exchange, etc, will increase the required $N \tau_c$ value. In order to calculate the confinement time one must know the transport coefficients of the plasma. The purpose of this work is to calculate the transport coefficients in a Tokamak plasma.

1.2 NEOCLASSICAL TRANSPORT THEORY

The orbits of particles in the magnetic field structure of a Tokamak can cause some significant changes in the transport coefficients of the plasma. The theory used to describe the transport in a toroidal plasma is referred to as the neoclassical theory. In a Tokamak the toroidal magnetic field is larger than the poloidal magnetic field. The field structure consists of helical line as shown in Figure 1. If a field line is followed around the torus many times, it will trace out a surface.

(except in the case that the field line exactly closes on itself, but there will be a field line very close to it which will not close on itself). This surface is called a magnetic field surface. The surfaces are described by a magnetic stream function Ψ . Ψ equal to a constant is the equation for a flux surface. In the cases considered here, the flux surfaces are nested and Ψ is independent of the angle θ , as is all scalars, i.e. axisymmetric. Figure 1 indicates the flux surface as well as the magnetic axis, which is the line interior to all flux surfaces. The magnetic axis is the line on which the poloidal magnetic field is zero.

As a particle moves along a magnetic field it experiences a change in the magnitude of the magnetic field. The magnetic field is largest on the part of the flux surface closest to the z axis. Particles with low velocity parallel to the magnetic field will be trapped in the outer part of the flux surface. These particles will oscillate between two turning points on the field line, points t_1 and t_2 in Figure 4a. The conservation of toroidal canonical momentum requires that the particles drift off of their flux surface during the oscillation. The effect of this drift is to produce an orbit that looks like a banana. Figure 4b is a projection of an orbit in the poloidal plane. ($\theta = \text{constant}$). These trapped particles can have a very significant effect on the transport in a Tokamak.

Figure 4a The turning point of trapped particle.

Figure 4b Projection of trapped particle orbit in the poloidal plane.

In order to illustrate the effect of trapped particles, an approximate form of the magnetic field will be used. This approximation is the large aspect ratio expansion. It is used in much of the theoretical work on Tokamaks. The aspect ratio is the ratio of the major radius R to the minor radius a . The major and minor radii are defined in Figure 5. The aspect ratio is assumed to be a large number, and the flux surfaces are assumed to be concentric circles. This approximation is not made in this thesis, it is only used in this section to illustrate the effect of trapped particles. In this approximation the magnetic field can be written as [24]

$$B_\theta = B_T = B_{T0} \left[1 - \frac{a}{R} \cos \phi \right] \quad (1-2)$$

$$B_\phi = B_p = B_{p0}(a) \left[1 - \frac{a}{R} \cos \phi \right] \quad (1-3)$$

$$B_z = 0 \quad (1-4)$$

The a, θ, ϕ coordinate system which is used in this section is given in Figure 5. Where B_{T0} is the value of the toroidal magnetic field on the magnetic axis, $B_{p0}(a)$ is some function of a which determines the poloidal magnetic field. The expansion parameter is the inverse aspect ratio, which is assumed to be small.

Figure 5 r, ϕ, θ coordinate system.

$$\epsilon = \frac{a}{R} \ll 1$$

(1-5)

The requirement that the system be magnetohydrodynamically stable, requires that the stability factor q satisfy [25]

$$q = \frac{B_T}{B_p} \frac{a}{R} > 1$$

(1-6)

In most experiments q is about 2 to 3. Since a/R is a small number, B_T must be larger than B_p .

$$B_T \gg B_p$$

(1-7)

The magnitude of the magnetic field can be assumed to be due to B_T only. The ratio of the magnitude of the magnetic field on the inside, $\phi = \pi$, to the magnitude on the outside, $\phi = 0$ is

$$\frac{B(\pi)}{B(0)} = (1 + 2\epsilon)$$

(1-8)

Using the conservation of magnetic moment, $\frac{mv_{\perp}^2}{2B}$, where m is the particle mass and $v_{\perp}$ is the perpendicular velocity, and the conservation of energy $\frac{mv^2}{2}$ it is easy to show that if the parallel velocity, $v_{\parallel}$, and the perpendicular velocity $v_{\perp}$ at $\phi = 0$ satisfy

$$\frac{v_{H0}}{v_{L0}} \leq \sqrt{2\epsilon}$$

(1-9)

then the particle is trapped [26]. The particles which exhibit the largest deviation from the flux surface (largest banana width), are trapped particles which are almost untrapped [27]. That is, the trapped particles go almost all the way around the minor radius of the torus. The untrapped particles have a much smaller average deviation from the flux surface [27]. The barely trapped particles correspond to the equality in equation 1.9. The average value of v_{H0} for these barely trapped particles can be approximated by:

$$v_{H0} \approx v_{th} \sqrt{2\epsilon}$$

(1-10)

where v_{th} is the average thermal velocity $\sqrt{\frac{2T}{m}}$. This is a good approximation in a Tokamak plasma since the distribution function is approximately a Maxwellian.

The trapped particles are important only when the collision frequency is low i.e; long mean free path (high temperature). In a toroidal plasma one can identify three different regimes of transport depending on the value of the collision frequency. These regimes are, going from high to low collision frequency, collisional or Pfirsch Schluter, plateau and banana regime. In order to define the transition values of the collision frequency, one must compare the mean free path to the field connection length, L . The connection length is the length along

a field line while following the field line for one period of ϕ , see figure 5. The pitch angle of the field line, measured in the magnetic flux surface is given approximately by the ratio B_p/B_T . Thus the connection length is given approximately by,

$$L = \pi \frac{B_T}{B_p} = qR \quad (1-11)$$

The collisional or "Finnish-Schluter" regime is defined by,

$$\lambda < L \quad (1-12)$$

Where λ is the meanfree path of an electron with the thermal velocity v_{th} . Here the electrons are being considered since they dominate the transport process in most cases. However, as will be seen the transition temperatures, which are being calculated, are independent of the particle mass. Thus the same results would be obtained if the ions were considered instead of the electrons. The collision frequency to use in the equation for the mean free path is the electron ion collision frequency, since this collisional process makes the largest contribution to the transport across the flux surface. The inequality (1-12) can be written in terms of the collision frequency as,

$$\nu > \nu_{cp} = \frac{v_{th}}{gR} \quad (1-13)$$

where ν_{cp} is the value of the collision frequency for the transition from the collisional regime to the plateau regime.

The plateau regime is defined by,

$$\nu_{pB} < \nu < \nu_{cp} \quad (1-14)$$

where ν_{pB} is the value of the collision frequency for the transition from the plateau regime to the banana regime. In the plateau regime the untrapped particles will travel the distance L without making a collision, but the trapped particles which have small parallel velocity will suffer a collision before going the distance L . ν_{pB} is the collision frequency at which the barely trapped particles make one collision in the distance L which results in it becoming an untrapped particle.

The banana regime is defined by,

$$\nu < \nu_{pB} \quad (1-15)$$

Since the trapped particles occupy a small region in velocity space, in the large aspect ratio limit, they do not have to diffuse very far in velocity space in order to make a transition from being trapped to untrapped. The velocity space is indicated in Figure 6. Since the angle in velocity space which a particle must diffuse in order to make a transition from trapped to untrapped is of order $\sqrt{2\epsilon}$, and since diffusion time is proportional to the square of distances (or angle), the effective collision frequency ν_{eff} for transition from trapped to untrapped is,

$$\nu_{eff} = \nu/c \quad (1-16)$$

The transition collision frequency between the plateau and the banana regime is obtained by setting the effective collision time, reciprocal of equation (1-16), to the transient time of the barely trapped particles. The transient time of the barely trapped particles is obtained by dividing the length L by the parallel velocity of the trapped particles, equation (1-10). The result is,

$$\nu_{PB} = \epsilon^{3/2} \frac{v_{th}}{qR} \quad (1-17)$$

Figure 6 Velocity Space

Using the electron ion collision frequency [28], the transition collision frequencies can be written in terms of a transition temperature. For a deuterium plasma they are:

$$T_{ep} = \sqrt{\frac{4\sqrt{\pi}}{3} \ln \Lambda e^4 N_e R} \quad (1-18)$$

$$T_{ea} = \sqrt{\frac{4\sqrt{\pi}}{3} \ln \Lambda e^4 N_e \frac{2R}{\epsilon^{3/2}}} \quad (1-19)$$

Where $\ln \Lambda$ is the Coulomb logarithm, e is the electron charge and N_e is the electron density. Note that approximately the same results are obtained for ions since the square root of the mass in the thermal velocity cancels the square root of mass in the collision frequency. Most experiments to date have operated in the collisional and plateau regime. A reactor would operate in the plateau and banana regime [27].

A Brief description of the order of magnitude of the transport coefficients in these three collision frequency regimes will be given. Also a summary of previous work will be given.

The first calculation of the particle transport in a torus in the collisional regime was given by Pfirsch and Schlüter [29]. They used a resistive magnetohydrodynamic model to calculate the order of magnitude effect due to the current flow parallel to the

magnetic field. They showed that the particle transport coefficient was of order, (see Appendix A)

$$D = D_{10} (1 + q^2) \quad (1-20a)$$

$$D_{10} = \frac{c^2 n_L P}{B^2} \quad (1-20b)$$

where D_{10} is the transport coefficient for a straight cylinder and n_L is the resistivity of the plasma perpendicular to the field. Since q is of order 2 to 3 in most experiments, the particle transport coefficient can be an order of magnitude larger than that of a straight cylinder.

Shafranov [30] calculated the thermal conductivity of a toroidal plasma in the collisional regime. The order of magnitude increase in the thermal conductivity is about the same as the particle transport coefficients. Also Maschke calculated the particle transport coefficient [31] for a toroidal plasma using a fluid model. Rutherford, [32] calculated the transport coefficients without the effect of temperature gradient. A rigorous calculation of the transport coefficients is given by Hazeltine and Hinton [33].

In the banana regime Galeev and Sagdeev [34] were the first to calculate the effect of trapped particles on the transport coefficients. They concluded that the barely trapped particles make the largest contribution to the transport. Using their value of the banana width,

$$\Delta x = \frac{\rho_e q}{\sqrt{B}} \quad (1-21)$$

where ρ_e is the electron gyroradius and the effective collision frequency for trapped particle they obtain, (see Appendix B)

$$D_{\perp} = D_{\perp 0} \left(\frac{R}{a} \right)^{3/2} q^2 \quad (1-22)$$

The increase is due to two effects, first the banana width is larger than a gyroradius, and secondly the effective collision frequency is larger than the collision frequency for scattering of 90° . Rutherford [32], Kovrizhnikh [35] and Frieman [36] also have calculated the transport coefficients in the banana regime. They all use the aspect ratio expansion and neglect the effect of electron - electron collisions. Rosenbluth, Hazeltine, and Hinton [24] have calculated the transport coefficients using the aspect ratio expansion and the full Fokker-Planck collision operator. Hinton and Rosenbluth [37] have extended the calculation of reference 24 to include the plateau as well as the banana regime. Hazeltine, Hinton, and Rosenbluth [38] have extended the calculation of reference 24 to finite aspect ratio but only circular flux surfaces. Glasser and Thompson [39] have presented a method for calculating the transport coefficients in the case of general shaped flux surfaces, but still make the aspect ratio expansion.

An order of magnitude calculation of the transport coefficients in the plateau regime is given by Kadomtsev

and since [16] their result is,

$$D_L = \frac{V_{th}}{gR} g^2 \rho_e^2 \quad (1-13)$$

Note that it is independent of the collision frequency. The calculation of Hinton and Rosenbluth [27] includes the plateau and the banana regime as well as the transition region.

Figure 7 is a graph showing the dependence of the transport coefficients on the collision frequency over the three collision frequency regimes [28].

1.2. NON-CIRCULAR FLUX SURFACES

There has been interest recently in the calculation of transport coefficients in magnetic field structures in which the flux surfaces are not circular, and also flux surfaces of low aspect ratio. The reason for the interest in these field structures can be seen from the Kruskal-Schwarzschild condition [25], which can be written as,

$$q = \frac{B_T}{B_P} \frac{l_P}{l_T} > 1 \quad (1-14)$$

l_P is the circumference of the projection of the torus in the poloidal plane ($\theta = \text{constant}$). l_T is the circumference of the magnetic axis. The quantities l_P and l_T are illustrated in Figure 8 for an

Figure 7 Neoclassical transport coefficient, D_L , and straight cylinder transport coefficient, D_{L0} , vs collision frequency and the ratio of D_L and D_{L0} vs collision frequency.

elliptical cross section torus. In a torus, the plasma pressure is balanced almost entirely by the poloidal magnetic field. The toroidal field is used to maintain magnetohydrodynamic stability. For economic reasons, the large toroidal magnetic field should be reduced as much as possible. There are two ways to reduce B_T , i.e. sized B_p . If the aspect ratio can be reduced from a circular flux surface $a/R = l_p/l_T$ then B_T can be reduced. Another way to increase l_p/l_T is to elongate the flux surface in the z direction as shown in Figure 2. Rigorous calculation shows that the reduction in B_T is not as much as indicated by this simple method since q must increase to maintain stability [2], but the advantage of elliptical shaped flux surfaces still exist. Another advantage of elliptical shaped flux surfaces is that the Ohmic heating is improved [2].

Most Tokamak fusion reactors studies include the use of a poloidal divertor [27]. A divertor is a magnetic field structure on the outer edge of the plasma designed to keep the plasma from coming in contact with the wall. The magnetic flux surfaces in such a reactor are non circular [40]. Thus, in order to model such a reactor, one needs to be able to calculate the transport coefficients of a plasma in a magnetic field of noncircular flux surfaces.

The method presented in this thesis makes no specific assumption on the shape of the magnetic flux surface.

Figure 8 Torus of elliptical cross section

The thermal conductivity for an elliptic closed flux surface has been calculated by Maschke [41] in the collisional regime. The particle and energy transport coefficients have been calculated in the collisional regime by McKride et al. [42]. Strauss [43] has calculated the particle transport coefficients for an elliptical shaped cross section assuming a large aspect ratio in the banana and plateau regimes. In the plateau regime the particle transport coefficient decreased with increasing elongation in the z direction [43]. In the banana regime calculations by Strauss [43] indicate no significant changes in the particle transport coefficient due to elliptical shape. There are no verifications of these results yet.

The purpose of this work is to give a more general method for calculating transport coefficients in a Tokamak. In this work no specific assumptions are made about the flux surface shape or the aspect ratio. The only constraint on the magnetic field is that it should have nested axisymmetric flux surfaces and contain no separatrix. This will enable one to calculate the transport coefficients in the actual magnetic field of a Tokamak experiment. The transport coefficients can be calculated for various flux surface shapes in order to find the optimum shape for a Tokamak reactor. Of course, no analytic expression for the transport coefficients can be given in these complicated magnetic fields. There are no specific assumptions on collision frequency regimes in this work. One restrictive

assumption that is valid in the case of a uniform temperature. In this case, the transport equations described in terms of the total particle density and total energy density. With this assumption, the electrostatic potential is eliminated from the transport problem. The electrostatic potential which appears in each of the particle and energy transport equations is cancelled when they are added to obtain the total particle and energy transport equations.

1.4 TRIAL FUNCTIONS FOR VARIATIONAL PRINCIPLE

The transport coefficients are calculated using a variational principle similar to the one given by Kadomtsev and Bernstein [44]. This variational principle differs from others [45], [46] in that it includes in the trial function not only the distribution function, but also the component of the electric field parallel to the magnetic field. The trial functions which are used are a series of functions. The variational principle, which is performed on each flux surface, calculates the coefficients in this series. The calculation involves the inversion of a large matrix. A computer code was written to do the evaluation of the matrix elements and perform the inversion. The matrix elements contain one space integral and two velocity integrals or in the case of the collision term, five velocity integrals. In a variational principle there is a great deal of freedom in the choice of trial functions. One can either use solutions from similar problems, or use an orthogonal series with "several" terms. (Where

be made in this matter, that is, whether to use the independent variable as energy and the magnetic moment. The latter one is chosen for the trial function into a one dimensional integral part. However, in this case the space and velocity parts of the integrals in the matrix elements of the collision principle can not be separated. Thus, one would have to do a three dimensional integral in each matrix element which would most likely have to be done numerically. With the velocity as the independent variable, the matrix elements can be separated into a one dimensional space integral which must be done numerically, and two one dimensional velocity integrals which can be done analytically for a good choice of trial functions.

The orthogonal functions which are used in this work are Hermite polynomials for the dependence on the velocity parallel to the magnetic field, and Laguerre polynomials for the dependence on the perpendicular velocity. Sine and cosine are used for the space dependence around the flux surface. With this choice of trial function all of the velocity integrals in the matrix elements can be done analytically; however, the large number of terms prevents an easy integration of the collision terms. A separate computer program was written to perform this calculation.

The transport coefficient calculated in this work, and all others referred to, are not local transport

coefficients; that is, giving the flux densities at a point due to gradients of thermodynamic forces at that point. The coefficients are global transport coefficients which give the flux crossing a closed surface in terms of the gradient of quantities which are functions of ψ and time only (flux surface quantities).

The results of the calculation in the collisional regime are in close agreement with coefficients calculated from the equation for Braginskii [28]. In the banana regime the computer code does not work well; this is not due to any fault of the theory, but is due to the choice of trial functions. The distribution function in the banana regime contains large derivatives in velocity space near the boundary between the trapped and untrapped part of the distribution function [34]. It is difficult for the trial function to represent this part of the velocity space. Trial functions with as many as 88 terms were tried.

A possibility better choice of trial functions for the calculation in the banana regime are the functions used by Meier, et. al. [47]. They are combinations of spherical harmonics and Laguerre polynomials. The collision terms are much more difficult to evaluate for these functions than for the functions used in this thesis. Beasley, et.al. [48] have been successful in using these functions to expand the distribution function in the banana regime

1.5 IMPURITY TRANSPORT

Another area of active research is the effect of impurities on the transport in a Tokamak. It is well known that impurities have an undesirable effect on the Tokamak plasma [2]. Low mass impurities such as oxygen appear in most Tokamak experiments. They cause energy loss from the plasma by means of radiation from low charge states. These impurities enter the plasma from the outside and they first encounter the low temperature collisional plasma at the boundary. Therefore, the description of the effect of the impurities on the transport in the collisional regime is important.

Previous work includes the work of Rutherford [49] who calculated the transport coefficients in the collisional regime. Rutherford assumed a large aspect ratio and large impurity to ion mass ratio. Also Samain [50] performed a similar calculation using a kinetic model. Connor [51] has calculated the transport coefficients for a three species plasma in the banana regime using a large aspect ratio expansion. Hinton and Moore [52] have done a similar calculation for the banana and plateau regime.

The previous extension of the theory to include three species of particles, electrons, ions, and neutral atoms, is only an extension of the previous theory. The previously used mass diffusion coefficient is not valid for the treatment of a plasma with one species of helium and oxygen. The effect of the presence of the neutral atoms is important since it is a factor of the order of 10 in the restrictive assumption in this work, that the ratio of ion density to neutral density is constant, independent of ψ . This is a factor of 10 in the calculation of the fluxes. They can be calculated in terms of the total particle density (electrons, ions and neutrals), and the total energy transport. The intention here is to show the enhancement of the diffusion coefficient in the presence of neutral atoms.

The calculation indicated that the particle transport coefficients can be increased significantly. For example, in a plasma of 1% oxygen, the particle transport coefficient is increased by a factor of about 4. In this work the same transport parameters that are used are the gradient of $NT^{-3/2}$, the gradient of temperature, T , and the gradient of D , where D is γ times the toroidal magnetic field, B_p . Any three linear independent combinations of the three transport coefficients can be used. This choice seems to be the most convenient. The coefficients are referred to as the first, second, or third coefficient if they are the gradient of $NT^{-3/2}$, T or D respectively. In the 1% oxygen plasma, the first particle

transport coefficient in the oxygen plasma is about a factor of four larger than the coefficient of a plasma with no oxygen. The second particle coefficient and the first energy coefficient are increased by a factor of about 2.5 on the outer flux surface. The second energy transport coefficient shows very little increase due to the presence of the impurity. The other transport coefficients which depend only on the parallel resistivity are reduced by about 30% for the 1.3% oxygen plasma. The increase in the first particle transport coefficient can be interpreted as a decrease in the perpendicular resistivity. The increase in the particle transport coefficient is due to the collisional scattering of the deuterons by the heavy oxygen ions which is larger than the deuteron-electron scattering, the normal process for particle transport in a two-species plasma. The reduction in the parallel resistivity is left unexplained.

1.6 OUTLINE

In Chapter 2 the magnetic and the electric fields are described. In Chapter 3 the ordering of the terms in Boltzmann's equation is described. In Chapter 4 the solutions to the zeroth order Boltzmann equation and part of the first order Boltzmann equation are given. The second order equation is averaged over velocity space to eliminate the second order part of the distribution

function. The resulting equation for the remaining part of the first order distribution function will be solved by a variational principle.

In Chapter 4 the transport equations are derived.

In Chapter 5 the variational principle is described.

The transport coefficients are given via the variational principle. In Chapter 6 a method is given for performing the variation with linear trial functions. In Chapter 7

a procedure is given for the calculation of the time evolution of the plasma using the transport coefficients

of this thesis. Transport coefficients are given in Chapter 8 for a three species plasma in the collisional regime.

A brief description of the computer code which performs the variation is given in Appendix C.

II DESCRIPTION OF MAGNETIC AND ELECTRIC FIELDS

2.1 MAGNETIC FIELD

In this work an axisymmetric torus is considered, that is no scalars are functions of the angle θ , see figure 3. The magnetic field can be written as,

$$\underline{B} = \nabla \times \left(\psi \frac{\hat{e}_\theta}{r} \right) + \frac{D}{r} \hat{e}_\theta \quad (2-1)$$

The unit vector in the θ direction is $\hat{e}_\theta$ and $\psi \frac{\hat{e}_\theta}{r}$ is the θ component of the magnetic vector potential. The first term in the above equation is the poloidal magnetic field.

$$\underline{B}_p = \nabla \times \left(\psi \frac{\hat{e}_\theta}{r} \right) = \nabla \psi \times \frac{\hat{e}_\theta}{r} \quad (2-2)$$

The other term is the toroidal magnetic field.

$$\underline{B}_T = \frac{D}{r} \hat{e}_\theta \quad (2-3)$$

It will be shown that D is a function of ψ only. The pressure balance equation is,

$$\nabla p = \frac{\underline{J} \times \underline{B}}{c} \quad (2-4)$$

where p is the pressure, $\underline{J}$ is the current, and c is the speed of

Figure 2-1 shows that P is constant on a Ψ surface. Taking the dot product of equation (2-4) with ∇P yields,

$$\underline{B} \cdot \nabla P = 0 \quad (2-5)$$

$$\underline{J} \cdot \nabla P = 0 \quad (2-6)$$

Which means the $\underline{B}$ and $\underline{J}$ are in the surface of constant P .

A useful expression for the curl $\underline{J}$ is obtained from Ampere's Law,

$$\frac{4\pi}{c} \underline{J} = \nabla \times \underline{B} \quad (2-7)$$

Substituting equation (2-1) for $\underline{B}$ into the above equation yields;

$$\underline{J} = \frac{c}{4\pi} \left\{ -r \hat{e}_\theta \cdot \nabla \cdot \left(\frac{\nabla \Psi}{r^2} \right) + \nabla D \times \frac{\hat{e}_\theta}{r} \right\} \quad (2-8)$$

In order to show that P and D are functions of Ψ only, substitute equations (2-1) for $\underline{B}$ into equation (2-5) and equation (2-8) for $\underline{J}$ into equation (2-6)

$$\nabla \Psi \times \frac{\hat{e}_\theta}{r} \cdot \nabla P = 0 \quad (2-9)$$

$$\nabla D \times \nabla P = 0$$

(2-10)

Since P is a function of θ , due to axisymmetry, ∇P is parallel to $\nabla \Psi$ and P is a function of Ψ only. Equation (2-10) then says ∇D is parallel to ∇P and since D is a function of θ , D must be only a function of Ψ and time.

$$P = P(\Psi, t)$$

(2-11)

$$D = D(\Psi, t)$$

(2-12)

In the problem considered here the flux surfaces are surfaces of constant Ψ , form nested tori as shown in figure (9). Ψ is a measure of the amount of poloidal magnetic flux between two flux surfaces. As an example consider the poloidal flux through the surface shown in figure (10). The poloidal flux through the surface is:

$$\Gamma_p = \int d\mathbf{A} \cdot \mathbf{B}_p$$

Where the integral is over the flux surface shown in the figure. Using the equation (2-2) for $\mathbf{B}_p$ and Stokes's law, the poloidal flux is;

$$\Gamma_p = 2\pi(\Psi_2 - \Psi_1)$$

(2-14)

Figure 9 Nested Flux Surfaces

Figure 10 Poloidal Magnetic Flux

where ψ_1 and ψ_2 are the values of ψ on the circuits (1) and (2) in figure (10), respectively.

Another equation for the current is obtained from the pressure balance equation,

$$J = bJ_{||} + \frac{c b \times \nabla P}{B} \quad (2-15)$$

where $J_{||}$ is the component of the current parallel to the unit vector in the direction of the B field, b . By substituting equation (2-15) in the left hand side of equation (2-8) and taking the dot product of the resulting equation with $\nabla \psi \times \hat{e}_\theta$ one can obtain an equation for $J_{||}$

$$J_{||} = \frac{c B}{4\pi} \frac{dD}{d\psi} + \frac{c D}{B} \frac{dP}{d\psi} \quad (2-16)$$

Also, a partial differential equation for the ψ function is obtained by taking the scalar product of equation (2-15) and (2-8) with $\hat{e}_\theta$ and combining,

$$\hat{\nabla} \cdot \left(\frac{\nabla \psi}{r^2} \right) = -4\pi \frac{dP}{d\psi} - \frac{D}{r^2} \frac{dD}{d\psi} = -\frac{4\pi}{rc} J_\theta \quad (2-17)$$

where J_θ is the θ component of the current.

2.2 ELECTRIC FIELD

The form of the electric field will now be given.

The electric field consist of two parts, the lowest order part is first order in ϵ and is determined by an electrostatic potential.

$$\mathbf{E}^{(1)} = -\nabla\Phi \quad (2-1)$$

Only the inductive part of the second order part of the electric field is needed in this problem, the second order part of the electrostatic potential is not needed. The inductive component of the electric field is determined by Maxwell's equation,

$$\nabla \times \mathbf{E}^{(2)} = \frac{\partial \mathbf{B}}{\partial t} \quad (2-2)$$

An expression for $\mathbf{E}_\theta^{(2)}$, the theta component of the electric field, is obtained by substituting equation (2-1)

for the magnetic field into equation (2-19), and taking the dot product with $\nabla\psi$, the result is,

$$E_{\theta}^{(1)} = \frac{-1}{cr} \frac{\partial\psi}{\partial t} \quad (2-20)$$

The partial derivative of ψ in the above equation can be given in terms of the flux surface velocity, $\underline{V}$. The total time derivative moving with the flux surface is defined by,

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \underline{V} \cdot \nabla\psi = 0 \quad (2-21)$$

therefore,

$$\underline{V} = - \frac{\nabla\psi}{|\nabla\psi|^2} \frac{\partial\psi}{\partial t} \quad (2-22)$$

A relation which will be used later will now be given. The quantity to be calculated is $(\underline{E} \times \underline{B}) \cdot \nabla\psi$.

Using the following equation,

$$(\underline{B} \times \underline{\nabla} \Psi) = r B^2 \hat{e}_\theta - D \underline{B} \quad (2-23)$$

and using equation (2-20) it can be shown that,

$$(\underline{E}^{(2)} \times \underline{B}) \cdot \underline{\nabla} \Psi = - D \underline{E}^{(2)} \cdot \underline{B} - \frac{B^2}{c} \frac{\partial \Psi}{\partial t} \quad (2-24)$$

This equation can be written in terms of the $\underline{E} \times \underline{B}$ drift and the flux surface velocity as,

$$(\underline{V} - \underline{v}_E) \cdot \underline{\nabla} \Psi = \frac{D c}{B^2} \underline{E}^{(2)} \cdot \underline{B} \quad (2-25)$$

where,

$$\underline{v}_E = c \frac{\underline{E}^{(2)} \times \underline{B}}{B^2} \quad (2-26)$$

Equations (2-24) and (2-25) will be used in Chapter 4 in the derivation of the transport equations.

III KINETIC EQUATION

3.1 BOLTZMANN'S EQUATION

In this section the linearization of the Boltzmann's equation is presented. This is done by an investigation of the magnitude of various terms in Boltzmann's equation. A solution of the zeroth order equation and part of the first order equation will be derived. A differential equation for the remaining part of the first order solution will be derived. The transport coefficients will be obtained from this equation by a variational principle given in Chapter 5.

The Boltzmann's equation for species α is,

$$\frac{\partial f_{\alpha}}{\partial t} + \mathbf{v} \cdot \nabla f_{\alpha} + \frac{Z_{\alpha} e}{m_{\alpha}} (\mathbf{E}^{(2)} - \nabla \Phi) \cdot \nabla_{\mathbf{v}} f_{\alpha} + \frac{Z_{\alpha} e}{m_{\alpha}} \frac{\mathbf{v} \times \mathbf{B}}{c} \cdot \nabla_{\mathbf{v}} f_{\alpha} = \left(\frac{\partial f_{\alpha}}{\partial t} \right)_c \quad (3-1)$$

Where f_{α} is the distribution function for the α species, $\mathbf{v}$ is the velocity variable, Z_{α} is the atomic number of the species (-1 for electrons), e is the charge on the electron, m_{α} is the mass of the α species. Landau's collision integral will be used for the collision term,

$$\left(\frac{\partial f_{\alpha}}{\partial t} \right)_c = - \nabla_{\mathbf{v}} \cdot (\mathbf{j}_{\alpha\alpha} + \mathbf{j}_{\alpha\beta}) \quad (3-2)$$

where,

$$\mathbf{j}_{\alpha\alpha} = \frac{2\pi \ln \Lambda e^4 Z_{\alpha}^4}{m_{\alpha}} \int d^3v' \nabla_{\mathbf{v}} \nabla_{\mathbf{v}'} |\mathbf{v} - \mathbf{v}'| \cdot \left(\frac{f_{\alpha}}{m_{\alpha}} \nabla_{\mathbf{v}'} f_{\alpha}' - \frac{f_{\alpha}'}{m_{\alpha}} \nabla_{\mathbf{v}} f_{\alpha} \right) \quad (3-3a)$$

$$j_{\alpha\beta} = \frac{2\pi m \Lambda^4 Z_\alpha^2 Z_\beta^2}{m_\alpha} \int d^3v \nabla_v \nabla_v |v-v'| \cdot \left(\frac{f_\alpha}{m_\beta} \nabla_v f'_\beta - \frac{f'_\beta}{m_\alpha} \nabla_v f_\alpha \right) \quad (3-3)$$

This form of the collision term conserves momentum, that is,

$$\int d^3v \frac{m_\alpha v}{2} \left(\frac{\partial f_\alpha}{\partial t} \right)_c + \int d^3v \frac{m_\beta v}{2} \left(\frac{\partial f_\beta}{\partial t} \right)_c = 0 \quad (3-4)$$

The energy transfer due to collisions between two species having a Maxwellian distribution function is,

$$\int d^3v \frac{m_\alpha v^2}{2} \left(\frac{\partial f_\alpha}{\partial t} \right)_c = - \int d^3v \frac{m_\beta v^2}{2} \left(\frac{\partial f_\beta}{\partial t} \right)_c = \frac{4\pi Z_\alpha^2 Z_\beta^2 e^4 N_\alpha N_\beta}{m_\beta T_\alpha} (T_\beta - T_\alpha) \left(\frac{m_\alpha}{2\pi T_\alpha} \right)^{1/2} \quad (3-5)$$

N_α is the particle density of the species α defined by the following integral.

$$N_\alpha = \int d^3v f_\alpha \quad (3-6)$$

and T_α is the temperature of the species α defined by,

$$T_\alpha = \frac{1}{N_\alpha} \int d^3v \frac{m_\alpha v^2}{3} f_\alpha \quad (3-7)$$

In the case considered in this work, the two species have equal temperature.

$$T_\alpha = T_\beta = T \quad (3-8)$$

Thus the collisional energy transfer due to the lowest order part of the distributions is zero.

EXPANSION OF BOLZMANN'S EQUATION

In order to make an estimation of the magnitude of the terms in Boltzmann's equation, it is convenient to make a temporary change of variables to a dimensionless time, space, and velocity coordinate, these are

$$t = \tau / S \quad (3-8a)$$

$$x = \xi L \quad (3-8b)$$

$$v = v_e \underline{v} \quad (3-8c)$$

where S is the characteristic frequency for change of microscopic quantities, L is the scale length of the plasma, or, equivalently, the minor radius, and v_e is the thermal velocity.

$$v_e = \sqrt{\frac{2T}{m}} \quad (3-9)$$

The collision term will be taken to be of the order of magnitude of the collision frequency ν , times the deviation from equilibrium.

$$C(f) = \nu(f - f_0) \quad (3-10)$$

where f_0 is the Maxwellian distribution function. Boltzmann's equation now becomes

$$\frac{\partial f}{\partial \tau} + \underline{v} \cdot \nabla_{\underline{v}} f + \frac{\partial f}{\partial \tau} = \nu(f - f_0) \quad (3-11)$$

The subscript α on f , m , and N , has been omitted in this section. This equation can be rewritten as,

$$\frac{1}{2} \nabla^2 f + \frac{1}{2} \nabla^2 f + \frac{e}{4} \nabla^2 f - \frac{e}{4} \nabla^2 f + \frac{1}{2} \nabla^2 f = \frac{N}{\omega} (f'' - f) \quad (1-12)$$

The gyroradius, a , is given by,

$$a = \frac{v_e}{\omega} \quad (1-13)$$

The gyrofrequency, ω , is given by,

$$\omega = \frac{ZeB}{mc} \quad (1-14)$$

An example of the values of the dimensionless ratios which appear in equation (1-12) will now be given. Consider a deuterium plasma of the following parameters; temperature of 2 kev, electron density of $2 \times 10^{15} \text{ cm}^{-3}$, magnetic field of 10^4 Gauss, and a minor radius of 100 cm. The gyroradii for the electrons and ions are respectively,

$$a_e = 1.5 \times 10^{-2} \text{ cm} \quad (1-15a)$$

$$a_i = 8.6 \times 10^{-1} \text{ cm} \quad (1-15b)$$

The electron and ion collision frequencies are respectively,

$$\nu_e = 1.0 \times 10^8 \text{ sec}^{-1} \quad (1-16a)$$

$$\nu_i = 1.2 \times 10^6 \text{ sec}^{-1} \quad (1-16b)$$

The electron and ion gyrofrequencies are respectively,

$$\omega_{ce} = 1.8 \times 10^9 \text{ sec}^{-1} \quad (3-17a)$$

$$\omega_{ci} = 4.8 \times 10^7 \text{ sec}^{-1} \quad (3-17b)$$

The electron and ion thermal velocities are respectively,

$$v_{te} = 2.7 \times 10^9 \text{ cm/sec} \quad (3-18a)$$

$$v_{ti} = 4.4 \times 10^8 \text{ cm/sec} \quad (3-18b)$$

The values of some of the ratios which appear in equation (3-12) are,

$$\omega_{ce}/L = 1.5 \times 10^{-4} \quad (3-19a)$$

$$\omega_{ci}/L = 8.6 \times 10^{-3} \quad (3-19b)$$

$$v_{te}/c = 9.0 \times 10^{-2} \quad (3-20a)$$

$$v_{ti}/c = 1.5 \times 10^{-3} \quad (3-20b)$$

$$\omega_{ce}/\omega_{pe} = 5.8 \times 10^{-4} \quad (3-21a)$$

$$\omega_{ci}/\omega_{pi} = 2.5 \times 10^{-2} \quad (3-21b)$$

Also the ratios of the gyrofrequencies to the plasma frequencies,

ω_p , are,

$$\omega_{ce}/\omega_{pe} = 5.3 \times 10^{-2} \quad (3-22a)$$

$$\omega_{ci}/\omega_{pi} = 4.1 \times 10^{-3} \quad (3-22b)$$

The plasma frequency is given by,

$$\omega_p = \sqrt{\frac{4\pi n N (ze)^2}{m}} \quad (3-23)$$

The values of these ratios suggest the following ordering, where ϵ is a small number.

$$q/L \approx \epsilon \quad (3-24)$$

$$v/R \approx \epsilon \quad (3-25)$$

$$v/c \approx \epsilon \quad (3-26)$$

$$R/\omega_p \approx \epsilon \quad (3-27)$$

In this ordering β , which is the ratio of the plasma pressure to the magnetic field pressure, is approximately,

$$\beta \approx \left(\frac{\omega_p}{2}\right)^2 \left(\frac{v}{c}\right)^2 \approx 1 \quad (3-28)$$

Thus this ordering is correct for modeling Tokamaks of finite beta.

An estimation is needed for S/R , which appears in the first term in equation (3-12). Assuming that the macroscopic quantities change by a diffusion process, then,

$$1/S \approx \frac{1}{a^2 \nu} L^2 \quad (3-30a)$$

or,

$$S/R \approx \frac{\nu}{R} \frac{a^2}{L^2} \quad (3-30b)$$

Using equation (3-24) and (3-25) this ratio is given the following ordering.

$$S/R \approx \epsilon^3 \quad (3-31)$$

An estimate of the ratio of the inductive electric field, $E^{(2)}$, to the magnetic field, B , is needed. An approximation of Maxwell's equation (2-18) is,

$$c \frac{E^{(2)}}{L} \approx -\dot{B} \quad (3-32)$$

Using the previously assigned ordering this ratio becomes,

$$\frac{E}{B} \approx \epsilon \quad (3-33)$$

An estimate of the ratio of the electrostatic field, $|\nabla\Phi|$, to the magnetic field is needed. If one equates the energy density in the electrostatic field to the kinetic energy density of the plasma, the following results are obtained.

$$|\nabla\Phi| \approx \sqrt{N m} v_t \quad (3-34)$$

This is, of course, an over estimation of the electrostatic field, since the largest part of the potential drop occurs across a thin sheath at the edge of the plasma. The thickness of this sheath is approximately the Debye length, λ_D .

$$\lambda_D = \frac{v_t}{\omega_p} = \sqrt{\frac{T}{4\pi N (Ze)^2}} \quad (3-35)$$

The potential drop across the main body of the plasma will be smaller than the total potential drop by a factor of about λ_D/L . Thus the approximate value of $|\nabla\Phi|$ is,

$$|\nabla\Phi| = \sqrt{N m} v_t \lambda_D / L \quad (3-36)$$

The needed ratio can be written as,

$$\frac{|\nabla\Phi|}{B} \approx \left(\frac{\omega_p}{\omega_c}\right)\left(\frac{v_t}{c}\right)\left(\frac{\lambda_D}{L}\right) \quad (3-37)$$

An estimation of the last factor in this equation can be obtained from,

$$\frac{a}{L} = \frac{v_t}{L\omega_c} = \frac{1}{L} \frac{v_t}{\omega_p} \frac{\omega_p}{\omega_c} = \frac{\lambda_D}{L} \frac{\omega_p}{\omega_{te}} \quad (3-38)$$

therefore,

$$\frac{\lambda_D}{L} = \frac{a}{L} \frac{\omega_c}{\omega_p} \approx \epsilon^2 \quad (3-39)$$

Using these results equation (3-37) can be written as,

$$\frac{|\nabla\Phi|}{B} \approx \epsilon^2 \quad (3-40)$$

Substituting equations (3-24), (3-25), (3-26), (3-27), (3-31), (3-33) and (3-40) into Boltzmann's equation (3-12) it becomes,

$$\epsilon \frac{\partial f}{\partial t} + \epsilon \mathbf{v} \cdot \nabla_{\mathbf{r}} f + \epsilon \frac{\mathbf{E} \cdot \nabla_{\mathbf{v}} f}{|E|} - \epsilon \frac{\nabla\Phi}{|\nabla\Phi|} \cdot \nabla_{\mathbf{v}} f + \frac{1}{2} \times \frac{1}{2} \cdot \nabla_{\mathbf{v}} f = \epsilon (f^{(0)} - f) \quad (3-41)$$

Thus an estimation of each term in Boltzmann's equation has been obtained.

3.3 SOLUTIONS OF THE ZEROth AND FIRST ORDER BOLTZMANN'S EQUATION

The distribution will now be written as an expansion in ϵ ,

$$f = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} \quad (3-42)$$

Equations (3-42) and (3-31) are substituted into equation (3-41), then by equating like powers of ϵ for the first three powers of ϵ , three equations are obtained. The zeroth equation is,

$$\frac{Z_A e}{m_A} \frac{\mathbf{v} \times \mathbf{B}}{c} \cdot \nabla_{\mathbf{v}} f_A^{(0)} = 0 \quad (3-43)$$

The 1st order equation is,

$$\mathbf{v} \cdot \nabla f_A^{(0)} - \frac{Z_A e}{m_A} \nabla \Phi \cdot \nabla_{\mathbf{v}} f_A^{(0)} + \frac{Z_A e}{m_A} \frac{\mathbf{v} \times \mathbf{B}}{c} \cdot \nabla_{\mathbf{v}} f_A^{(0)} = \left(\frac{\partial f_A^{(0)}}{\partial t} \right)_c \quad (3-44)$$

and the 2nd order equation is,

$$\mathbf{v} \cdot \nabla f_A^{(1)} - \frac{Z_A e}{m_A} \nabla \Phi \cdot \nabla_{\mathbf{v}} f_A^{(1)} + \frac{Z_A e}{m_A} \mathbf{E} \cdot \nabla_{\mathbf{v}} f_A^{(0)} + \frac{Z_A e}{m_A} \frac{\mathbf{v} \times \mathbf{B}}{c} \cdot \nabla_{\mathbf{v}} f_A^{(2)} = \left(\frac{\partial f_A^{(1)}}{\partial t} \right)_c \quad (3-45)$$

The velocity coordinates that are used are illustrated in figure 11a and 11b. The velocity variable is divided into a component parallel to the magnetic field, $u_{\underline{b}}$, and a component perpendicular to the magnetic field, $\underline{w}$.

$$\underline{v} = u_{\underline{b}} + \underline{w} \quad (3-46)$$

$$u = \underline{v} \cdot \underline{b} \quad (3-47)$$

The perpendicular velocity is written as,

$$\underline{w} = w [\hat{e}_1 \cos \phi + \hat{e}_2 \sin \phi] \quad (3-48)$$

where w is the magnitude of $\underline{w}$. The unit vectors

$\hat{e}_1$, $\hat{e}_2$ and the angle ϕ are defined by the following equations and illustrated in figure 11a and 11b.

$$\tan \phi = \frac{\hat{e}_2 \cdot \underline{v}}{\hat{e}_1 \cdot \underline{v}} \quad (3-49)$$

$$\hat{e}_1 = \frac{\nabla \psi}{|\nabla \psi|} \quad (3-50)$$

$$\hat{e}_2 = \frac{\underline{b} \times \nabla \psi}{|\nabla \psi|} \quad (3-51)$$

Figure 11a Velocity coordinate system

Figure 11b Velocity $\underline{w}$ and angle ϕ

Making the change to this velocity system the kinetic equations (3-43), (3-44) and (3-45) become,

Zeroth order

$$\frac{\underline{w} \times \underline{B}}{c} \cdot \frac{\partial f_a^{(0)}}{\partial \underline{w}} = 0 \quad (3-52)$$

First order

$$\underline{v} \cdot \frac{\partial f_a^{(1)}}{\partial \underline{x}} - \frac{Z_a e}{m_a} \underline{\nabla \Phi} \cdot \underline{b} \frac{\partial f_a^{(1)}}{\partial u} + \frac{Z_a e}{m_a} \frac{\underline{v} \times \underline{B}}{c} \cdot \frac{\partial f_a^{(1)}}{\partial \underline{w}} = 0 \quad (3-53)$$

Second order

$$\begin{aligned} \underline{v} \cdot \left(\frac{\partial f_a^{(1)}}{\partial \underline{x}} + \frac{\partial f_a^{(1)}}{\partial u} \underline{v} \cdot \underline{\nabla b} - u \frac{\partial \underline{E}^{(1)}}{\partial \underline{w}} \cdot \underline{\nabla b} \right) + \frac{Z_a e}{m_a} \underline{E}^{(1)} \cdot \underline{b} \frac{\partial f_a^{(1)}}{\partial u} \\ - \frac{Z_a e}{m_a} \underline{\nabla \Phi} \cdot \frac{\partial f_a^{(1)}}{\partial \underline{w}} + \frac{Z_a e}{m_a} \frac{\underline{v} \times \underline{B}}{c} \cdot \frac{\partial f_a^{(2)}}{\partial \underline{w}} = \left(\frac{\partial f_a^{(1)}}{\partial t} \right)_c \end{aligned} \quad (3-54)$$

The 0th order equation becomes,

$$\frac{\partial f_a^{(0)}}{\partial \phi} = 0 \quad (3-55)$$

Therefore, $f_a^{(0)}$ is a function of u and w^2 only.

$$f_a^{(0)} = f_a(u, w^2) \quad (3-56)$$

one ordering is such that the collision frequency satisfies equation (3-25). In order that this ordering be complied with, the zeroth order distribution function must satisfy the following equation:

$$\left(\frac{\partial f_a^{(0)}}{\partial t} \right)_c = 0 \quad (3-57)$$

If $f_a^{(0)}$ satisfies the following equation:

$$\frac{f_a^{(0)}}{m_a} \nabla_v f_a^{(0)'} - \frac{f_a^{(0)'}}{m_a} \nabla_v f_a^{(0)} = (\text{scalar}) X_v (v - v') \quad (3-58a)$$

$$\frac{f_a^{(0)'}}{m_\beta} \nabla_v f_\beta^{(0)'} - \frac{f_\beta^{(0)'}}{m_\alpha} \nabla_v f_a^{(0)} = (\text{scalar}) X (v - v') \quad (3-58b)$$

and since,

$$\nabla_v \nabla_v |v - v'| \cdot (v - v') = \frac{I(v - v')^2 - (v - v')(v - v') \cdot (v - v')}{|v - v'|^3} \cdot (v - v') = 0 \quad (3-59)$$

the collision operator given by equations (3-2a), (3-2b), and (3-2c) will be zero, thus satisfying equation (3-57).

Equations (3-53a) and (3-53b), are satisfied if,

$$\nabla f_{\alpha}^{(0)} = (\text{constant}) m_{\alpha} \nabla f_{\alpha}^{(0)} \quad (3-60)$$

The solution to this equation is a "Maxwellian."

$$f_{\alpha}^{(0)} = N_{\alpha} \left(\frac{m_{\alpha}}{2\pi T} \right)^{3/2} e^{-\frac{m_{\alpha}}{2T} v^2} \quad (3-61)$$

Equations (3-6) and (3-7) have been used to determine the factors in the above equation.

The first order equation can be written as,

$$f_{\alpha}^{(0)} \{b u + w\} \cdot \left\{ \nabla \ln N_{\alpha} + \frac{3}{2} \nabla \ln T + \frac{m_{\alpha}}{2T} (u^2 + w^2) \nabla \ln T \right\} + \quad (3-62)$$

$$\frac{Z_{\alpha} e}{T} f_{\alpha}^{(0)} (b u + w) \cdot \nabla \Phi = \frac{Z_{\alpha} e B}{m_{\alpha} c} \frac{\partial f_{\alpha}^{(1)}}{\partial \phi}$$

The requirement that $f_a^{(u)}$ be a single valued function of ϕ leads to the following equation.

$$b \cdot \left\{ \nabla \ln N + \frac{3}{2} \nabla \ln T + \frac{m_a}{2T} (u^2 + w^2) \nabla \ln T \right\} + \frac{Z_{ac}}{T} b \cdot \nabla \Phi = 0 \quad (3-63)$$

This equation must be true for any values of u and w . It will be satisfied for all u and w if all three of the following equations are satisfied.

$$b \cdot \nabla \ln N = 0 \quad (3-64)$$

$$b \cdot \nabla \ln T = 0 \quad (3-65)$$

$$b \cdot \nabla \Phi = 0 \quad (3-66)$$

Using the equation for b equation (3-64) becomes,

$$\nabla \Phi \cdot (\nabla N \times \nabla \Psi) = 0 \quad (3-67)$$

Or since ∇N and $\nabla \Psi$ do not have any ϕ component equation (3-67) becomes.

$$\nabla N \times \nabla \Psi = 0 \quad (3-68)$$

So N must be a function of Ψ and t only, and similarly for T and Φ .

$$N = N(\Psi, t) \quad (3-69)$$

$$T = T(\Psi, t) \quad (3-70)$$

$$\Phi = \Phi(\psi, t) \quad (3-71)$$

Using these results and,

$$\underline{v} \cdot \nabla \Psi = w |\nabla \Psi| \cos \phi \quad (3-72)$$

the first order equation becomes,

$$N_\alpha \frac{\partial f_\alpha^{(1)}}{\partial \phi} = f_\alpha^{(0)} |\nabla \Psi| w \cos \phi \left\{ \frac{\partial \ln N_\alpha}{\partial \psi} - \frac{3}{2} \frac{\partial \ln T}{\partial \psi} + \frac{m_\alpha}{2T} v^2 \frac{\partial \ln T}{\partial \psi} + \frac{Z_\alpha e}{T} \frac{\partial \Phi}{\partial \psi} \right\} \quad (3-73)$$

The solution to this equation is,

$$f_\alpha^{(1)} = f_\alpha^{(0)} \chi_{\alpha c} + f_\alpha^{(0)} \chi_\alpha \quad (3-74)$$

The first term on the right hand side of this equation is the integral of equation (3-73).

$$\chi_{\alpha c} = \frac{w |\nabla \Psi|}{N_\alpha} \sin \phi \left\{ \frac{\partial \ln N_\alpha}{\partial \psi} - \frac{3}{2} \frac{\partial \ln T}{\partial \psi} + \frac{m_\alpha}{2T} v^2 \frac{\partial \ln T}{\partial \psi} + \frac{Z_\alpha e}{T} \frac{\partial \Phi}{\partial \psi} \right\} \quad (3-75)$$

The second term on the right hand side of equation (3-74) is an integration constant which is independent of ϕ .

$$\chi_\alpha = \chi_\alpha(u, w) \quad (3-76)$$

An equation for χ_α will be given in the next section.

Equation (3-75) can also be written as,

$$\chi_{\alpha c} = \underline{v} \cdot d_\alpha = \underline{w} \cdot d_\alpha \quad (3-77)$$

where the vector d_α is given by,

$$d_\alpha = \hat{e}_2 \frac{|\nabla \Psi|}{N_\alpha} \left\{ \frac{\partial \ln N_\alpha}{\partial \psi} - \frac{3}{2} \frac{\partial \ln T}{\partial \psi} + \frac{m_\alpha}{2T} v^2 \frac{\partial \ln T}{\partial \psi} + \frac{Z_\alpha e}{T} \frac{\partial \Phi}{\partial \psi} \right\} \quad (3-78)$$

3.4 DERIVATION OF THE EQUATION FOR χ_a

An equation for χ_a can be obtained from the 2nd order kinetic equation. This is done by averaging the 2nd order kinetic equation over the velocity angle ϕ and taking advantage of the single valued nature of $f_a^{(2)}$.

The 2nd order kinetic equation after putting in the expression for $f_a^{(1)}$ becomes,

$$\begin{aligned} & \underline{v} \cdot \left\{ \frac{\partial(f_a^{(2)} \chi_a)}{\partial \underline{x}} + \frac{\partial(f_a^{(2)} \chi_a)}{\partial u} \underline{v} \cdot \frac{\partial \underline{b}}{\partial \underline{x}} - u \frac{\partial(f_a^{(2)} \chi_a)}{\partial \underline{w}} \cdot \frac{\partial \underline{b}}{\partial \underline{x}} + \underline{v} \cdot \frac{\partial(f_a^{(2)} \chi_a)}{\partial \underline{x}} \right\} + \\ & \frac{\underline{z}_a e}{m_a} E^{(2)} \frac{\partial f_a^{(2)}}{\partial u} - \frac{\underline{z}_a e}{m_a} \underline{v} \cdot \underline{\nabla} \underline{b} \cdot \frac{\partial f_a^{(1)}}{\partial \underline{w}} + \frac{\underline{z}_a e}{m_a} E^{(1)} \frac{\partial f_a^{(2)}}{\partial \underline{w}} - \chi_a \frac{\partial f_a^{(2)}}{\partial \phi} = \frac{\partial f_a^{(2)}}{\partial t} \end{aligned} \quad (3-80)$$

The following symbol will be used for the ϕ average of any function g .

$$\langle g \rangle_\phi = \frac{1}{2\pi} \int_0^{2\pi} g d\phi \quad (3-81)$$

The average of the 1st term in equation (3-80) is

$$\langle \underline{v} \cdot \underline{\nabla}(f_a^{(2)} \chi_a) \rangle_\phi = \langle u \underline{b} \cdot \underline{\nabla} f_a^{(2)} \chi_a \rangle_\phi + \langle \underline{w} \cdot \underline{\nabla} f_a^{(2)} \chi_a \rangle_\phi \quad (3-82)$$

$f_a^{(2)} \chi_a$ is independent of ϕ and the 2nd term on the right hand side averaged to zero since $\underline{w}$ contains $\sin \phi$ and $\cos \phi$. Therefore,

$$\langle \underline{v} \cdot \underline{\nabla}(f_a^{(2)} \chi_a) \rangle_\phi = u f_a^{(2)} \underline{b} \cdot \underline{\nabla} \chi_a \quad (3-83)$$

The 2nd term in equation (3-80) is,

$$\left\langle \frac{\partial(f_a^{(0)}\chi_a)}{\partial u} \cdot \underline{v} \cdot \frac{\partial \underline{b}}{\partial \underline{x}} \cdot \underline{v} \right\rangle_\phi = \frac{\partial f_a^{(0)}\chi_a}{\partial u} \omega^2 \underline{\hat{e}}_1 \cdot \underline{\nabla} \underline{b} \cdot \underline{\hat{e}}_1 \langle \cos^2 \phi \rangle_\phi + \frac{\partial(f_a^{(0)}\chi_a)}{\partial u} \omega^2 \underline{\hat{e}}_2 \cdot \underline{\nabla} \underline{b} \cdot \underline{\hat{e}}_2 \langle \sin^2 \phi \rangle_\phi \quad (3-84)$$

Since $\underline{b} \cdot \underline{\nabla} \underline{b} \cdot \underline{b} = 0$,

$$\underline{\hat{e}}_1 \cdot \underline{\nabla} \underline{b} \cdot \underline{\hat{e}}_1 + \underline{\hat{e}}_2 \cdot \underline{\nabla} \underline{b} \cdot \underline{\hat{e}}_2 = \underline{\underline{I}} : \underline{\nabla} \underline{b} = \underline{\nabla} \cdot \underline{b} \quad (3-85)$$

where $\underline{\underline{I}}$ is the unit tensor. Equation (3-84) now becomes,

$$\left\langle \frac{\partial(f_a^{(0)}\chi_a)}{\partial u} \cdot \underline{v} \cdot \frac{\partial \underline{b}}{\partial \underline{x}} \cdot \underline{v} \right\rangle_\phi = f_a^{(0)} \left\{ \frac{\partial \chi_a}{\partial u} - \frac{m_a}{T} u \chi_a \right\} \underline{\nabla} \cdot \underline{b} \frac{\omega^2}{2} \quad (3-86)$$

By a similar procedure the 3rd term in equation (53) becomes,

$$\left\langle u \frac{\partial(f_a^{(0)}\chi_a)}{\partial w} \cdot \underline{\nabla} \underline{b} \cdot \underline{v} \right\rangle_\phi = f_a^{(0)} \left\{ \frac{\partial \chi_a}{\partial w} - \frac{m_a}{T} w \chi_a \right\} \underline{\nabla} \cdot \underline{b} \frac{uw}{2} \quad (3-87)$$

The 4th term in equation (3-80), after averaging becomes,

$$\left\langle \underline{v} \cdot \underline{\nabla} (f_a^{(0)} \underline{d}_a) \cdot \underline{v} \right\rangle_\phi = f_a^{(0)} \left\{ (u^2 - \frac{\omega^2}{2}) \underline{b} \cdot \underline{\nabla} \underline{d} \cdot \underline{b} + \frac{\omega^2}{2} \underline{\nabla} \cdot \underline{d} \right\} \quad (3-88)$$

Since $\underline{d} \cdot \underline{b} = 0$, it can be shown that,

$$\underline{\nabla} \underline{d}_a \cdot \underline{b} = -\underline{\nabla} \underline{b} \cdot \underline{d}_a \quad (3-89)$$

$$\nabla \cdot \underline{d}_\alpha = -\underline{d}_\alpha \cdot \nabla \ln B - \underline{b} \cdot \nabla \underline{b} \cdot \underline{d}_\alpha \quad (3-90)$$

Using the above results, the 4th term in equation (3-80) becomes,

$$\langle \underline{v} \cdot \nabla (f_\alpha^{(0)} \underline{d}_\alpha) \cdot \underline{v} \rangle_\phi = -f_\alpha^{(0)} \left\{ u^2 \underline{b} \cdot \nabla \underline{b} + \frac{w^2}{2} \nabla \ln B \right\} \cdot \underline{d}_\alpha \quad (3-91)$$

The 5th term in equation (3-80) is independent of ϕ , therefore,

$$\left\langle \frac{Z_{\alpha e}}{m_\alpha} \underline{E} \cdot \underline{b} \frac{\partial f_\alpha^{(0)}}{\partial u} \right\rangle_\phi = -\frac{Z_{\alpha e}}{T} u f_\alpha^{(0)} \underline{E} \cdot \underline{b} \quad (3-92)$$

The 6th term in equation (3-80) after using equation (3-71) for $\nabla \Phi$ becomes,

$$\frac{Z_{\alpha e}}{m_\alpha} \nabla \Phi \cdot \frac{\partial f_\alpha^{(0)}}{\partial \underline{w}} = \frac{Z_{\alpha e}}{m_\alpha} \frac{\partial \Phi}{\partial \psi} \left\{ \cos \phi \frac{\partial}{\partial w} (f_\alpha^{(0)} \chi_\alpha + f_\alpha^{(0)} \underline{w} \cdot \underline{d}_\alpha) - \frac{\sin \phi}{w} \frac{\partial}{\partial \phi} (f_\alpha^{(0)} \chi_\alpha + f_\alpha^{(0)} \underline{w} \cdot \underline{d}_\alpha) \right\} \quad (3-93)$$

Remembering that $\underline{w} \cdot \underline{d}_\alpha$ contains a $\sin \phi$ factor, the ϕ dependence of the four terms on the right hand side of equation (3-93) are $\sin \phi$, $\cos \phi \sin \phi$, $\sin \phi \cos \phi$ and $\sin \phi \cos \phi$, all of which average to zero, therefore,

$$\left\langle \frac{Z_{\alpha e}}{m_\alpha} \nabla \Phi \cdot \frac{\partial f_\alpha^{(0)}}{\partial \underline{w}} \right\rangle_\phi = 0 \quad (3-94)$$

The 7th term in equation (3-80) also averages to zero since $f_{\alpha}^{(1)}$ is independent of ϕ and the w contains a $\sin\phi$ and $\cos\phi$.

$$\left\langle \frac{Ze}{m_{\alpha}} \mathbb{E} \cdot \frac{\partial f_{\alpha}^{(1)}}{\partial \mathbf{w}} \right\rangle_{\phi} = 0 \quad (3-95)$$

The last term on the left hand side of equation (3-80) is zero due to the single valued nature of $f_{\alpha}^{(1)}$.

$$\int_0^{2\pi} \frac{\partial f_{\alpha}^{(1)}}{\partial \phi} d\phi = f_{\alpha}^{(1)}(0) - f_{\alpha}^{(1)}(2\pi) = 0 \quad (3-96)$$

The ϕ averaged 2nd order kinetic equation can now be written

as.

$$f_{\alpha}^{(2)} \left\{ u_b \cdot \nabla \chi_a + \frac{w}{2} \left(u \frac{\partial \chi_a}{\partial u} - u \frac{\partial \chi_a}{\partial w} \right) \cdot \nabla \cdot \mathbf{b} - d_{\alpha} \left(u_b^2 \cdot \nabla b + \frac{w^2}{2} \nabla \ln b \right) - \frac{Ze}{T} \mathbf{e} \cdot \mathbb{E} \cdot \mathbf{b} \right\} = \left\langle \frac{\partial f_{\alpha}^{(1)}}{\partial t} \right\rangle_{\phi} \quad (3-97)$$

Introducing some new notation equation (3-97) can be written as,

$$H_{\alpha} \chi_{\alpha} + K_{\alpha\alpha} \chi_{\alpha} + K_{\alpha\beta} \chi_{\beta} = h_{\alpha N} \left\{ \frac{\partial \ln N}{\partial \Psi} - \frac{3}{2} \frac{\partial \ln T}{\partial \Psi} + \frac{Z_{\alpha} e}{T} \frac{\partial \Phi}{\partial \Psi} \right\} + h_{\alpha T} \frac{\partial \ln T}{\partial \Psi} - C_{\alpha} E_{\alpha}^{(1)} \quad (3-98)$$

$$H_{\alpha} \chi_{\alpha} = f_{\alpha}^{(0)} \left\{ u_{\beta} \cdot \nabla \chi_{\alpha} + \frac{w}{2} \left(w \frac{\partial \chi_{\alpha}}{\partial u} - u \frac{\partial \chi_{\alpha}}{\partial w} \right) \nabla \cdot \mathbf{b} \right\} \quad (3-99)$$

$$K_{\alpha\alpha} \chi_{\alpha} + K_{\alpha\beta} \chi_{\beta} = - \left\langle \left(\frac{\partial f_{\alpha}^{(1)}}{\partial t} \right) \right\rangle_{\phi} \quad (3-100)$$

$$h_{\alpha N} = - \nabla \Psi \cdot \nabla_{\mathbf{B}\alpha} f_{\alpha}^{(0)} \quad (3-101)$$

$$h_{\alpha T} = - \nabla \Psi \cdot \nabla_{\mathbf{B}\alpha} \frac{m_{\alpha} v^2}{2T} f_{\alpha}^{(0)} \quad (3-102)$$

$$C_{\alpha} = - \frac{Z_{\alpha} e}{T} u f_{\alpha}^{(0)} \quad (3-103)$$

The K_s are the collision operators which are derived in Appendix

E.

$$K_{\alpha\alpha} \chi_{\alpha} = \frac{\Gamma_{\alpha}}{m_{\alpha}} \left\{ \frac{1}{2\pi} \int d\phi \nabla_{\mathbf{v}} \cdot \left[Z_{\alpha}^2 e^2 \int d^3 v' f_{\alpha}^{(0)} f_{\alpha}^{(0)'} \nabla_{\mathbf{v}} \nabla_{\mathbf{v}'} |\mathbf{v} - \mathbf{v}'| \cdot \nabla_{\mathbf{v}} \chi_{\alpha}' - Z_{\alpha}^2 e^2 \int d^3 v' f_{\alpha}^{(0)} f_{\alpha}^{(0)'} \nabla_{\mathbf{v}} \nabla_{\mathbf{v}'} |\mathbf{v} - \mathbf{v}'| \cdot \nabla_{\mathbf{v}} \chi_{\alpha} - Z_{\beta}^2 e^2 \int d^3 v' f_{\alpha}^{(0)} f_{\beta}^{(0)'} \nabla_{\mathbf{v}} \nabla_{\mathbf{v}'} |\mathbf{v} - \mathbf{v}'| \cdot \nabla_{\mathbf{v}} \chi_{\alpha} \right] \right\} \quad (3-104)$$

$$K_{\alpha\beta} \chi_{\beta} = \frac{\Gamma_{\alpha} Z_{\beta}^2 e^2}{m_{\beta}} \frac{1}{2\pi} \int d\phi \nabla_{\mathbf{v}} \cdot \left[\int d^3 v' f_{\alpha}^{(0)} f_{\beta}^{(0)'} \nabla_{\mathbf{v}} \nabla_{\mathbf{v}'} |\mathbf{v} - \mathbf{v}'| \cdot \nabla_{\mathbf{v}} \chi_{\beta}' \right] \quad (3-105)$$

$$\Gamma_{\alpha} = \frac{2\pi \ln \Lambda Z_{\alpha}^2 e^2}{m_{\alpha}} \quad (3-106)$$

The following result of Appendix F has been used.

$$K_{\alpha\alpha} \chi_{\alpha} + K_{\alpha\beta} \chi_{\beta} = 0 \quad (3-107)$$

Where H_{α} will be called the orbit operator, $K_{\alpha\alpha}$ is the self collision operator, and $K_{\alpha\beta}$ is the cross collision operator due to collisions between the α and β species. The magnetic drift velocity, $\underline{v}_{B\alpha}$, is given by,

$$\underline{v}_{B\alpha} = \frac{b}{\mathcal{N}_{\alpha}} \times \left\{ \frac{\omega^2}{2} \underline{\nabla} \ln B + u^2 \underline{b} \cdot \underline{\nabla} \underline{b} \right\} \quad (3-108)$$

It would be difficult to solve equation (3-98). It is a partial differential integral equation in four dimensions, two velocities, u and w and two space dimensions, ψ and the coordinate around the flux surface, λ . As will be shown in the next chapter, χ_{α} itself does not appear in the transport equation. The quantities which appear in the transport equation are integrals over u , w , and λ of χ_{α} . These integrals are the transport coefficients. In chapter 6 a variational principle will be given which computes the transport coefficients.

IV TRANSPORT EQUATIONS

4.1 MOMENT EQUATIONS

In this chapter the transport equations are derived. The moment equations are presented. They are then averaged over a flux surface to yield equations for the transport of particles, energy and toroidal magnetic flux across a flux surface.

The moment equations which are derived by multiplying Boltzmann's equation by 1 , $m\mathbf{v}$, $\frac{mv^2}{2}$ and $\frac{m\mathbf{v}^2}{2}\mathbf{v}$ and then integrating over velocity space are,

$$\frac{dN_\alpha}{dt} - \mathbf{v} \cdot \nabla N_\alpha + \nabla \cdot (N_\alpha \mathbf{u}_\alpha) = 0 \quad (4-1)$$

$$\frac{d}{dt}(N_\alpha m_\alpha \mathbf{u}_\alpha) - \mathbf{v} \cdot \nabla (N_\alpha m_\alpha \mathbf{u}_\alpha) + \nabla \cdot \mathbf{P}_\alpha - Z_\alpha e N_\alpha \left\{ \mathbf{E}^{(N)} \cdot \nabla \mathbf{\Phi} + \frac{\mathbf{u}_\alpha \times \mathbf{B}}{c} \right\} = \int d^3v m_\alpha \mathbf{v} \left(\frac{\partial f_\alpha}{\partial t} \right)_c \quad (4-2)$$

$$\frac{d}{dt} \left(\frac{3}{2} N_\alpha T \right) - \mathbf{v} \cdot \nabla \left(\frac{3}{2} N_\alpha T \right) + \nabla \cdot \mathbf{q}_\alpha - Z_\alpha e N_\alpha \left\{ \mathbf{E}^{(N)} \cdot \nabla \mathbf{\Phi} \right\} \cdot \mathbf{u}_\alpha = \int d^3v \frac{m_\alpha v^2}{2} \left(\frac{\partial f_\alpha}{\partial t} \right)_c \quad (4-3)$$

$$\frac{d\mathbf{q}_\alpha}{dt} - \mathbf{v} \cdot \nabla \mathbf{q}_\alpha + \nabla \cdot \mathbf{R}_\alpha - \frac{Z_\alpha e}{m_\alpha} \left\{ \mathbf{E}^{(N)} \cdot \nabla \mathbf{\Phi} \right\} \cdot \left\{ \mathbf{p}_\alpha + \frac{3}{2} N_\alpha T \mathbf{I} \right\} + \frac{Z_\alpha e}{m_\alpha c} \mathbf{B} \times \mathbf{q}_\alpha = \int d^3v \frac{m_\alpha v^2}{2} \mathbf{v} \left(\frac{\partial f_\alpha}{\partial t} \right)_c \quad (4-4)$$

Equation (4-1) is the continuity equation, equation (4-2)

is the momentum equation, equation (4-3) is the energy equation,

and equation (4-4) is the equation of motion of the heat flow vector. The flux surface velocity is $\underline{V}$, as defined by equation (2-22). The fluid velocity of the α species is defined by,

$$\underline{u}_\alpha = \frac{1}{N_\alpha} \int d^3v \underline{v} f_\alpha \quad (4-5)$$

The zeroth order distribution function $f_\alpha^{(0)}$, does not contribute to $\underline{u}_\alpha$ since it is an even function of $\underline{u}$.

The pressure tensor $\underline{P}_\alpha$ is defined by,

$$\underline{P}_\alpha = \int d^3v m_\alpha \underline{v} \underline{v} f_\alpha \quad (4-6)$$

The $\chi_{\alpha c}$ part of the distribution function does not make any contribution to the pressure tensor since it is an odd function of ϕ . The pressure tensor can be written as,

$$\underline{P}_\alpha = \{N_\alpha T + P_{\parallel\alpha}\} \underline{\underline{I}} + \{P_{1\alpha} - P_{2\alpha}\} \underline{b} \underline{b} \quad (4-7a)$$

where,

$$P_{1\alpha} = \int d^3v \frac{m_\alpha w^2}{2} f_\alpha^{(0)} \chi_\alpha \quad (4-7b)$$

$$P_{2\alpha} = \int d^3v \frac{m_\alpha u^2}{2} f_\alpha^{(0)} \chi_\alpha \quad (4-7c)$$

the heat flow vector $\underline{q}_\alpha$ is defined by,

$$\underline{q}_\alpha = \int d^3v \frac{m_\alpha v^2}{2} \underline{v} f_\alpha \quad (4-8)$$

The zeroth order distribution function, $f_\alpha^{(0)}$, does not contribute to $\underline{q}_\alpha$ since it is an even function of $\underline{v}$. The tensor $\underline{R}_\alpha$ is defined by,

$$\underline{R}_\alpha = \int d^3v \frac{m_\alpha v^2}{2} \underline{v} \underline{v} f_\alpha \quad (4-9)$$

The $\chi_{\alpha c}$ part of the distribution function does not make any contribution to $\underline{R}_\alpha$ since it is an odd function of ϕ . This tensor can be written as,

$$\underline{R}_\alpha = \left\{ \frac{5}{2} \frac{N_\alpha T^2}{m_\alpha} + r_{\alpha\alpha} \right\} + \{ r_{\parallel\alpha} \quad r_{\perp\alpha} \} \underline{b} \underline{b} \quad (4-10a)$$

where,

$$r_{\parallel\alpha} = \int d^3v \frac{m_\alpha v^2}{2} u^2 f_\alpha^{(c)} \chi_\alpha \quad (4-10b)$$

$$r_{\perp\alpha} = \int d^3v \frac{m_\alpha v^2}{2} \frac{w^2}{2} f_\alpha^{(c)} \chi_\alpha \quad (4-10c)$$

The same ordering used in the kinetic equation will be applied to the moment equations. The $\underline{u}_\alpha$ consists of two parts. The first order part is the result of using $f_\alpha^{(1)}$ in equation (4-5). The second order part is the result of using $f_\alpha^{(2)}$ in equation (4-5). In the pressure tensor, equation (4-7a), the $N_\alpha T$ term which comes from $f_\alpha^{(0)}$ is zeroth order in ϵ . The other terms in the pressure tensor are due to $f_\alpha^{(1)}$ and are first order in ϵ . The ordering of the vector $\underline{g}_\alpha$ and the tensor $\underline{P}_\alpha$ are similar to $\underline{u}_\alpha$ and $\underline{P}_\alpha$.

The first order part of $\underline{u}_\alpha$ and $\underline{g}_\alpha$ are calculated using $f_\alpha^{(0)} \chi_\alpha$ and $f_\alpha^{(0)} \chi_{\alpha c}$. The results of these calculations, which are presented in Appendix G are,

$$\underline{u}_\alpha^{(1)} = \frac{b}{N_\alpha} \left\{ d^3 f_\alpha^{(0)} \chi_\alpha u + \frac{b \times \nabla \psi}{N_\alpha m_\alpha \Omega_\alpha} \left\{ \frac{d(N_\alpha T)}{d\psi} + Z_\alpha e N_\alpha \frac{d\Phi}{d\psi} \right\} \right\} \quad (4-11)$$

$$\underline{g}_\alpha^{(1)} = \frac{b}{2} \frac{m_\alpha}{2} \left\{ d^3 u v^2 f_\alpha^{(0)} \chi_\alpha + \frac{5}{2} \frac{b \times \nabla \psi}{m_\alpha \Omega_\alpha} \left\{ \frac{d(N_\alpha T)}{d\psi} + Z_\alpha e N_\alpha T \frac{d\Phi}{d\psi} \right\} \right\} \quad (4-12)$$

Note that the lowest order part of $\underline{u}_\alpha$ and $\underline{g}_\alpha$ as given in equation (4-11) and (4-12), are vectors in the flux surface. Thus the equilibration of particle density and temperature on

a flux surface is a fast process compared to the characteristic time for change of particle density and temperature on adjacent flux surfaces. This is consistent with the statement that the flux surfaces are surfaces of constant particle density and temperature.

4.2 DERIVATION OF PARTICLE TRANSPORT EQUATION

The continuity equation (4-1) will now be integrated over the interior of a flux surface. The momentum equation (4-2) will be used to eliminate the second order part of u_α from the integrated continuity equation. The volume integrals over Ψ are from Ψ to Ψ_0 where Ψ_0 is the value of Ψ on the magnetic axis. Ψ_0 is the maximum value of Ψ in the system. The volume element is written as,

$$d^3 = d\Psi \frac{d^2 r}{|\nabla\Psi|} \quad (4-13)$$

The surface element is,

$$dS = \frac{d^2 r}{|\nabla\Psi|} \nabla\Psi \quad (4-14)$$

Note that $d\underline{S}$ is an inward vector. The continuity equation is integrated over the flux surface.

$$\int d^3r \frac{dN_\alpha}{dt} = \int d^3r \underline{V} \cdot \underline{\nabla} N_\alpha - \int d^3r \underline{\nabla} \cdot (N_\alpha \underline{u}_\alpha) \quad (4-15)$$

Using Gauss's law on the last term yields,

$$\int d^3r \frac{dN_\alpha}{dt} = \int d^3r \underline{V} \cdot \underline{\nabla} N_\alpha + \int \frac{d^3r}{|\underline{\nabla}\psi|} \underline{\nabla}\psi \cdot \underline{u}_\alpha N_\alpha \quad (4-16)$$

The lowest order part of this equation is,

$$\int \frac{d^3r}{|\underline{\nabla}\psi|} \underline{\nabla}\psi \cdot \underline{u}_\alpha^{(0)} N_\alpha = 0 \quad (4-17)$$

which yields no useful information since the first order part of $\underline{u}_\alpha$ as given by equation (4-11) satisfies this equation and,

$$\nabla \Psi \cdot \underline{u}_\alpha^{(1)} = 0 \quad (4-18)$$

The lowest order momentum equation is,

$$\nabla(N_\alpha T) = Z_\alpha e N_\alpha \left\{ \frac{\underline{u}_\alpha^{(1)} \times \underline{B}}{c} - \nabla \Phi \right\} \quad (4-19)$$

If equation (4-19) for both species, α and β are summed the resulting equation is the pressure balance equation (2-4). The electrostatic potential drops out due to the quasineutrality condition, which is,

$$Z_\alpha e N_\alpha + Z_\beta e N_\beta = 0 \quad (4-20)$$

The next order momentum equation is,

$$\nabla \cdot \underline{P}_\alpha - Z_\alpha e N_\alpha \left\{ \underline{E} + \frac{\underline{u}_\alpha^{(1)} \times \underline{B}}{c} \right\} = \int d^3 \underline{u}_\alpha \underline{u}_\alpha \cdot \left(\frac{\partial \underline{F}_\alpha}{\partial t} \right)_c \quad (4-21)$$

An expression for $\nabla\psi \cdot \underline{g}_a^{(2)}$ which is needed in equation (4-16) is obtained by taking the cross product of equation (4-21) with $\underline{\beta}$ then a dot product with $\nabla\psi$.

$$N_a \nabla\psi \cdot \underline{g}_a^{(2)} = \frac{(\underline{b} \times \nabla \cdot \underline{p}_a)}{m_a \mathcal{N}_a} \cdot \nabla\psi + c N_a \frac{\underline{\varepsilon}^{(n)} \times \underline{\beta}}{\beta^2} \cdot \nabla\psi + \nabla\psi \cdot \left(d^3 \underline{v} \times \underline{b} \frac{(\partial f_r)}{\partial t} \right) \quad (4-22)$$

Substituting equation (4-22) into equation (4-16) and using equation (2-24) for the $\underline{\varepsilon}^{(n)} \times \underline{\beta}$ term yields,

$$\begin{aligned} \int d^3r \frac{dN_a}{dt} - \int d^3r \underline{v} \cdot \nabla N_a + \int \frac{d^2r}{|\nabla\psi|} N_a \frac{\partial\psi}{\partial t} &= \int \frac{d^2r}{|\nabla\psi|} \frac{(\underline{b} \times \nabla \cdot \underline{p}_a)}{m_a \mathcal{N}_a} \cdot \nabla\psi \\ &- \int \frac{d^2r}{|\nabla\psi|} \frac{c DN_a}{\beta^2} \underline{\varepsilon}^{(n)} \cdot \underline{\beta} - \int \frac{d^2r}{|\nabla\psi|} \nabla\psi \cdot \underline{b} \times \left(d^3 \underline{v} \cdot \left(\frac{\partial f_r}{\partial t} \right) \right) \end{aligned} \quad (4-23)$$

The left hand side of this equation can be simplified. Starting with the following,

$$\frac{d}{dt} \int d^3r N_a = \int d^3r \underline{v} \cdot \nabla N_a + \int d^3r \frac{dN_a}{dt} \quad (4-24a)$$

Integrate the first term on the right hand side by parts.

$$\frac{d}{dt} \left(\int d^3 N_\alpha \right) = - \int d^3 \underline{V} \cdot \nabla N_\alpha + \int dS \cdot \underline{V} N_\alpha + \int d^3 \frac{dN_\alpha}{dt} \quad (4-24b)$$

this can be written as,

$$\frac{d}{dt} \left(\int d^3 N_\alpha \right) = - \int d^3 \underline{V} \cdot \nabla N_\alpha + \int \frac{d^2}{|\nabla \psi|} N_\alpha \frac{\partial \psi}{\partial t} + \int d^3 \frac{dN_\alpha}{dt} \quad (4-24c)$$

Using equation (4-24c) in equation (4-24) the equation for the time rate of change of the number of particles enclosed by a fixed surface becomes

$$\begin{aligned} \frac{d}{dt} \left\{ \int d^3 N_\alpha \right\} &= \int \frac{d^2}{|\nabla \psi|} \frac{\underline{b} \times \nabla \underline{P}_\alpha}{m_\alpha N_\alpha} \cdot \nabla \psi - \int \frac{d^2}{|\nabla \psi|} e D N_\alpha \frac{\underline{E} \cdot \underline{B}}{B^2} \\ &\quad - \int \frac{d^2}{|\nabla \psi|} \nabla \psi \cdot \frac{\underline{b}}{N_\alpha} \times \left(d^3 \underline{V} \cdot \left(\frac{\partial \underline{f}_\alpha}{\partial t} \right) \right) \end{aligned} \quad (4-25)$$

The first term on the right hand side can be put into a different form. Using the pressure tensor given in equation (4-7a), part of the integrand of this term becomes,

$$(b \times \nabla \cdot \underline{P}) \cdot \underline{\psi} = \nabla \psi \cdot (b \times \nabla \underline{P}) + (\rho_{11} - \rho_{22}) \nabla \psi \cdot b \times b \cdot \nabla b \quad (4-26)$$

Using the following equation for the first term on the right hand side,

$$\frac{b}{B} \times \nabla \underline{P} = \underline{P} \nabla \times \left(\frac{b}{B} \right) - \nabla \times \left(\underline{P} \frac{b}{B} \right) \quad (4-27)$$

and a vector identity given in Appendix E,

$$\nabla \times \left(\frac{b}{B} \right) = \frac{b \cdot \nabla \times b}{B^2} + \frac{b}{B} \times (\nabla \ln B + b \cdot \nabla b) \quad (4-28)$$

equation (4-26) becomes,

$$(b \times \nabla \cdot \underline{P}) \cdot \underline{\psi} = \left\{ \underline{P} \nabla \ln B + \rho_{11} b \cdot \nabla b \right\} \cdot \nabla \psi + \left\{ \underline{P} \cdot \frac{b}{B} \right\} B \cdot \nabla \psi \quad (4-29)$$

On dividing the above equation by $m_a \Omega_a$ it becomes the integrand of the first term on the right hand side of equation (4-25). The surface integral involving the last term in equation (4-29) can be changed into the volume integral of the divergence of $\vec{r} \cdot \vec{e}_r$ and is thus zero. Also using the integral expression for $\vec{\rho}_{La}$ and $\vec{\rho}_{La}$ the pressure gradient term in the transport equation (4-25) becomes,

$$\left(\frac{d^2}{dt^2} \left\{ \frac{\vec{\rho} \times \vec{V} \cdot \vec{\rho}}{m_a \Omega_a} \right\} \cdot \vec{V} \Psi \right) = \left(\frac{d^2}{dt^2} \right) \left(d^3 v_{Ba} \cdot \vec{V} \Psi + \dots \right) \chi_a \quad (4-30)$$

where $\vec{B}_{Ba}$ is the magnetic drift given by equation (3-108).

The particle transport equation can now be written as,

$$\frac{1}{c} \left(d^3 N_a \right) = \left(\frac{d^2}{dt^2} \right) \left(d^3 v_{Ba} \cdot \vec{V} \Psi + \dots \right) \chi_a - e D N_a \left(\frac{d^2}{dt^2} \right) \frac{\vec{B}}{B^2}$$

$$\frac{d^2}{dt^2} \left(\vec{V} \cdot \vec{\rho} \cdot \frac{\vec{\rho}}{\Omega_a} \times \left(d^3 v \left(\frac{d^2}{dt^2} \right) \right) \right) \quad (4-31)$$

A physical explanation of the terms in this equation will

be given. The left hand side is the time rate of change of

the number of particles enclosed in a flux surface. The first

term on the right hand side of equation (4-31) is the flux of

particles crossing the boundary of the volume due to magneti drift. A method for evaluating this term by use of a variational principle will be presented in the next chapter.

For the purpose of understanding the second term on the right hand side of equation (4-31) it is rewritten using equation (2-26) as,

$$N_n \int \frac{d^3r}{10^4} (\mathbf{v}_e \cdot \mathbf{V}) \cdot \nabla \psi$$

Thus, this term is the flux of particles due to the $\mathbf{E} \times \mathbf{B}$ drift. This term is also evaluated by the variational principle to be described in the next chapter.

The last term in equation (4-31) is the $\mathbf{F} \times \mathbf{B}$ drift where $\mathbf{F}$ is a drag due to collisions. Self collisions do not contribute to this term. The only part of the distribution function which contributes to this term is the δf_{α} which is known, equation (4-75). The velocity integral of this term is evaluated in Appendix I.

4. DERIVATION OF THE ENERGY TRANSPORT EQUATION

Now in order to obtain an energy transport equation, the energy equation (4-3) will be integrated over the volume enclosed by a closed surface.

$$\int d^3r \frac{d}{dt} \left(\frac{3}{2} N_a T \right) = \int d^3r \nabla \cdot \nabla \left(\frac{3}{2} N_a T \right) + \int \frac{d^3r}{|\nabla \Psi|} \nabla \Psi \cdot \mathbf{g}_a$$

$$+ d^3r \frac{3}{2} e N_a \left\{ \mathbf{E} \cdot \nabla \Phi \right\} \cdot \mathbf{u}_a + \int d^3r \int d^3v \frac{m_a v^2}{2} \left(\frac{\partial f_a}{\partial t} \right) \quad (4-32)$$

Here Gauss's law has been used to change the volume integral of $\nabla \cdot \mathbf{g}_a$ into a surface integral. Since the collision term is zero due to the assumption of equal species temperature, the lowest order energy equation is,

$$\int \frac{d^3r}{|\nabla \Psi|} \nabla \Psi \cdot \mathbf{g}_a^{(1)} = 0 \quad (4-33)$$

This equation does not yield any useful information since due to the form of $\mathbf{g}_a$, given in equation (4-12), it can be seen that,

$$\nabla \Psi \cdot \mathbf{g}_a^{(1)} = 0 \quad (4-34)$$

The calculation of the $\nabla \Psi \cdot \mathbf{g}_a^{(2)}$ term in the next order of equation (4-32) is done in a manner similar to the calculation of $\nabla \Psi \cdot \mathbf{g}_a^{(1)}$. The next flow vector equation (4-9) is first crossed with $\mathbf{r}$, then dotted with $\nabla \Psi$. Equation (4-11a) (4-11b) and (4-11c) are used for $\mathbf{g}_a$, the result is,

$$\begin{aligned}
 \frac{d}{dt} \left\{ \int d\tau \frac{1}{2} N_A T \right\} &= \frac{3}{2} N_A T \left(\frac{d^2}{d\psi^2} \right) \cdot D \frac{E \cdot B}{B^2} + N_A T \left(\frac{d^2}{d\psi^2} \right) \nabla \Psi \cdot \nabla \\
 &+ \left(\frac{d^2}{d\psi^2} \right) \left(\nabla^2 \Psi \cdot \nabla \cdot \mathbf{v}_{\perp} \right) \frac{m_A v^2}{2} f_{\perp} - \left(\frac{d^2}{d\psi^2} \right) \left(\nabla^2 \Psi \cdot \frac{b}{\Omega_A} \times \mathbf{v} \right) \frac{(\partial f_{\perp}}{\partial t_c)} \frac{m_A v^2}{2} \\
 &+ \left(\frac{d^2}{d\psi^2} \right) \left(\frac{1}{\psi} \right) \left(N_A T \right) \left(\nabla \cdot \mathbf{v}_{\perp} \right) - \left(\frac{d^2}{d\psi^2} \right) \left(\frac{1}{\psi} \right) \left(N_A T \right) \left(\nabla \cdot \mathbf{v}_{\perp} \right) \left(\frac{b}{\Omega_A} \times \mathbf{v} \right) \\
 &+ \left(\frac{d^2}{d\psi^2} \right) \left(\frac{1}{\psi} \right) \left(N_A T \right) \left(\nabla \cdot \mathbf{v}_{\perp} \right) \left(\frac{b}{\Omega_A} \times \mathbf{v} \right) + \left(\frac{d^2}{d\psi^2} \right) \left(\frac{1}{\psi} \right) \left(N_A T \right) \left(\nabla \cdot \mathbf{v}_{\perp} \right) \left(\frac{b}{\Omega_A} \times \mathbf{v} \right) \left(\frac{\partial f_{\perp}}{\partial t_c} \right) \frac{m_A v^2}{2}
 \end{aligned}$$

(4-37)

A physical explanation will now be given for each term

in this equation. The left hand side is the time rate of change of the thermal energy enclosed in a flux surface. The first four terms on the right hand side are terms due to the transport of energy across the surface. The remaining terms are due to volume sources of energy.

The first term on the right hand side of equation (4-37)

is the energy flux due to $E \times B$ drift. This term is similar to the first term on the right hand side of equation (4-31):

The $\frac{3}{2} N_A T$ factor is due to the fact that the energy flow across a surface is the energy density, $\frac{3}{2} N_A T$, times a velocity plus the pressure gradient, which is $N_A T$ times a velocity. Thus, the energy flow is the sum of these two contributions.

which is the enthalpy, $\frac{5}{2} N k T$, times a velocity.

The second term on the right hand side of equation (4-37) is the pressure times volume change work done on the plasma by the motion of the flux surface.

The third term on the right hand side of equation (4-37) is the transport of energy across the flux surface due to magnetic drift.

The fourth term on the right hand side of equation (4-37) is the transport of energy across a flux surface due to $\mathbf{F} \times \mathbf{B}$ drift where $\mathbf{F}$ is a drag force due to collisions.

The fifth term on the right hand side of equation (4-37) is the change in the energy content of the flux surface due to $\mathbf{E} \times \mathbf{B}$ drift against a pressure gradient.

The sixth term on the right hand side of equation (4-37) is the Joule heating due to current resulting from the magnetic drift of charge against the electrostatic potential.

The seventh term on the right hand side of equation (4-37) is similar to the sixth except that the drift is due to a drag force cross $\mathbf{B}$ drift.

The eighth term on the right hand side of equation (4-37) is the Joule heating due to current parallel to the $\mathbf{B}$ field.

The last term on the right hand side of equation (4-37) is the direct transfer of energy to the species α due to collisions with the other species.

4.4 QUASINEUTRALITY

It will be shown in this section that the diffusion is quasineutral, that is, if the quasineutrality condition (4-20) is satisfied initially, it will be satisfied for all later time. The transport equation for the charge is obtained by multiplying the particle transport equation (4-25) by the particle's charge. The transport equation for the total charge is obtained by adding the charge transport equation for the α and β species. This equation is,

$$\frac{d}{dt} \left\{ \int d^3r [Z_\alpha e N_\alpha + Z_\beta e N_\beta] \right\} = c \left\langle \frac{\mathbf{b} \times \nabla \cdot \mathbf{P}}{B} \cdot \nabla \psi \right\rangle + c \left\langle \frac{\mathbf{b} \times \nabla \cdot \mathbf{P}}{B} \cdot \nabla \psi \right\rangle - \left\{ Z_\alpha e N_\alpha + Z_\beta e N_\beta \right\} c D \left\langle \frac{\nabla^2 B}{B^2} \right\rangle - c \left\langle \nabla \psi \cdot \frac{\mathbf{b}}{B} \times \left\{ \int d^3v m_\alpha \mathbf{v} \left(\frac{\partial f_\alpha}{\partial t} \right)_c + \int d^3v m_\beta \mathbf{v} \left(\frac{\partial f_\beta}{\partial t} \right)_c \right\} \right\rangle \quad (4-38)$$

Here a new notation has been introduced for a weighted average over a flux surface. This average for any function

g is,

$$\langle g \rangle = \frac{\int g \mathbf{b} \cdot \nabla \psi d^3r}{\int \mathbf{b} \cdot \nabla \psi d^3r}$$

(4-39)

The third term on the right hand side of equation (4-38) is zero due to the quasineutrality condition being satisfied initially. The fourth term is zero due to the conservation of momentum of the collision term as shown in equation (3-3). The sum of the first and second term will now be shown to be zero. Adding the momentum equation (4-21) for species α and β one obtains, after taking the cross product with $\underline{b}$ and the dot product with $\nabla\psi$,

$$\begin{aligned} \underline{b} \times \nabla \underline{P}_\alpha \cdot \nabla \psi - \underline{b} \times \nabla \underline{P}_\beta \cdot \nabla \psi &= \{Z_\alpha e N_\alpha + Z_\beta e N_\beta\} \underline{b} \times \underline{E}^{(1)} \cdot \nabla \psi \\ + \underline{b} \times \frac{\underline{J} \times \underline{B}}{c} \cdot \nabla \psi + \underline{b} \times \left\{ \int d^3 v m_\alpha \underline{v} \left(\frac{\partial f_\alpha}{\partial t} \right) + \int d^3 v m_\beta \underline{v} \left(\frac{\partial f_\beta}{\partial t} \right) \right\} \cdot \nabla \psi \end{aligned} \quad (4-40)$$

The first term on the right hand side of equation (4-40) is zero due to the quasineutrality condition. The second term is zero since $\nabla \psi \cdot \underline{B} = 0$ and $\nabla \psi \cdot \underline{J} = 0$. The third term is zero due to the conservation of momentum by the collision term. Therefore equation (4-38) reduces to,

$$\frac{d}{dt} \left\{ \int d^3 v [Z_\alpha e N_\alpha + Z_\beta e N_\beta] \right\} \quad (4-41)$$

which is the desired result.

4.5 TRANSPORT EQUATION FOR THE TOTAL NUMBER OF PARTICLES

Now an equation for the time rate of change of the total number of particles enclosed in a flux surface will be given. Equation (4-31) for the α species and equation (4-31) for the β species are added to give,

$$\frac{d}{dt} \left\{ \int_V d^3 N \right\} = - (\chi_\alpha, h_{\alpha N}) - (\chi_\beta, h_{\beta N}) - c DN \left\langle \frac{\mathbf{E} \cdot \mathbf{B}}{B^2} \right\rangle - \left\langle \nabla \psi \cdot \frac{\mathbf{B}}{B} \times \int_V d^3 \left\{ \frac{v}{n_\alpha} \left(\frac{\partial f_\alpha}{\partial t} \right)_c + \frac{v}{n_\beta} \left(\frac{\partial f_\beta}{\partial t} \right)_c \right\} \right\rangle \quad (4-42)$$

where,

$$N = N_\alpha + N_\beta \quad (4-43)$$

Here the following notation for an inner product has been introduced,

$$(\chi, h) = \left(\frac{d^3}{4\pi} \right) \int_V \chi h \quad (4-44)$$

This inner product is the surface average of the velocity space average of two functions χ and h .

4.6 PRESSURE TRANSPORT EQUATION

A transport equation for the pressure will now be derived. This is done by adding equation (4-37) for the species α and the same equation for the species β . Several terms in the resulting equation cancel. The sixth term on the right hand side of equation (4-37) makes no contribution to the pressure transport equation due to conditions (4-30) and (4-40). The seventh term on the right hand side of equation (4-37) makes no contribution to the pressure transport equation due to the conservation of momentum of the collision operator, equation (3-4). Thus the two terms in equation (4-37) which depend on the electrostatic potential, Φ , make no contribution to the pressure transport equation. The last term in equation (4-37) makes no contribution to the pressure transport equation due to the conservation of energy of the collision operator, equation (3-5). The eighth term in equation (4-37) combines to give a term containing the parallel current $J_{||}$.

$$J_{11} = Z_{\alpha c} \int d^3 u f_{\alpha}^{(1)} \chi_{\alpha} + Z_{\beta c} \int d^3 u f_{\beta}^{(1)} \chi_{\beta} \quad (4-45)$$

Equation (2-16) is used for J_{11} in the pressure transport equation. The resulting equation for the time rate of change of the integral of the pressure is,

$$\begin{aligned} \frac{d}{dt} \left\{ \int d^3 u \frac{p}{2} \right\} = & -T(\chi_{\alpha}, h_{\alpha T}) - T(\chi_{\beta}, h_{\beta T}) - \frac{5}{2} P_c D \left\langle \frac{\mathbf{E} \cdot \mathbf{B}}{B^2} \right\rangle \\ & + P \left\langle \mathbf{V} \cdot \nabla \Psi \right\rangle - \left\langle \int d^3 u \nabla \Psi \cdot \frac{\mathbf{B}}{B} \times \mathbf{V} \left\{ \frac{m_{\alpha}}{N_{\alpha}} \left(\frac{d\mathbf{f}_{\alpha}}{dt} \right) + \frac{m_{\beta}}{N_{\beta}} \left(\frac{d\mathbf{f}_{\beta}}{dt} \right) \right\} \frac{\mathbf{V}^2}{2} \right\rangle \\ & + \int d^3 u \frac{dp}{d\Psi} \mathbf{V} \cdot \nabla \Psi + \int d^3 u \frac{c}{4\pi} \frac{dD}{d\Psi} \mathbf{E} \cdot \mathbf{B} \end{aligned} \quad (4-46)$$

where,

$$P = N_{\alpha} T_{\alpha} + N_{\beta} T_{\beta} \quad (4-47)$$

4.7 TRANSPORT EQUATION FOR $P N^{5/3}$

In this section an equation will be derived for the time rate of change of the quantity, $P N^{5/3}$. This is presented as a substitute for the pressure transport equation. The problem with the pressure transport equation

is the presence of the terms which depend on the flux surface velocity, V . These terms cannot be written as the products of transport coefficients and gradients. The flux surface velocity is determined by a complicated differential integral equation which depends on the time rate of change of the pressure. This equation for V is given in Appendix J. In chapter 7 it will be shown that the transport equation for $PN^{-5/3}$ can be used instead of the pressure transport equations in the calculation of the evolution of the plasma properties and magnetic field.

The transport equation for $PN^{-5/3}$, like the particle transport equation (4-42), does not depend on the flux surface velocity. The total particle density enclosed in a flux surface and $PN^{-5/3}$ are adiabatic invariants, that is, if there is no diffusion then these quantities are independent of time. The logarithm of the quantity $PN^{-5/3}$ is proportional to the entropy per particle [53]. In the next section a transport equation for another adiabatic invariant, the toroidal magnetic flux enclosed in a flux surface, will be given.

The quantity to be calculated in this section is,

$$\frac{d(PN)}{dt} = N^{-5/3} \left\{ \frac{1}{3} \frac{dP}{dt} - \frac{5}{3} \frac{P}{N} \frac{dN}{dt} \right\}$$

Equations for $\frac{dP}{dt}$ and $\frac{dN}{dt}$ are obtained by taking $\frac{d}{d\psi}$ of equation (4-42) and (4-45) respectively. Using the relation,

$$\frac{d}{d\psi} \int_{\psi}^{\psi_0} d\psi F(\psi) = -F(\psi) \quad (4-49)$$

equations (4-46) and (4-42) become,

$$\begin{aligned} \frac{d}{dt} \left\{ \int \frac{d^3v}{(2\pi)^3} \frac{3}{2} \rho \right\} &= \frac{d}{d\psi} \left\{ T(\chi_a, h_{aT}) + T(\chi_p, h_{pT}) + \frac{5}{2} P_c D \left\langle \frac{\underline{E} \cdot \underline{B}}{B^2} \right\rangle \right. \\ &\quad \left. + \left\langle \int d^3v \nabla \psi \cdot \underline{b} \times \underline{v} \left[\frac{m_a}{\Omega_a} \left(\frac{\partial f_a}{\partial t} \right) + \frac{m_p}{\Omega_p} \left(\frac{\partial f_p}{\partial t} \right) \right] \frac{v^2}{2} \right\rangle \right\} \\ &\quad - P \frac{d}{d\psi} \langle \nabla \psi \cdot \underline{v} \rangle + \frac{c}{4\pi} \frac{dD}{d\psi} \langle \underline{E} \cdot \underline{B} \rangle \end{aligned} \quad (4-50)$$

$$\left\{ \frac{d}{dt} \int_V \frac{d^3 r}{|\nabla \psi|} N \right\} = \frac{d}{d\psi} \left\{ (Z_o, h_{zv}) + (X_p, h_{p\phi}) + cDN \left\langle \frac{\underline{E} \cdot \underline{E}}{B^2} \right\rangle \right. \\ \left. \left\langle \left\{ d^3 v \nabla \psi \cdot \underline{b} \times \underline{v} \left[\frac{1}{\mathcal{L}_\alpha} \left(\frac{\partial f_\alpha}{\partial t} \right)_c - \frac{1}{\mathcal{L}_p} \left(\frac{\partial f_p}{\partial t} \right)_c \right] \right\} \right\rangle \right\} \quad (4-51)$$

The left hand side of equation (4-50) involves a time derivative of an integral over a flux surface. In order to get an expression for this quantity in terms of the flux surface velocity, start with the following equation,

$$\frac{d}{dt} \int_V d^3 r = \frac{d}{dt} \int_{\psi}^{\psi_0} d\psi \int \frac{d^3 r}{|\nabla \psi|} = - \int dS \cdot \underline{V} \quad (4-52)$$

After taking the derivative of this equation with respect to ψ one obtains,

$$\frac{d}{dt} \langle 1 \rangle = \frac{d}{d\psi} \langle \underline{V} \cdot \nabla \psi \rangle \quad (4-53)$$

Substituting equation (4-53) in (4-50) and (4-51) they become,

$$\begin{aligned} \frac{dP}{dt} \langle 1 \rangle = & \frac{2}{3} \frac{d}{d\psi} \left\{ T(\chi_a, h_{a+}) + T(\chi_a, h_{a-}) + \frac{2}{3} P_c D \left\langle \frac{E^2}{B^2} \right\rangle \right. \\ & + \left. \left\langle \int d^3v \nabla \psi \cdot \mathbf{b} \times \mathbf{v} \left[\frac{m_a}{\chi_a} \left(\frac{\partial f_a}{\partial t} \right) + \frac{m_p}{\chi_p} \left(\frac{\partial f_p}{\partial t} \right) \right] \frac{v^2}{2} \right\rangle \right\} \\ & - \frac{5}{3} P \frac{d}{d\psi} \langle \mathbf{V} \cdot \nabla \psi \rangle + \frac{2}{3} \frac{D}{4\pi} \frac{dD}{d\psi} \left\langle \frac{E^2}{B^2} \right\rangle \quad (4-54) \end{aligned}$$

$$\begin{aligned} \frac{dN}{dt} \langle 1 \rangle = & \frac{d}{d\psi} \left\{ (\chi_a, h_{a+}) + (\chi_p, h_{p+}) + DN \left\langle \frac{E^2}{B^2} \right\rangle \right. \\ & + \left. \left\langle \int d^3v \nabla \psi \cdot \mathbf{b} \times \mathbf{v} \left[\frac{1}{\chi_a} \left(\frac{\partial f_a}{\partial t} \right) + \frac{1}{\chi_p} \left(\frac{\partial f_p}{\partial t} \right) \right] \right\rangle \right\} \\ & - N \frac{d}{d\psi} \langle \mathbf{V} \cdot \nabla \psi \rangle \quad (4-55) \end{aligned}$$

If equations (4-54) and (4-55) are substituted into equation (4-48), the resulting transport equation for $PN^{-5/2}$ is,

Fig. C-3. Mina tangential velocity response with trial $R_{dT} = 3690 \Omega$.
 $T_n = 40.6$ sec.

$$\Gamma = \frac{1}{4\pi} \int d^3r \frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \right) \right) \quad (4-10)$$

Using the time derivative in equation (4-10) is given by,

$$\frac{d\Gamma}{dt} = \frac{1}{8\pi} \int d^3r \frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \right) \right) + \frac{1}{8\pi} \int d^3r \left[\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \right) \right] + \frac{1}{8\pi} \int d^3r \left(\frac{\partial}{\partial t} \right) \quad (4-11)$$

An expression for the second term in the first term can be obtained by using the vector identity, $\nabla \cdot (\nabla \times \mathbf{A}) = 0$. Using equation (4-11), the result is,

$$\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \right) = -\frac{1}{2} \nabla \cdot (\nabla \times \nabla \times \mathbf{A}) \quad (4-12)$$

Stokes' law can now be used on this term. Also, the first and third terms in equation (4-11) can be combined into a term which is the divergence of $\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \right)$, and Stokes' law is used on this term also, the result is,

$$\frac{dP}{dt} = \frac{1}{2\pi} \left(\frac{d^2}{dr^2} \frac{\partial \Psi}{\partial t} \frac{D}{r^2} + \frac{c}{2\pi} \left(\frac{d^2}{dr^2} \left(\frac{e_0}{r} \right) \right) \right)$$

Using,

$$\frac{e_0}{r} \times \nabla \Psi = \nabla \phi - \underline{B} \quad (4-51)$$

the theta component of the electric field in the 1st term in equation (4-51) cancels the theta term on the right hand side, thus equation (4-51) becomes,

$$\frac{dP}{dt} = - \frac{c}{2\pi} \left(\frac{d^2}{dr^2} \right) \underline{E} \cdot \underline{B} \quad (4-52)$$

or,

$$\frac{d}{dt} \left\{ \left(\frac{d^2}{dr^2} \right) \frac{D}{r^2} \right\} = -c \left(\underline{E}^{(2)} \cdot \underline{B} \right) \quad (4-53)$$

In the next chapter the right hand side of equation (4-64) will be written in terms of transport coefficients times gradients.

The toroidal magnetic flux can be written in terms of an integral over Ψ of the magnetohydrodynamic stability factor, q . This quantity was given in the Introduction by equation (1-6) and (1-9); however, these equations are approximations only appropriate for the large aspect ratio expansion. The quantity q can be defined in terms on the magnetic field structure. If one integrates $d\theta$ along a field line for one period of the λ coordinate then divides by 2π one obtains,

$$q = \frac{1}{2\pi} \int_{\text{field line}} d\theta \quad (4-65)$$

The coordinate λ is described in Appendix K, it is the coordinate around the flux surface in the $\theta =$ constant plane.

The equation for the field line in the flux surface is,

$$\frac{h_\theta d\theta}{B_r} = \frac{h_\lambda d\lambda}{B_p} \quad (4-66)$$

The scale factors are given in Appendix Y.

$$h_\theta = r \quad (4-67)$$

$$h_\lambda = \mathcal{S} B_p \quad (4-68)$$

The quantity $\mathcal{S}$ is also given in Appendix X. Equation (4-66) is substituted into equation (4-65),

$$q = \frac{1}{2\pi} \int_0^{\mathcal{S}} \frac{D}{r^2} d\lambda \quad (4-69)$$

Using an equation from Appendix K which gives the integral over λ in terms of the flux surface average, equation (4-69) becomes,

$$q = \frac{D}{(2\pi)^2} \left\langle \frac{1}{r^2} \right\rangle = \frac{1}{2\pi} \frac{d\Gamma}{d\psi} \quad (4-70)$$

The transport equation (4-64) can be now written as,

$$\frac{d}{dt} \left\{ \int_{\psi} \psi \right\} = - (2\pi)^2 c \langle E \cdot B \rangle$$

(4-71)

ψ is the value of ψ on the magnetic axis.

This is the transport equation for ψ . It is presented here because ψ is a commonly quoted parameter for Tokamaks.

4.9 SUMMARY OF TRANSPORT EQUATIONS

There are now three transport equations which do not depend on the flux surface velocity. These are, the equation for the time rate of change of the total number of particles enclosed in a flux surface, equation (4-42), the equation for the time rate of change of the quantity $PN^{-5/3}$, equation (4-56), and the time rate of change of the toroidal magnetic flux enclosed in a flux surface, equation (4-64). These equations are repeated here.

$$\frac{d}{dt} \left\{ \int d^3x N \right\} = - \langle \chi_a, h_{an} \rangle - \langle \chi_p, h_{pn} \rangle - c D N \left\langle \frac{\mathbf{E} \cdot \mathbf{B}}{B^2} \right\rangle - \left\langle \nabla \Psi \cdot \mathbf{b} \times \left(d^3x \left[\frac{\mathbf{v}}{N_a} \left(\frac{\partial f_a}{\partial t} \right)_c + \frac{\mathbf{v}}{N_p} \left(\frac{\partial f_p}{\partial t} \right)_c \right] \right) \right\rangle \quad (4-52)$$

$$\begin{aligned} \frac{d(PN^{-5/3})}{dt} &= \frac{N^{-5/3}}{\langle 1 \rangle} \left[\frac{2}{3} \frac{d}{d\psi} \left\{ T(\chi_a, h_{an}) + T(\chi_p, h_{pn}) + \frac{5}{2} P_c D \left\langle \frac{\mathbf{E} \cdot \mathbf{B}}{B^2} \right\rangle \right. \right. \\ &\quad \left. \left. - \left\langle d^3x \nabla \Psi \cdot \mathbf{b} \times \mathbf{v} \left[\frac{m_a}{N_a} \left(\frac{\partial f_a}{\partial t} \right)_c + \frac{m_p}{N_p} \left(\frac{\partial f_p}{\partial t} \right)_c \right] \frac{v^2}{2} \right\} \right\} \right. \\ &\quad \left. - \frac{5}{3} T \frac{d}{d\psi} \left\{ \langle \chi_a, h_{an} \rangle + \langle \chi_p, h_{pn} \rangle + c D N \left\langle \frac{\mathbf{E} \cdot \mathbf{B}}{B^2} \right\rangle + \right. \right. \\ &\quad \left. \left. \left\langle d^3x \nabla \Psi \cdot \mathbf{b} \times \mathbf{v} \left[\frac{1}{N_a} \left(\frac{\partial f_a}{\partial t} \right)_c + \frac{1}{N_p} \left(\frac{\partial f_p}{\partial t} \right)_c \right] \right\} \right\} + \frac{c}{\pi} \frac{dD}{d\psi} \left\langle \frac{\mathbf{E} \cdot \mathbf{B}}{B^2} \right\rangle \right] \quad (4-53) \end{aligned}$$

$$\frac{d}{dt} \left\{ \int d^3x \frac{D}{r^2} \right\} = -c \langle \mathbf{E} \cdot \mathbf{B} \rangle \quad (4-54)$$

In the next chapter it will be shown that all the terms on the right hand side of these equations can be written in terms of transport coefficients and gradients of various quantities. In Chapter 6 a method will be presented for the calculation of the transport coefficients.

TRANSPORT COEFFICIENTS

5.1 ELIMINATION OF THE ELECTROSTATIC POTENTIAL

In this chapter a method for evaluating the terms on the right hand side of equations (4-42), (4-55) and (4-63) will be presented.

First the electrostatic potential will be eliminated from the equations for χ_a , (3-98). The set of equations which must be solved for χ_a , χ_p and E_b are,

$$H_a \chi_a + K_{aa} \chi_a + K_{ap} \chi_p + C_a E_b = h_{aN} \left\{ \frac{dL_N}{d\psi} - \frac{3}{2} \frac{dL_T}{d\psi} + \frac{Z_e d\phi}{T d\psi} \right\} + h_{aT} \frac{dT}{d\psi} \quad (5-1a)$$

$$H_p \chi_p + K_{pp} \chi_p + K_{pa} \chi_a + C_p E_b = h_{pN} \left\{ \frac{\partial L_N}{\partial \psi} - \frac{3}{2} \frac{\partial L_T}{\partial \psi} + \frac{Z_e \partial \phi}{T \partial \psi} \right\} + h_{pT} \frac{\partial T}{\partial \psi} \quad (5-1b)$$

$$Z_a e \int d^3 v f_a^{(0)} \chi_a + Z_p e \int d^3 v f_p^{(0)} \chi_p = J_{||} \quad (5-1c)$$

Equation (5-1c) which is the equation for the current parallel to the magnetic field is included in this set of equations because χ_a and χ_p must be consistent with the fact that $J_{||}$ is known from the given magnetic field configuration, equation (2-16). For convenience the following notation has been used in equations (5-1a) and (5-1b).

$$E_b = E \cdot b$$

(5-2)

For the case considered here, that is, a uniform electron temperature, the electrostatic potential can be eliminated from the problem by the following change of variables.

$$g_\alpha = \chi_\alpha - \frac{uD}{T N_\alpha} Z_\alpha e \frac{d\Phi}{d\rho}$$

Equation (5-3) is substituted into equation (5-1a), (5-1b) and (5-1c). Results of Appendix A are used, they are,

$$K_\alpha \left(\frac{uD}{T N_\alpha} Z_\alpha e \right) + K_{\alpha\beta} \left(\frac{uD}{T N_\beta} Z_\beta e \right) = 0 \quad (5-4)$$

$$H_\alpha \left(\frac{uD}{T N_\alpha} \right) = h_{\alpha n} \quad (5-5)$$

Equation (5-1c) becomes,

$$T_\alpha = Z_\alpha e \left(d^3 v f_\alpha'' \right) g_\alpha + Z_\beta e \left(d^3 v f_\beta'' \right) g_\beta + \frac{D}{B} \left\{ Z_\alpha e N_\alpha + Z_\beta e N_\beta \right\} \quad (5-6)$$

The last term in equation (5-6) is zero due to the quasineutrality condition. Equation (5-1a), (5-1b) and (5-1c) become,

$$H_a z_a^- + K_{aa} z_a^+ + K_{ap} z_p^+ = h_{av} G_1 + h_{av} G_2 \quad (5-7a)$$

$$H_a z_a^+ + K_{aa} z_a^- + K_{ap} z_p^- + C_a E_b = 0 \quad (5-7b)$$

$$H_p z_p^- + K_{pp} z_p^+ + K_{pa} z_a^+ = h_{pv} G_1 + h_{pv} G_2 \quad (5-7c)$$

$$H_p z_p^+ + K_{pp} z_p^- + K_{pa} z_a^- + C_p E_b = 0 \quad (5-7d)$$

$$Z_a \int \text{div} f_a^{(u)} z_a^- + Z_p \int \text{div} f_p^{(u)} z_p^- = h_0 G_3 + h_p [G_1 + \frac{5}{2} G_2] \quad (5-7e)$$

In these equations the distribution functions f_a and f_p have been divided into a part which is an even function of u , f_a^+ and f_p^+ , and a part which is an odd function of u , f_a^- and f_p^- .

$$f_a = f_a^+ + f_a^- \quad (5-8)$$

$$f_p = f_p^+ + f_p^- \quad (5-9)$$

Also $J_{||}$ has been written as,

$$J_{||} = h_0 \frac{d \ln D}{d \psi} + h_p \frac{d \ln P}{d \psi} = h_0 G_3 + h_p [G_1 + \frac{5}{2} G_2] \quad (5-10)$$

where,

$$h_0 = \frac{D_c B}{4 \pi} \quad (5-11a)$$

$$h_p = \frac{D_c P}{B} \quad (5-11b)$$

The following notation has been introduced for the gradients,

$$G_1 = \frac{d \ln P}{d \psi} - \frac{5}{2} \frac{d \ln T}{d \psi} \quad (5-12)$$

$$G_2 = \frac{d \ln T}{d \psi} \quad (5-13)$$

$$G_3 = \frac{d \ln D}{d \psi} \quad (5-14)$$

The electrostatic potential has been eliminated from the equations for j_α , j_β and E_b , equations (5-7a), (5-7b), (5-7c), (5-7d) and (5-7e).

The quantities in the transport equations which depend on χ_α can be written in terms of g_α as,

$$(\chi_\alpha, h_{\alpha N}) = (g_\alpha, h_{\alpha N}) \quad (5-15a)$$

$$(\chi_\alpha, h_{\alpha T}) = (g_\alpha, h_{\alpha T}) \quad (5-15b)$$

This result is due to the fact that χ_α and g_α differ only by a term which is an odd function of u and $h_{\alpha N}$ and $h_{\alpha T}$ are even functions of u .

5.2 VARIATIONAL PRINCIPLE

Now a functional will be given which is an extremum when the set of equations, (5-7a), (5-7b), (5-7c), (5-7d) and (5-7e) are satisfied. The functional is,

$$L\{\xi_\alpha, \xi_\rho, \varepsilon, h\} = M\{\xi_\alpha, \xi_\rho, \varepsilon\} - 2(\xi_\alpha^+, h_\alpha) - 2(\xi_\rho^+, h_\rho) - 2\left\langle \varepsilon \frac{T_H}{T} \right\rangle \quad (5-16)$$

where,

$$\begin{aligned}
 m_{\alpha\beta} = & (Z_{\alpha}^+, K_{\alpha\alpha} Z_{\alpha}^+) - (Z_{\alpha}^-, K_{\alpha\alpha} Z_{\alpha}^-) + (Z_{\alpha}^+, K_{\alpha\beta} Z_{\beta}^+) \\
 & - (Z_{\alpha}^-, K_{\alpha\beta} Z_{\beta}^-) + 2(Z_{\alpha}^+, H_{\alpha} Z_{\alpha}^-) + (Z_{\beta}^+, K_{\beta\beta} Z_{\beta}^+) \\
 & - (Z_{\beta}^-, K_{\beta\beta} Z_{\beta}^-) + 2(Z_{\beta}^+, H_{\beta} Z_{\beta}^-) + (Z_{\beta}^-, K_{\beta\alpha} Z_{\alpha}^-) \\
 & + 2(Z_{\beta}^+, H_{\beta} Z_{\alpha}^-) - (Z_{\alpha}^-, Z_{\beta}^-, \epsilon) - (Z_{\alpha}^-, Z_{\beta}^-, \epsilon)
 \end{aligned} \quad (5-17)$$

where,

$$h_{\alpha} = h_{\alpha\alpha} G_1 + h_{\alpha\beta} G_2 \quad (5-18)$$

and similarly for h_{β} .

The variation of the functional (5-16) is taken with respect to Z_{α}^+ , Z_{α}^- , Z_{β}^+ , Z_{β}^- and ϵ . The result of these variations is,

$$\begin{aligned}
 \delta L = & 2(\delta Z_{\alpha}^+, K_{\alpha\alpha} Z_{\alpha}^+ + K_{\alpha\beta} Z_{\beta}^+ + H_{\alpha} Z_{\alpha}^- - h_{\alpha}) \\
 & - 2(\delta Z_{\alpha}^-, K_{\alpha\alpha} Z_{\alpha}^- + K_{\alpha\beta} Z_{\beta}^- + H_{\alpha} Z_{\alpha}^+ + C_{\alpha} \epsilon) \\
 & + 2(\delta Z_{\beta}^+, K_{\beta\beta} Z_{\beta}^+ + K_{\beta\alpha} Z_{\alpha}^+ + H_{\beta} Z_{\beta}^- - h_{\beta}) \\
 & - 2(\delta Z_{\beta}^-, K_{\beta\beta} Z_{\beta}^- + K_{\beta\alpha} Z_{\alpha}^- + H_{\beta} Z_{\beta}^+ + C_{\beta} \epsilon) \\
 & - 2(Z_{\alpha}^-, Z_{\beta}^-, \epsilon)
 \end{aligned} \quad (5-19)$$

For the functional $\mathcal{L}$ to be an extremum, the variations $\delta \mathcal{L}$ must be zero. Thus, equations (5-7a), (5-7b), (5-7c), (5-7d) and (5-7e) must all be satisfied. Therefore, at the extremum of the functional (5-16) the functions ξ_α , ξ_β and E are ξ_α , ξ_β and E_0 respectively. In deriving equation (5-16) use has been made of the following relations given in Appendix A.

$$(\xi_\alpha, K_{\alpha\beta} \xi_\beta) = (\xi_\beta, K_{\beta\alpha} \xi_\alpha) \quad (5-17)$$

$$(\xi_\alpha, K_{\alpha\beta} \xi_\beta) = (\xi_\beta, K_{\beta\alpha} \xi_\alpha) \quad (5-18)$$

$$(\xi_\alpha, H_\alpha \chi_\alpha) = -(\chi_\alpha, H_\alpha \xi_\alpha)$$

The value of the functional (5-16) at its extremum is obtained by substituting equations (5-7a), (5-7b), (5-7c), (5-7d) and (5-7e) into equation (5-16). The result is,

$$\mathcal{L}(\xi_\alpha, \xi_\beta, E) = \mathcal{L}(\xi_\alpha, \xi_\beta, E_0) = E_0 \quad (5-19)$$

The left hand side of this equation is to be read as "the extremum of $\mathcal{L}$ with respect to ξ_α , ξ_β and E ". Equation (5-23) can also be written as,

$$\mathcal{L}(\xi_\alpha, \xi_\beta, E) = -(\chi_\alpha, h_\alpha) - (\chi_\beta, h_\beta) - (E, b) \quad (5-20)$$

The value of the functional at its extremum as given by equation (5-24) can be shown to be the negative of part of the entropy production rate due to collisions averaged over a flux surface. The entropy production rate due to collisions, $\frac{\partial S}{\partial t}$ averaged over a flux surface is [54],

$$\left\langle \frac{\partial S}{\partial t} \right\rangle = - \int \frac{d^3v}{|\nabla\psi|} \left(d^3v \left(\frac{\partial}{\partial t} \right)_c [f_a^0 \ln f_a + f_p \ln f_p] \right) \quad (5-25a)$$

or,

$$\left\langle \frac{\partial S}{\partial t} \right\rangle = - \left(\frac{d^2}{|\nabla\psi|} \right) \int d^3v \left\{ \left(\frac{\partial f_a}{\partial t} \right)_c (\ln f_a^0 + 1) + \left(\frac{\partial f_p}{\partial t} \right)_c (\ln f_p + 1) \right\} \quad (5-25b)$$

Substituting the expression for f_a , equation (3-42) and using equations (3-57) and (3-74) yields,

$$\begin{aligned} \left\langle \frac{\partial S}{\partial t} \right\rangle = & - \left(\frac{d^2}{|\nabla\psi|} \right) \int d^3v \left[\left(\frac{\partial f_a}{\partial t} \right)_c \left\{ \ln \left[N_a \left(\frac{m_a}{2\pi T} \right)^{3/2} \right] + \frac{m_a v^2}{2T} + \chi_a + \chi_{ac} + 1 \right\} \right. \\ & \left. + \left(\frac{\partial f_p}{\partial t} \right)_c \left\{ \ln \left[N_p \left(\frac{m_p}{2\pi T} \right)^{3/2} \right] + \frac{m_p v^2}{2T} + \chi_p + \chi_{pc} + 1 \right\} \right] \quad (5-26) \end{aligned}$$

The logarithm term and the term with 1 are zero due to the fact that the collision integral conserves particles, that is,

$$\int d^3v \left(\frac{\partial f_a}{\partial t} \right)_c = 0 \quad (5-27)$$

The two terms in equation (5-26) containing m_e^2 are zero due to the fact that the collision operator conserves energy, that is,

$$\int d^3v \frac{m_e v^2}{2} \left(\frac{\partial f_e}{\partial t} \right) + \int d^3v \frac{m_p v^2}{2} \left(\frac{\partial f_p}{\partial t} \right) = 0 \quad (5-26)$$

After using equation (3-100) for the collision term, equation (5-26) becomes,

$$\left\langle \frac{\partial S}{\partial t} \right\rangle = (\chi_a, K_{aa} \chi_a) + (\chi_a, K_{ap} \chi_p) + (\chi_a, K_{as} \chi_s) + (\chi_s, K_{sa} \chi_a) + (\chi_{ac}, K_{aa} \chi_{ac}) + (\chi_{ac}, K_{ap} \chi_{pc}) - (\chi_{ac}, K_{ap} \chi_p) + (\chi_{ac}, K_{sa} \chi_s) \quad (5-29)$$

If equation (5-1a) and (5-1b) are substituted into equation (5-29) and then using equation (5-1c) to combine the χ_a and χ_{ac} terms, equation (5-29) becomes,

$$\left\langle \frac{\partial S}{\partial t} \right\rangle = (\chi_a, h_a) + (\chi_p, h_p) + \left\langle E \cdot b \frac{J_a}{T} \right\rangle \quad (5-30)$$

where $\left\langle \frac{\partial S}{\partial t} \right\rangle$ is the entropy production rate due to the χ_a and χ_{pc} distribution function, the last four terms in equation (5-29). In obtaining this result the following equation has been used,

$$(\chi_a, H_a \chi_a) = 0 \quad (5-31)$$

This is obtained from equation (K-14) of Appendix K. By

comparing equation (5-30) and (5-23), and using (5-14) and (5-15) it is easy to see that,

$$\frac{\partial}{\partial \xi} \mathcal{L} \{ \xi_a, \xi_b, \xi_c, h \} = - \left\{ \left\langle \frac{\partial S}{\partial \xi} \right\rangle - \left\langle \frac{\partial S_c}{\partial \xi} \right\rangle \right\} \quad (5-32)$$

Thus when $\mathcal{L}$ is an extremum the entropy production rate due to collisions of the χ_a and χ_b distribution functions is an extremum. The extremum is not a maximum or a minimum but a saddle point as can be seen from equation (5-17), the terms which are second order in ξ^+ and ξ^- have different signs. The variational principle could be reformulated so that the functional is the total entropy production rate due to collisions, in which case a trial functions for both χ_a and χ_{ac} would be used in the variational principle. But since χ_{ac} is known this is not necessary.

5.3 OBTAINING TRANSPORT COEFFICIENTS FROM THE VARIATIONAL PRINCIPLE

The quantities needed in the transport equations are,

$$\left(\frac{1}{N} \frac{\partial}{\partial \xi_a} \right) + \left(\frac{1}{N} \frac{\partial}{\partial \xi_b} \right) + N_c D \left(\frac{\partial}{\partial \xi_c} \right) \quad (5-33)$$

$$(\chi_a^T, h_{aT}) + (\chi_\beta^T, h_{\beta T}) + \frac{5}{2} N_A \chi_a^T \frac{E}{kT}$$

$$\frac{D_c}{4\pi i} \langle \frac{E}{kT} \rangle$$

(5-11)

(5-12)

In order that the transport equations be written in their usual form, the above quantities must be written in terms of a sum of transport coefficients times the three gradients, equations (5-11), (5-12) and (5-13). This will be done in this section.

Note that the set of equations for $\hat{J}_a$, $\hat{J}_\beta$ and E_c , equations (5-7a), (5-7b), (5-7c), (5-7d) and (5-7e) are linear in $\hat{J}_a$, $\hat{J}_\beta$ and E_c . Also the three gradients which appear on the right hand side of these equations are multiplied by known quantities, h_{aT} , h_{aT} , $h_{\beta T}$, $h_{\beta T}$, h_T and h_0 . Thus the solution to this set of equations can be written as the solution to the homogeneous equations plus the solutions to the inhomogeneous equations.

$$\hat{J}_a = \hat{J}_{a0} + \hat{J}_{a1} C_1 + \hat{J}_{a2} C_2 + \hat{J}_{a3} C_3 \quad (5-14)$$

$$\hat{J}_\beta = \hat{J}_{\beta 0} + \hat{J}_{\beta 1} C_1 + \hat{J}_{\beta 2} C_2 + \hat{J}_{\beta 3} C_3 \quad (5-15)$$

$$E_c = E_{c0} + E_{c1} C_1 + E_{c2} C_2 + E_{c3} C_3 \quad (5-16)$$

The homogeneous equations are,

$$H_a g_{a0}^+ + K_{aa} g_{a0}^- + \cancel{K_{ap} g_{p0}^-} + C_a E_{b0} = 0 \quad (5-39a)$$

$$H_a g_{a0}^- + K_{aa} g_{a0}^+ + K_{ap} g_{p0}^+ = 0 \quad (5-39b)$$

$$H_\beta g_{\beta 0}^+ + K_{\beta\beta} g_{\beta 0}^- + K_{\beta\alpha} g_{\alpha 0}^- + C_\beta E_{b0} = 0 \quad (5-39c)$$

$$H_\beta g_{\beta 0}^- + K_{\beta\beta} g_{\beta 0}^+ + K_{\beta\alpha} g_{\alpha 0}^+ = 0 \quad (5-39d)$$

$$Z_{ac} \int d^3 f_a^{(0)} g_{a0}^- u + Z_{pe} \int d^3 f_p^{(0)} g_{p0}^- u = 0 \quad (5-39e)$$

It is shown in Appendix N that a solution to this set of equations is of the form,

$$g_{a0}^- = m_a (5 + 8v^2) \quad (5-40a)$$

$$g_{a0}^+ = 0 \quad (5-40b)$$

$$g_{\beta 0}^+ = m_\beta (5 + 8v^2) \quad (5-40c)$$

$$g_{\beta 0}^- = 0 \quad (5-40d)$$

$$E_{b0} = 0 \quad (5-40e)$$

where δ and γ are constants.

The set of equations for β_{a1} , $\beta_{\beta 1}$ and E_{b1} , which will be called the "N set" of equations is,

$$H_{\alpha} \beta_{a1}^{+} + K_{\alpha\alpha} \beta_{a1}^{-} + K_{\alpha\beta} \beta_{\beta 1}^{-} + C_{\alpha} E_{b1} = 0 \quad (5-41a)$$

$$H_{\alpha} \beta_{a1}^{-} + K_{\alpha\alpha} \beta_{a1}^{+} + K_{\alpha\beta} \beta_{\beta 1}^{+} = h_{\alpha N} \quad (5-41b)$$

$$H_{\beta} \beta_{\beta 1}^{+} + K_{\beta\beta} \beta_{\beta 1}^{-} + K_{\beta\alpha} \beta_{a1}^{-} + C_{\beta} E_{b1} = 0 \quad (5-41c)$$

$$H_{\beta} \beta_{\beta 1}^{-} + K_{\beta\beta} \beta_{\beta 1}^{+} + K_{\beta\alpha} \beta_{a1}^{+} = h_{\beta N} \quad (5-41d)$$

$$E_{b1} \left(\int \psi_{\alpha}^{(0)} \psi_{\alpha}^{(0)} d\tau - \int \psi_{\beta}^{(0)} \psi_{\beta}^{(0)} d\tau \right) = h_p \quad (5-41e)$$

The set of equations for β_{a2} , $\beta_{\beta 2}$ and E_{b2} , which will be called the "T set" of equations is,

$$H_{\alpha} \beta_{a2}^{+} + K_{\alpha\alpha} \beta_{a2}^{-} + K_{\alpha\beta} \beta_{\beta 2}^{-} + C_{\alpha} E_{b2} = 0 \quad (5-42a)$$

$$H_{\alpha} \beta_{a2}^{-} + K_{\alpha\alpha} \beta_{a2}^{+} + K_{\alpha\beta} \beta_{\beta 2}^{+} = h_{\alpha T} \quad (5-42b)$$

$$H_{\beta} \beta_{\beta 2}^{+} + K_{\beta\beta} \beta_{\beta 2}^{-} + K_{\beta\alpha} \beta_{a2}^{-} + C_{\beta} E_{b2} = 0 \quad (5-42c)$$

$$H_{\rho} \gamma_{\rho 2}^{-} + K_{\rho \beta} \gamma_{\rho 2}^{+} + K_{\rho \alpha} \gamma_{\rho 2}^{-} = h_{\rho} \quad (5-42d)$$

$$Z_{\alpha e} \left(d^3 f_{\alpha}^{(0)} \gamma_{\alpha \rho} u + Z_{\rho e} \left(d^3 f_{\rho}^{(0)} \gamma_{\rho 2} u - Z_{\rho h} \right) \right) \quad (5-42e)$$

The set of equations for $\gamma_{\rho 2}$, $\gamma_{\rho 3}$ and $\gamma_{\rho \alpha}$, which will be called the "D set" of equations is,

$$H_{\alpha} \gamma_{\alpha 3}^{-} + K_{\alpha \rho} \gamma_{\alpha 3}^{+} + K_{\alpha \beta} \gamma_{\alpha 3}^{-} + C_{\alpha} E_{\alpha 3} = 0 \quad (5-43a)$$

$$H_{\alpha} \gamma_{\alpha 3}^{-} + K_{\alpha \rho} \gamma_{\alpha 3}^{+} + K_{\alpha \beta} \gamma_{\alpha 3}^{-} = 0 \quad (5-43b)$$

$$H_{\beta} \gamma_{\beta 3}^{-} + K_{\beta \rho} \gamma_{\beta 3}^{+} + K_{\beta \alpha} \gamma_{\beta 3}^{-} + \rho E_{\beta 3} = 0 \quad (5-43c)$$

$$H_{\rho} \gamma_{\rho 3}^{-} + K_{\rho \beta} \gamma_{\rho 3}^{+} + K_{\rho \alpha} \gamma_{\rho 3}^{-} = 0 \quad (5-43d)$$

$$Z_{\alpha e} \left(d^3 f_{\alpha}^{(0)} \gamma_{\alpha 3} u + Z_{\rho e} \left(d^3 f_{\rho}^{(0)} \gamma_{\rho 3} u - h_{\rho} \right) \right) \quad (5-43e)$$

If equations (5-36), (5-37) and (5-38) are substituted into the quantities (5-33), (5-34) and (5-35), they become,

$$+ NcD \left(\frac{E_B}{B} \right) \quad (5-46)$$

$$(\beta_1^T, h_{\beta 1}) + \frac{1}{2} NcD \left(\frac{E_B}{B} \right) = D_{21} G + D_{22} G_2 + D_{23} G_3 \quad (5-47)$$

$$\frac{Dc}{4\pi T} \langle E \cdot B \rangle = D_{31} G_1 + D_{32} G_2 + D_{33} G_3 \quad (5-48)$$

where the transport coefficients are given by;

$$D_{11} = (\beta_{a1}^T, h_{a1}) + (\beta_{\beta 1}^T, h_{\beta 1}) + NcD \left(\frac{E_{b1}}{B} \right) \quad (5-47a)$$

$$D_{12} = (\beta_{a2}^T, h_{a1}) + (\beta_{\beta 2}^T, h_{\beta 1}) + NcD \left(\frac{E_{b1}}{B} \right) \quad (5-47b)$$

$$D_{13} = (\beta_{a3}^T, h_{a1}) + (\beta_{\beta 3}^T, h_{\beta 1}) + NcD \left(\frac{E_{b1}}{B} \right) \quad (5-47c)$$

$$D_{21} = (\beta_{a1}^T, h_{aT}) + (\beta_{\beta 1}^T, h_{\beta T}) + \frac{1}{2} NcD \left(\frac{E_{b1}}{B} \right) \quad (5-48a)$$

$$D_{22} = (\beta_{a2}^T, h_{aT}) + (\beta_{\beta 2}^T, h_{\beta T}) + \frac{1}{2} NcD \left(\frac{E_{b2}}{B} \right) \quad (5-48b)$$

$$D_{23} = (\beta_{a3}^T, h_{aT}) + (\beta_{\beta 3}^T, h_{\beta T}) + \frac{1}{2} NcD \left(\frac{E_{b3}}{B} \right) \quad (5-48c)$$

$$D_{31} = \frac{1}{4\pi} \langle E_{b1} B \rangle \quad (5-49a)$$

$$D_{32} = \frac{1}{4\pi} \langle E_{b2} B \rangle \quad (5-49b)$$

$$D_{33} = \frac{1}{4\pi} \langle E_{b3} B \rangle \quad (5-49c)$$

Now the procedure for calculating the D_{ij} will be presented. First the method for calculating the diagonal coefficients will be given, that is, the coefficients with $i=j$. As an example D_{11} will be calculated. To calculate this coefficient one starts with the functional (5-16) and only the ψ_i which were multiplied by G_i are kept in the functional. This new functional is,

$$\mathcal{F}\{\xi_a, \xi_p, \xi, \eta\} = M\{\xi_a, \xi_p, \xi\} - 2\langle \xi_a^\dagger, h_{aa} \rangle - 2\langle \xi_p^\dagger, h_{pp} \rangle - 2\langle \xi, \frac{h_p}{T} \rangle \quad (5-50)$$

If the variation of this functional with respect to $\xi_a^\dagger$,

ξ_a^- , $\xi_p^\dagger$, ξ_p^- and ξ is set to zero, the resulting set of equations is the set given by equations (5-41a),

(5-41b), (5-41c), (5-41d) and (5-41e). Thus the functions

which extremize the functional (5-50) are the functions

$\xi_a^\dagger$, ξ_a^- , $\xi_p^\dagger$, ξ_p^- and ξ . The value of the functional at its extremum is,

$$\sum_{\alpha, \beta, \epsilon}^{\text{extr}} \mathcal{L}\{\xi_{\alpha}, \xi_{\beta}, \epsilon, h_1\} = -\frac{1}{2} \lambda_{\alpha, \alpha} - \frac{1}{2} \lambda_{\beta, \beta} - \frac{1}{2} \lambda_{\epsilon, \epsilon} \quad (5-51)$$

Thus the transport coefficient D_{11} is given by,

$$D_{11} = - \sum_{\alpha, \beta, \epsilon}^{\text{extr}} \mathcal{L}\{\xi_{\alpha}, \xi_{\beta}, \epsilon, h_1\} \quad (5-52)$$

Thus the transport coefficient D_{11} is the value of the functional (5-50) at its extremum. Similarly the other diagonal transport coefficients are,

$$D_{22} = - \sum_{\alpha, \beta, \epsilon}^{\text{extr}} \mathcal{L}\{\xi_{\alpha}, \xi_{\beta}, \epsilon, h_2\} \quad (5-53)$$

$$D_{33} = - \sum_{\alpha, \beta, \epsilon}^{\text{extr}} \mathcal{L}\{\xi_{\alpha}, \xi_{\beta}, \epsilon, h_3\} \quad (5-54)$$

The functional used in the calculation of these two transport coefficients are,

$$\mathcal{L}\{\xi_{\alpha}, \xi_{\beta}, \epsilon, h_2\} = M\{\xi_{\alpha}, \xi_{\beta}, \epsilon\} - 2(\xi_{\alpha}, h_{2T}) - 2(\xi_{\beta}, h_{2T}) - 2\left(\epsilon, \frac{5}{2} \frac{h_2}{T}\right) \quad (5-55)$$

$$\mathcal{L}\{\xi_{\alpha}, \xi_{\beta}, \epsilon, h_3\} = M\{\xi_{\alpha}, \xi_{\beta}, \epsilon\} - 2\left(\epsilon, \frac{h_3}{T}\right) \quad (5-56)$$

The calculation of the off-diagonal transport coefficients, that is D_{ij} where $i \neq j$, is more complicated. The procedure used to calculate these transport coefficients starts with a functional including two of the n_i with one being multiplied by a parameter α . This functional used for calculating the D_{12} transport coefficient is,

$$\begin{aligned} \mathcal{F}\{\xi_a, \xi_b, \xi, \alpha h_1 + h_2\} = & \mathcal{F}\{\xi_a, \xi_b, \xi\} - \alpha \langle \xi_a^+ \xi_b \rangle - \langle \xi^+ \alpha h_1 \rangle \\ & - \langle \xi^+ h_2 \rangle - \langle \xi^+ \frac{\alpha h_1}{T} \rangle - \langle \xi^+ \frac{h_2}{T} \rangle \end{aligned} \quad (5-57)$$

If the variation of the above functional with respect to ξ_a , ξ_b , and ξ is set to zero, the following set of equations are obtained,

$$H_a \xi_a^+ + K_{aa} \xi_a^- + K_{ab} \xi_b^- + C_a E_b = 0 \quad (5-58a)$$

$$H_a \xi_a^- + K_{aa} \xi_a^+ + K_{ab} \xi_b^+ = \alpha h_{an} + h_{aT} \quad (5-58b)$$

$$H_b \xi_b^+ + K_{bb} \xi_b^- + K_{ba} \xi_a^- + C_b E_b = 0 \quad (5-58c)$$

$$H_b \xi_b^- + K_{bb} \xi_b^+ + K_{ba} \xi_a^+ = \alpha h_{bn} + h_{bT} \quad (5-58d)$$

$$Z_{ae} \int d^3 v \xi_a^{(a)} \xi_a^- u + Z_{be} \int d^3 v \xi_b^{(a)} \xi_b^- u = \alpha h_p + \frac{5}{2} n_p \quad (5-58e)$$

Due to the fact that this set of equations is a linear combination of the set of equations (5-41a), (5-41b), (5-41c), (5-41d) and (5-41e) and the equation set (5-42a), (5-42b), (5-42c), (5-42d) and (5-42e), the solution of equations (5-58a), (5-58b), (5-58c), (5-58d) and (5-58e) can be written as,

$$j_{\alpha} = j_{\alpha 1} + a j_{\alpha 2} \quad (5-59a)$$

$$j_{\beta} = j_{\beta 1} + a j_{\beta 2} \quad (5-59b)$$

$$E_b = E_{b1} + a E_{b2} \quad (5-59c)$$

The value of the functional (5-55) at its extremum is,

$$\begin{aligned} \text{extr}_{j_{\alpha}, j_{\beta}, E} \{ \langle j_{\alpha}, j_{\beta}, E, a h_1 + h_2 \rangle \} = & -a \langle j_{\alpha 2}, h_{\alpha N} \rangle - a^2 \langle j_{\alpha 1}, h_{\alpha N} \rangle \\ & - \langle j_{\alpha 2}, h_{\alpha T} \rangle - a \langle j_{\alpha 1}, h_{\alpha T} \rangle - a \langle j_{\beta 2}, h_{\beta N} \rangle - a^2 \langle j_{\beta 1}, h_{\beta N} \rangle \\ & - \langle j_{\beta 2}, h_{\beta T} \rangle - a \langle j_{\beta 1}, h_{\beta T} \rangle - a \langle E_{b2}, \frac{h_{\beta}}{T} \rangle - a^2 \langle E_{b1}, \frac{h_{\beta}}{T} \rangle \\ & - \langle E_{b2}, \frac{5}{2} \frac{h_{\beta}}{T} \rangle - a \langle E_{b1}, \frac{5}{2} \frac{h_{\beta}}{T} \rangle \end{aligned} \quad (5-60)$$

Now taking $\frac{\partial}{\partial a}$ of equation (5-60) and setting $a = 0$.

$$\frac{\partial}{\partial a} \frac{\text{entr}}{\xi_a \xi_\beta \xi} \mathcal{L} \left\{ \xi_a, \xi_\beta, \xi, ah_1 + h_2 \right\} \bigg|_{a=0} = - \left(\gamma_{a2}^+, h_{a2} \right) - \left(\gamma_{a1}^+, h_{a1} \right) - \left(\gamma_{\beta 2}^+, h_{\beta 2} \right) - \left(\gamma_{\beta 1}^+, h_{\beta 1} \right) - \left\langle E_{a2} \frac{h_p}{T} \right\rangle - \left\langle E_{\beta 1} \frac{h_p}{T} \right\rangle \quad (5-61)$$

Using the following Onsager relation given in Appendix 3,

$$\left(\gamma_{a1}^+, h_{a1} \right) + \left(\gamma_{\beta 1}^+, h_{\beta 1} \right) + \left\langle E_{\beta 1} \frac{h_p}{T} \right\rangle = \left(\gamma_{a2}^+, h_{a2} \right) + \left(\gamma_{\beta 2}^+, h_{\beta 2} \right) + \left\langle E_{a2} \frac{h_p}{T} \right\rangle \quad (5-62)$$

equation (5-61) becomes,

$$\frac{\partial}{\partial a} \frac{\text{entr}}{\xi_a \xi_\beta \xi} \mathcal{L} \left\{ \xi_a, \xi_\beta, \xi, ah_1 + h_2 \right\} \bigg|_{a=0} = -2 \left\{ \left(\gamma_{a2}^+, h_{a2} \right) + \left(\gamma_{\beta 2}^+, h_{\beta 2} \right) + \left\langle E_{a2} \frac{h_p}{T} \right\rangle \right\} \quad (5-63)$$

The transport coefficient given by equation (5-47b) is,

$$D_2 = -\frac{1}{2} \frac{\partial}{\partial a} \frac{\text{entr}}{\xi_a \xi_\beta \xi} \mathcal{L} \left\{ \xi_a, \xi_\beta, \xi, ah_1 + h_2 \right\} \quad (5-64)$$

The functionals used in equations (5-68), (5-69) and (5-70) are given respectively by equations (5-50), (5-55) and (5-56). The functional used in equation (5-71) is given by equation (5-57). The functionals used in equations (5-72) and (5-73) are given by the following equations,

$$\begin{aligned} \mathcal{F}_{\alpha, \beta, \gamma} \{ \xi, \eta_2 + \eta_3 \} = M \{ \xi, \xi, \xi \} - 2 \langle \xi_{\alpha}^+, \eta_{2T} \rangle - 2 \langle \xi_{\beta}^+, \eta_{3T} \rangle \\ - 2 \langle \xi \frac{5}{2} a \frac{h_p}{T} \rangle - 2 \langle \xi \frac{h_D}{T} \rangle \end{aligned} \quad (5-74)$$

$$\begin{aligned} \mathcal{F}_{\alpha, \beta, \gamma} \{ \xi, \eta_3 + \eta_4 \} = M \{ \xi, \xi, \xi \} - 2 \langle \xi_{\alpha}^+, \eta_{4T} \rangle - 2 \langle \xi_{\beta}^+, \eta_{3T} \rangle \\ - 2 \langle \xi \frac{h_D}{T} \rangle \end{aligned} \quad (5-75)$$

The particle and energy transport equations contain two other terms which can be written as coefficients times gradients. These are the two terms due to the $\chi'_{\alpha c}$, they are, from the particle and energy transport equation respectively,

$$\left\langle \nabla \cdot \left(\frac{3}{2} \frac{v}{\mathcal{N}} \left(\frac{\partial f_{\alpha}}{\partial t_c} + \frac{\partial f_{\beta}}{\partial t_c} \right) \right) \right\rangle = D_{\chi N1} G_1 + D_{\chi N2} G_2 \quad (5-76)$$

$$\left\langle \nabla \cdot \left(\frac{3}{2} \frac{v^2}{2} \left(\frac{v m_{\alpha}}{\mathcal{N}} \left(\frac{\partial f_{\alpha}}{\partial t_c} + \frac{v m_{\beta}}{\mathcal{N}} \left(\frac{\partial f_{\beta}}{\partial t_c} \right) \right) \right) \right) \right\rangle = D_{\chi E1} G_1 + D_{\chi E2} G_2 \quad (5-77)$$

The $\frac{d}{dt}$ on the right hand side of the above equation is evaluated in Appendix 7.

The particle, energy and toroidal flux transport equations can now be written as,

$$\frac{d}{dt} \left(\frac{3}{2} N \right) = - \dot{N} \quad (5-76)$$

$$\frac{d}{dt} \left(\frac{3}{2} N \right) = - \dot{E} + \dot{E}_E \quad (5-77)$$

$$\frac{d}{dt} \left(\frac{3}{2} \frac{D}{V} \right) = - \dot{D} \quad (5-78)$$

where the fluxes, $\dot{N}$, $\dot{E}$, $\dot{D}$, are given by,

$$\dot{N} = \Gamma_{N1} G_1 + \Gamma_{N2} G_2 + \Gamma_{N3} G_3 \quad (5-81)$$

$$\dot{E} = D_{E1} G_1 + D_{E2} G_2 + D_{E3} G_3 \quad (5-82)$$

$$\dot{D} = \Gamma_{D1} G_1 + \Gamma_{D2} G_2 + \Gamma_{D3} G_3 \quad (5-83)$$

The transport coefficients in these equations are combinations of the coefficients given in equations (5-68) to (5-73) and the coefficients given in equations (5-76) and (5-77).

$$D_N = D_{11} + D_{N1} \quad (5-84)$$

$$D_{N2} = D_{22} + D_{N2} \quad (5-85)$$

$$D_{N3} = D_{33} \quad (5-86)$$

$$D_{E1} = T D_{21} + D_{XE1} \quad (5-87)$$

$$D_{E2} = T D_{22} + D_{XE2} \quad (5-88)$$

$$D_{E3} = T D_{23} \quad (5-89)$$

$$D_{D1} = \frac{4\pi T}{D} D_{31} \quad (5-90)$$

$$D_{D2} = \frac{4\pi T}{D} D_{32} \quad (5-91)$$

$$D_{D3} = \frac{4\pi T}{D} D_{33} \quad (5-92)$$

The source term in the energy equation is given by,

$$S_E = \left\{ \lambda \psi \right\} \frac{dP}{d\psi} (V \cdot \nabla \psi) + \frac{2}{q_n} e_s \cdot \nabla \psi + P_s \nabla^2 \psi \quad (5-33)$$

The transport equation for PN^{s_3} , equation (4-56) can be written as,

$$\frac{d}{dt} \{ PN^{s_3} \} = \frac{1}{\langle n \rangle N^{s_3}} \left\{ \frac{2}{3} \frac{d\mathcal{E}}{d\psi} - \frac{5}{3} T \frac{d\mathcal{E}_w}{d\psi} + \frac{5\mathcal{E}_s}{\langle n \rangle} \nabla^2 \psi \right\} \quad (5-34)$$

The method for computing the transport coefficients have been given in this chapter. The terms in the transport equations (4-42), (4-45), (4-55) and (4-63) which depend on χ_a , χ_p and $E \cdot B$ have been written in terms of extremums of functionals, equations (5-68), (5-69), (5-70), (5-71), (5-72) and (5-73). The terms in these equations which depend on χ_{ac} and χ_{pc} are given in Appendix I. Note that there are still terms in the pressure transport equation (5-79), S_E , which depends on the flux surface velocity and can not be written in terms of transport coefficients. However as will be shown in Chapter 7 equation (5-94) can be used in place of the pressure transport equation

in the description of the time evolution of the plasma properties and the magnetic field. In the next chapter some details will be given for performing the variational principle for linear trial functions.

In this chapter an outline will be given of the procedure for calculating the transport coefficients using the functional described in the previous chapter. In order to obtain results from a variational principle a trial function is substituted into the functional and the extremum of the functional is calculated. The trial functions contain a number of parameters which are determined by setting the derivative of the functional with respect to these parameters to zero. In the case that will be considered here the trial functions are linear functions of these parameters. For these linear trial functions the variational principle reduces to a simple, but large, matrix inversion problem.

An important aspect of any calculation based on a variational principle is the choice of trial functions. There seems to be two approaches to this problem. One is to use intuition or knowledge of similar problems to obtain a trial function, or to use a series expansion using functions which are complete in the space involved in the calculation. In the former method one runs the risk of unconsciously imposing a constraint on the problem which could result in an inaccurate solution. In the latter method one may find that a large number

of terms are needed in the series to obtain an accurate approximation to the solution. In this work a series expansion will be used in the calculation of the diffusion coefficients.

The series used in this calculation is a finite series of which each term in the series is separated into a product of a function of space and two functions of velocity. Sine and cosine are used for the dependence on λ , the space coordinate around the flux surface. Hermite polynomials are used for the dependence on the parallel velocity component, u , and Laguerre polynomials are used for the dependence on the perpendicular velocity component, w .

In the implementation of the variational principle there are a number of inner products, as defined in equation (4-44), which contain the terms in the trial functions and various operators of the kinetic equation, (3-99), (3-100), (3-101), (3-102), (3-103), (3-104) and (3-105). These inner products will become matrix elements of a matrix inversion problem which yields the transport coefficients. In order for the calculation of the transport coefficients to be practical, the evaluation of these matrix elements should be as easy as possible. The most difficult matrix elements to evaluate are the collision matrix elements. The reason for this choice of polynomials

is that for these polynomials it is possible to analytically evaluate the velocity integral part of the collision matrix element. The velocity integrals in the collision matrix elements are four dimensional. To do such integrals numerically would require a very large amount of computer time. The analytical evaluation of the collision matrix is very complicated, see Appendix D. The computer is used to evaluate the analytical expression for the velocity integral in these matrix elements. Without the ability to perform the collision integrals analytically, the calculation of the transport coefficients by this method would be impractical. As an example, the time to do a numerical integration of one of these collision matrix elements requires about five hours of time on a CDC 3400 computer, but the evaluation of the analytic expression requires less than a minute on the same computer.

There are five trial functions in this calculation, the even and odd part of the electron and ion distribution functions and the electric field parallel to the magnetic field. These trial functions are written as,

$$f_e^{+p} = \sum_{j=1}^{NEE} A_j^{EE}(\psi) Y_j^{EE}(\lambda, u, w) \quad (6-1)$$

$$f_e^{-} = \sum_{j=1}^{NEO} A_j^{EO}(\psi) Y_j^{EO}(\lambda, u, w) \quad (6-2)$$

$$Y_{\sigma}^{I\sigma} = \sum_{j=0}^{N_{I\sigma}} Y_{\sigma}^{I\sigma}(j) Y_{\sigma}^{I\sigma}(\lambda, u, w) \quad (6-3)$$

$$Y_{\sigma}^{I\sigma} = \sum_{j=0}^{N_{I\sigma}} Y_{\sigma}^{I\sigma}(j) Y_{\sigma}^{I\sigma}(\lambda, u, w) \quad (6-4)$$

$$Y_{\sigma}^{I\sigma} = \sum_{j=0}^{N_{I\sigma}} Y_{\sigma}^{I\sigma}(j) Y_{\sigma}^{I\sigma}(\lambda) \quad (6-5)$$

$$Y_{\sigma}^{I\sigma}(\lambda, w) = Y_{\sigma}^{I\sigma}(\lambda) Y_{\sigma}^{I\sigma}(\frac{u}{v_{\sigma}}) L_{\sigma}^{I\sigma}(\frac{w^2}{v_{\sigma}^2}) \quad (6-6)$$

$$Y_{\sigma}^{I\sigma}(\lambda, w) = Y_{\sigma}^{I\sigma}(\lambda) Y_{\sigma}^{I\sigma}(\frac{u}{v_{\sigma}}) L_{\sigma}^{I\sigma}(\frac{w^2}{v_{\sigma}^2}) \quad (6-7)$$

$$Y_{\sigma}^{I\sigma}(\lambda, w) = Y_{\sigma}^{I\sigma}(\lambda) Y_{\sigma}^{I\sigma}(\frac{u}{v_{\sigma}}) L_{\sigma}^{I\sigma}(\frac{w^2}{v_{\sigma}^2}) \quad (6-8)$$

$$Y_{\sigma}^{I\sigma}(\lambda, w) = Y_{\sigma}^{I\sigma}(\lambda) Y_{\sigma}^{I\sigma}(\frac{u}{v_{\sigma}}) L_{\sigma}^{I\sigma}(\frac{w^2}{v_{\sigma}^2}) \quad (6-9)$$

$$Y_{\sigma}^{I\sigma}(\lambda, w) = Y_{\sigma}^{I\sigma}(\lambda) Y_{\sigma}^{I\sigma}(\frac{u}{v_{\sigma}}) L_{\sigma}^{I\sigma}(\frac{w^2}{v_{\sigma}^2}) \quad (6-10)$$

... it is a bit bulky, it is in a manner which can be easily converted ... The Hermite polynomial of order ... The Laguerre polynomial of order L is L_L . The ... the ... of order L . The ... (4-1) ... (4-2) consist of four letters.

The last letter in these subscripts is the running index, the orders are functions of l . The second refers to the type of particle, e for electrons, and i for ions. The third letter refers to the parity of the function with respect to α , 1 for even, and 2 for odd. The velocity dependency in equations (6-6) to (6-9) has been normalized to the thermal velocity, given by equation (3-9), v_{te} is the electron thermal velocity, and v_{ti} is the ion thermal velocity.

The A_{α} in equations (6-1) to (6-5) are the Fourier coefficients which are determined by the variational principle. The variational principle is performed in each flux surface, thus the A_{α} are functions of ψ .

The procedure for calculating the A 's proceeds as follows: put the expressions for the trial function (6-1) to (6-5) into the functional $\mathcal{L}$, equation (4-18), then take the derivative of $\mathcal{L}$ with respect to A_{α}^{ee} , A_{α}^{ei} , A_{α}^{ie} , A_{α}^{ii} , and set each of these derivatives to zero. This yields a complete set of equations for the A 's. The set of equations can be written as matrix equation.

$$C \cdot A = h$$

(6-11)

The vector A contains the A_{α} ; equation (6-1) to (6-5).

$$A = \left\{ \begin{array}{c} A_{EE}^{EE} \\ \vdots \\ A_{NEE}^{EE} \\ \hline A_{EO}^{EO} \\ \vdots \\ A_{NEO}^{EO} \\ \hline A_{IE}^{IE} \\ \vdots \\ A_{NIE}^{IE} \\ \hline A_{IO}^{IO} \\ \vdots \\ A_{NIO}^{IO} \\ \hline A_{IF}^{IF} \\ \vdots \\ A_{NIF}^{IF} \end{array} \right\}$$

(6-12)

The matrix $\underline{C}$ is given by,

$$\underline{C} = \left\{ \begin{array}{ccccc} \underline{C}_{EE}^{EE} & \underline{H}_{EE}^{EE} & \underline{C}_{IE}^{EE} & 0 & 0 \\ \underline{H}_{EO}^{EE} & \underline{C}_{EO}^{EO} & 0 & -\underline{C}_{IO}^{EO} & -\underline{F}_{EO}^{EO} \\ \underline{C}_{IE}^{IE} & 0 & \underline{C}_{IE}^{IE} & \underline{H}_{IO}^{IE} & 0 \\ 0 & \underline{C}_{EO}^{IO} & \underline{H}_{IO}^{IE} & \underline{C}_{EO}^{IO} & -\underline{F}_{EO}^{EO} \\ 0 & \underline{H}_{IO}^{IO} & 0 & -\underline{F}_{IO}^{IO} & 0 \end{array} \right\} \quad (6-13)$$

The elements of the sub-matrixes of the matrix (6-13) are given by the following equations,

$$C_{EEI}^{EEK} = (Y_K^{EE}, K_{ee} Y_I^{EE}) \quad (6-14)$$

$$C_{EOI}^{EOK} = (Y_K^{EO}, K_{ee} Y_I^{EO}) \quad (6-15)$$

$$C_{IEI}^{EEK} = (Y_K^{EE}, K_{ei} Y_I^{IE}) \quad (6-16)$$

$$C_{IOI}^{EOK} = (Y_K^{EO}, K_{ei} Y_I^{IO}) \quad (6-17)$$

$$H_{EOI}^{EEK} = (Y_K^{EE}, H_e Y_I^{EO}) \quad (6-18)$$

$$C_{IEI}^{IEK} = (Y_K^{IE}, K_{ii} Y_I^{IE}) \quad (6-19)$$

$$C_{IOI}^{IOK} = (Y_K^{IO}, K_{ii} Y_I^{IO}) \quad (6-20)$$

$$C_{EEI}^{IEK} = (Y_K^{IE}, K_{ie} Y_I^{EE}) \quad (6-21)$$

$$C_{EOI}^{IOK} = (Y_K^{IO}, K_{ie} Y_I^{EO}) \quad (6-22)$$

$$H_{IOI}^{IEK} = (Y_K^{IE}, H_i Y_I^{IO}) \quad (6-23)$$

$$F_I^{EDK} = (C_K Y_K^{ED}, Y_I^E) \quad (E-14)$$

$$F_I^{IDK} = (C_K Y_K^{ID}, Y_I^I) \quad (E-15)$$

The indexes I and K in the above expressions identify respectively, the column and row of the position in the matrix $\underline{C}$ of the various elements.

The vector $\underline{h}$ is given by the following equation.

$$\underline{h} = h_1 \underline{G}_1 + h_2 \underline{G}_2 + h_3 \underline{G}_3 \quad (E-26)$$

where,

$$\underline{h}_i = \left(\begin{array}{c} h_i^{EN} \\ \vdots \\ h_{NEE}^{EN} \\ \hline 0 \\ \vdots \\ 0 \\ \hline h_i^{IN} \\ \vdots \\ h_{NIE}^{IN} \\ \hline 0 \\ \vdots \\ 0 \\ \hline h_i^P \\ \vdots \\ h_{NIP}^P \end{array} \right) \quad (E-27)$$

The third term on the right hand side of equation (4-38) is zero due to the quasineutrality condition being satisfied initially. The fourth term is zero due to the conservation of momentum of the collision term as shown in equation (3-3). The sum of the first and second term will now be shown to be zero. Adding the momentum equation (4-21) for species α and β one obtains, after taking the cross product with $\underline{b}$ and the dot product with $\nabla\psi$,

$$\begin{aligned} \underline{b} \times \nabla \underline{P}_\alpha \cdot \nabla \psi - \underline{b} \times \nabla \underline{P}_\beta \cdot \nabla \psi &= \{Z_\alpha e N_\alpha + Z_\beta e N_\beta\} \underline{b} \times \underline{E}^{(1)} \cdot \nabla \psi \\ + \underline{b} \times \frac{\underline{J} \times \underline{B}}{c} \cdot \nabla \psi + \underline{b} \times \left\{ \int d^3 v m_\alpha \underline{v} \left(\frac{\partial f_\alpha}{\partial t} \right) + \int d^3 v m_\beta \underline{v} \left(\frac{\partial f_\beta}{\partial t} \right) \right\} \cdot \nabla \psi \end{aligned} \quad (4-40)$$

The first term on the right hand side of equation (4-40) is zero due to the quasineutrality condition. The second term is zero since $\nabla \psi \cdot \underline{B} = 0$ and $\nabla \psi \cdot \underline{J} = 0$. The third term is zero due to the conservation of momentum by the collision term. Therefore equation (4-38) reduces to,

$$\frac{d}{dt} \left\{ \int d^3 v [Z_\alpha e N_\alpha + Z_\beta e N_\beta] \right\} \quad (4-41)$$

which is the desired result.

The dotted lines in the equations for $\underline{A}$, $\underline{h}_1$, $\underline{h}_2$ and $\underline{h}_3$ indicate the division of these vectors into sub-vectors. There are NEE , NEO , NEE , NIO and NE terms in the first, second, third, fourth and fifth sub-vectors respectively. The terms in the first, second, third, fourth and fifth sub-vector corresponds to the even electron, odd electron, even ion, odd ion and electric field trial function respectively. The components of the vectors $\underline{h}_1$, $\underline{h}_2$ and $\underline{h}_3$ are given by,

$$\underline{h}_J^{EN} = \left(\underline{Y}_J^{EE}, \underline{h}_{eN} \right) \quad (6-30)$$

$$\underline{h}_J^{IN} = \left(\underline{Y}_J^{IE}, \underline{h}_{iN} \right) \quad (6-31)$$

$$\underline{h}_J^{ET} = \left(\underline{Y}_J^{EE}, \underline{h}_{et} \right) \quad (6-32)$$

$$\underline{h}_J^{IT} = \left(\underline{Y}_J^{IE}, \underline{h}_{it} \right) \quad (6-33)$$

$$\underline{h}_J^P = \left\langle \underline{Y}_J^F, \underline{h}_P \right\rangle \quad (6-34)$$

$$\underline{h}_J^D = \left\langle \underline{Y}_J^F, \underline{h}_D \right\rangle \quad (6-35)$$

The $\underline{A}$ is obtained by the inversion of the matrix equation (6-11),

$$\underline{A} = \underline{C}^{-1} \cdot \underline{h} \quad (6-36)$$

where $\underline{C}^{-1}$ is the inverse of the matrix $\underline{C}$.

An approximation of the electron distribution function, ion distribution function and the parallel electric field can be obtained by substituting the components of the $\underline{A}$ vector as given by equation (3-36) into equations (6-1) to (6-5). However the distribution functions themselves are not the quantities that are needed in the transport equations. The transport coefficients as given by equations (5-68) to (5-73) are the quantities that are needed.

The extremum of the functional as given by equation (5-24) is approximated by the dot product of $\underline{A}$ and $\underline{h}$.

$$\lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon \rightarrow 0}} \mathcal{L}\{\xi_\alpha, \xi_\beta, \xi, h\} \cong -\underline{A} \cdot \underline{h} \quad (6-37)$$

The approximation in this equation becomes an equality as the dimensions of the vectors $\underline{A}$ and $\underline{h}$ approach infinity. The accuracy of the approximation in equation (6-37) can be checked by progressively adding more terms

the trial functions until the change in the result is less than the desired accuracy. If equation (6-36) is substituted into equation (6-37) the following result is obtained.

$$\text{extr}_{\{z_1, z_2, z_3\}} \mathcal{L}\{z_1, z_2, z_3, h\} \cong -\underline{h} \cdot \underline{C}^{-1} \cdot \underline{h} \quad (6-38)$$

The diagonal transport coefficients as given by equations (6-66), (6-69) and (6-70) are given by,

$$D_{ii} = h_i \cdot \underline{C}^{-1} \cdot h_i \quad (6-39)$$

where i is 1, 2 or 3.

For the off-diagonal transport coefficients, the extremization of a functional of the form given by equation (6-50) results in,

$$\underline{C} \cdot \underline{A} = a h_i + h_j \quad (6-40)$$

where i and j are 1, 2 or 3. The value of the functional at its extremum is approximated by,

$$\text{extr}_{\{z_1, z_2, z_3\}} \mathcal{L}\{z_1, z_2, z_3, h_i + h_j\} \cong -[\underline{C}^{-1} \cdot (a h_i + h_j)] \cdot (a h_i + h_j) \quad (6-41)$$

Carrying out the operations indicated in equation (6-41),

that is, taking the derivative with respect to a of equation (6-41) then setting a to zero yields,

$$D_{ij} = -\frac{1}{2} \frac{\partial}{\partial a} \left[\frac{\text{extr}}{\sum_p \epsilon} \left\{ \sum_p \epsilon \right\} \left\{ \sum_p \epsilon, \frac{ah_1 + h_2}{2} \right\} \right]_{a=0} \quad (6-42)$$

or,

$$D_{ij} \approx \frac{1}{2} \left\{ h_1 \cdot C \cdot h_2 + h_j \cdot C \cdot h_i \right\} \quad (6-43)$$

This completes the description of the calculation of the transport coefficients. Note that if $i=j$ then equation (6-43) is the same as equation (6-39), thus equation (6-43) can be used to compute both the diagonal and off-diagonal transport coefficients. Also note that equation (6-43) satisfies the Onsager relations as given in Appendix 0, that is, $D_{ij} = D_{ji}$.

VII OUTLINE OF PLASMA EVOLUTION CALCULATION

In this chapter an outline is given of three different methods of calculating the evolution of the plasma properties in a Tokamak. No actual calculations of this type are performed in this thesis. This is done to indicate how the transport coefficients should be used.

The first method which will be described uses the transport equation for the particle density, the pressure, and the toroidal magnetic flux, equations (4-42), (4-45) and (4-53) respectively. The pressure equation contains a term which depends on the flux surface velocity, i.e., $\partial\psi/\partial t$. In order to compute the pressure at the next time step one could guess at a value of $\partial\psi/\partial t$ and use this guess to compute the pressure. Using this pressure and the new value of D equation (2-17) can be used to compute a new value of ψ . This is, of course, a guess at the new ψ since a guess was used to compute the new ψ . Comparing the new value of ψ with the old, a better guess at $\partial\psi/\partial t$ is hopefully obtained. There is no guarantee that this method will converge. If it does converge then all the quantities have been advanced to the new time.

In the second method, $\partial\psi/\partial t$ is computed from an elliptic differential integral equation. This equation is derived in Appendix J. It is derived by taking the time derivative of the equation for ψ equation, (2-17),

and using the transport equations for P and D .

This equation for $\frac{\partial \psi}{\partial t}$ is,

$$\begin{aligned} \nabla \cdot \left(\frac{\nabla \epsilon_0}{r^2} \right) - 4\pi \frac{d}{d\psi} \left\{ \frac{5P}{3\langle 1 \rangle} \frac{d\langle \epsilon_0 \rangle}{d\psi} \right\} - \frac{1}{r^2} \frac{d}{d\psi} \left\{ \frac{D^2}{\langle 1/r^2 \rangle} \frac{d\langle \epsilon_0 \rangle}{d\psi} \right\} + \epsilon_0 \left\{ 4\pi \frac{d^2 P}{d\psi^2} + \frac{1}{2r^2} \frac{d^2 D^2}{d\psi^2} \right\} = \\ \frac{4\pi}{c} \frac{d}{d\psi} \left\{ \frac{2}{3\langle 1 \rangle} \frac{d\epsilon_0}{d\psi} + \frac{DG_3 \epsilon_0}{6\pi \langle 1 \rangle} \right\} + \frac{1}{r^2} \frac{d}{d\psi} \left\{ -\frac{D}{c \langle 1/r^2 \rangle} \frac{d\epsilon_0}{d\psi} \right\} \quad (7-1) \end{aligned}$$

where,

$$\epsilon_0 = r E_\theta = -\frac{1}{c} \frac{\partial \psi}{\partial t} \quad (7-2)$$

This equation is to be solved over the plasma region.

The rate of change of the magnetic flux linking the torus determines the boundary value of ϵ_0 . It is a constant on the outer flux surface. After $\frac{\partial \psi}{\partial t}$ is computed from this equation, it is used in the transport equation to calculate the new values of pressure, particle density, and toroidal magnetic flux at the new time step. The only problem with this method is the fact that equation (7-1) would be a very difficult one to solve, since it is an integral equation.

The third method uses the transport equations for the three quantities given in Chapter 4, which do not

depend on $\frac{\partial \Psi}{\partial t}$. Introducing new abbreviations for these quantities, they are,

$$Q_1 = \int d^3 r N \quad (7-3)$$

$$Q_2 = PN^{-5/3} \quad (7-4)$$

$$Q_3 = \int d^3 r \frac{D}{r^2} \quad (7-5)$$

The transport equation for these quantities do not depend on the flux surface velocity.

$$\frac{dQ_1}{dt} = -\frac{I}{N} \quad (7-6)$$

$$\frac{dQ_2}{dt} = \frac{N^{-5/3}}{\langle 1 \rangle} \left\{ \frac{2}{3} \frac{dQ_2}{d\Psi} - \frac{5}{2} T \frac{dQ_2}{d\Psi} + \frac{2}{3} \frac{DQ_2}{4\pi} \right\} \quad (7-7)$$

$$\frac{dQ_3}{dt} = -\frac{I}{D} \quad (7-8)$$

Where the fluxes, J_N , J_E , and J_D are written in terms of the diffusion coefficients and gradients as,

$$J_N = \sum_{j=1}^3 D_{Nj} G_j \quad (7-9)$$

$$J_E = \sum_{j=1}^3 D_{Ej} G_j \quad (7-10)$$

$$J_D = \sum_{j=1}^3 D_{Dj} G_j \quad (7-11)$$

The particle density, pressure and D can be written in terms of the Q_i 's as,

$$N = \frac{-1}{\langle i \rangle} \frac{dQ_i}{d\psi} \quad (7-12)$$

$$P = Q_2 \left[\frac{-1}{\langle i \rangle} \frac{dQ_i}{d\psi} \right]^{5/3} \quad (7-13)$$

$$D = \frac{-1}{\langle i \rangle} \frac{dQ_3}{d\psi} \quad (7-15)$$

the new equilibrium which prevails after the diffusion is determined from a variational principle given by [1] and Kulsrud [56]. In this variational principle the quantity which is minimized is the potential energy, W .

$$W = \int d^3r \left[\frac{B^2}{8\pi} + \frac{3}{2}P \right] \quad (7-15)$$

must be placed on this variation otherwise the equilibrium is not a minimum. These constraints are that Q_1 , Q_2 and Q_3 be given. Thus the constraints yields the constraints on the equilibrium calculation. This completes the set of equations.

The calculation would proceed as follows. The initial conditions for $B(r)$ and $D(r)$ which are consistent with the initial Q_i via equation (2-17). One can then specify the temperature or particle density and the boundary condition consists of specifying the temperature, particle density, D and $\frac{\partial \psi}{\partial r}$ for all time. The initial data are used to solve the equations (7-3), (7-4), and (7-5), (7-7) and (7-8) for B , D , ψ and $\frac{\partial \psi}{\partial r}$. The new values of the Q_i are then calculated from the equations (7-3), (7-10) and (7-11) and the calculation is given by this work.

any other theory. The minimization of W , with respect to the Ψ function yields the new Ψ function. The new values of Q_1 , Q_2 and Q_3 are used as constraints during the minimization.

VIII. RESULTS

8.1 TWO SPECIES PLASMA

In this chapter some results of the calculation of the transport coefficients will be presented.

In this section the result for a two species plasma will be given. In the next section some transport coefficients will be given for a three species plasma.

The results presented in this section are for a simple two species plasma, electrons and deuterons. The results are compared with transport coefficients calculated from the equations of Braginskii [28] given in Appendix P.

In the collision regime most of the transport coefficients are in agreement with the expected results. There is some disagreement between the results of this work and that of Braginskii [28] for the first and second energy transport coefficients, equation (5-87) and (5-88).

These two transport coefficients differ by about 20%.

All of the other transport coefficients, equations (5-84), (5-85), (5-86), (5-89), (5-90), (5-91) and (5-92), are within 1/2% of the coefficients from Braginskii's

equations. The transport coefficients calculated from Braginskii's equations are given by equations (P-24), (P-25), (P-26), (P-27), (P-28), (P-29), (P-30), (P-31), (P-32), (P-33), (P-34) and (P-35).

This is for only seven terms in the electron and deuteron trial functions and only three terms in the electric field trial function.

The assumptions made in this work differs from those of Braginskii only in that no mass ratio expansion is made in this work. In the equations of Braginskii the ratio of the mass of the electron to the mass of the deuteron is assumed to be a small number. The reason for not making this assumption in this work is to make it possible to easily extend the calculation to a three species plasma where the mass ratio expansion is not justified. This difference is not expected to lead to the size of the discrepancy present with the two energy transport coefficients. This discrepancy is left unexplained.

In the banana regime the variational principle does not work well. In this regime the transport coefficient shows no convergence with respect to an increasing number of parameters in the trial function. Trial functions with as many as 88 parameters have been tried. Perhaps with more parameters in the trial function it would converge. Even if the results did converge for more parameters, say 1000, it would not be a very useful program due to the large amount of computer time it would take. In the variational principle one must invert a matrix whose dimensions are the total number of parameters in the trial functions.

The most likely reason for the lack of convergence in the banana regime is that the trial functions in velocity space can not properly represent the boundary

between trapped and untrapped particles. The trapped particles are very important in the banana regime. For large aspect ratio (major radius divided by minor radius), the trapped particles occupy a small region in velocity space. There is a large derivative of the distribution function in velocity space at the boundary between the trapped and untrapped particles, see Galeev and Sagdeev[34] It is difficult for trial functions in velocity space to represent this large derivative and maintain the smooth shape in the trapped and untrapped regions. The reason for choosing each term in the series expression for the trial function as a product of a function of space and a function of velocity was to separate the velocity and space integral in each of the matrix elements of the variational principle. Each matrix element can be written as the product of a space integral, which is done numerically, and a two dimensional velocity integral which can be done analytically. Another choice of trial function would be a function of energy and magnetic moment. In this case one has a three dimensional integral which most likely would have to be done numerically. The former method was tried in this work and does not work well in the banana regime.

8.2 THREE SPECIES PLASMA

The transport equations for a three species plasma, electrons and two types of ions, are given in Appendix Q. As pointed out in this appendix, these results are correct only for a plasma in which the ratio of the two ion densities are independent of ψ . This is a very restrictive assumption, but it should show the general dependence of the transport coefficients on the concentration of impurities. In this section the subscript 1 refers to electrons, 2 to ions and 3 to the other species of ions which will be referred to as the impurities. In the collisional regime the terms in the transport equation due to the magnetic drift, like (h_{aV}, χ_a) and (h_{aT}, χ_a) , are small and can be neglected. The variational principle is only needed to calculate the electric field terms. The transport coefficients can be written as,

$$D_m = c^2 P N \eta_L \delta_1 \left\langle \frac{|\nabla \psi|^2}{B^2} \right\rangle + c^2 P N \eta_N \delta_3 D^2 \left\langle \frac{1}{B^2} \right\rangle \quad (8-1)$$

$$D_{N2} = c^2 P N \eta_L \delta_2 \left[\frac{5}{2} - \frac{3 Z_{ef}}{2(1+Z_{ef})} \right] \left\langle \frac{|\nabla \psi|^2}{B^2} \right\rangle + \frac{5}{2} c^2 P N \eta_N \delta_3 D^2 \left\langle \frac{1}{B^2} \right\rangle \quad (8-2)$$

$$D_{N3} = c^2 D^3 N \frac{\eta_{\parallel} \delta_3}{4\pi} \langle 1 \rangle \quad (8-3)$$

$$D_{E1} = c^2 \rho^2 \eta_{\perp} \delta_4 \left[\frac{5}{2} - \frac{3z_{ef}}{2(1+z_{ef})} \right] \left\langle \frac{|v_{\parallel}|^2}{B^2} \right\rangle + c^2 \rho^2 \eta_{\parallel} \delta_3 \left\langle \frac{1}{B^2} \right\rangle \quad (8-4)$$

$$D_{E2} = c^2 \rho^2 \eta_{\perp} \delta_5 \left[\left(\frac{5}{2} \right)^2 - \frac{30z_{ef}}{2(1+z_{ef})} + \frac{z_{ef}^2 \alpha_2 + z_{ef} \sqrt{2\mu_{ef}}}{(1+z_{ef})^2} \right] \left\langle \frac{|v_{\parallel}|^2}{B^2} \right\rangle \\ + \left(\frac{5}{2} \right)^2 c^2 \rho^2 \eta_{\parallel} \delta_3 D^2 \left\langle \frac{1}{B^2} \right\rangle \quad (8-5)$$

$$D_{E3} = \frac{5}{2} \rho c^2 D^2 \frac{\eta_{\parallel} \delta_3}{4\pi} \langle 1 \rangle \quad (8-6)$$

$$D_{D1} = c^2 \rho D \eta_{\parallel} \delta_3 \langle 1 \rangle \quad (8-7)$$

$$D_{D2} = \frac{5}{2} c^2 \rho D \eta_{\parallel} \delta_3 \langle 1 \rangle \quad (8-8)$$

$$D_{D3} = c^2 D \frac{\eta_{\parallel} \delta_3}{4\pi} \langle B^2 \rangle \quad (8-9)$$

where,

$$z_{ef} = \frac{z_2 N_2 + z_3 N_3}{N_2 + N_3} \quad (8-10)$$

$$\mu_{eff} = \frac{\frac{m_1}{m_i} N_1 + \frac{m_2}{m_i} N_2}{N_1 + N_2} \quad (8-11)$$

$$\eta_{\perp} = \frac{4\sqrt{2}\pi}{3} \frac{\ln \Lambda \sqrt{m_i} e^2}{N_i T^{3/2}} (Z_1^2 N_1 + Z_2^2 N_2) \quad (8-12)$$

$$\eta_{\parallel} = \eta_{\perp} \alpha_i \quad (8-13)$$

If one starts with Braginskii's transport coefficients as given in Appendix N and substitute Z_{eff} , equation (8-10), and μ_{eff} , equation (8-11), for Z and the mass ratio, μ respectively, one would obtain equation (8-1) to (8-9) with all of the δ 's set to one. The effective Z is Z_{eff} and the effective mass ratio is μ_{eff} . The perpendicular and parallel resistivity are given by $\eta_{\perp}$ and $\eta_{\parallel}$ respectively. The perpendicular resistivity differs from the usual equation in that two Z 's, Z_1 and Z_2 , are included. Values of the δ 's will be presented in the following paragraphs, they should be thought of as deviations of the actual transport coefficients from the simple extension of Braginskii's results, ($\delta_i = 1$), to three species plasma. The α_1 and α_2 given by Braginskii are in the following table.

Z	α_1	α_2
1	.5129	4.664
2	.4408	3.957
3	.3965	3.721
4	.3752	3.604
∞	.2949	3.25

The concentration of the plasma is determined by the parameter α ,

$$\alpha = \frac{Z_2 N_2}{N_1} \quad (8-14)$$

which is the fraction of the positive charge carried by the third species of particles.

As $\alpha \rightarrow 1$ or 0 the δ'_2 should go to 1. This is the case except for the two δ'_2 in the energy transport equations, δ_4 and δ_5 .

In the limit $\alpha \rightarrow 0$ δ_1 and δ_2 converge to within 0.1% of 1.0. δ_3 converges to a value of about 1.005, this is for seven terms in the trial function. δ_4 and δ_5 do not converge to 1.0. This indicates the discrepancy between these two energy transport coefficients and those of Braginskii. δ_4 converges to .857 and δ_5 converges to .817.

Some of the deviations, δ_1 , δ_2 , δ_4 and δ_5 , can be calculated exactly. One of the deviations, δ_3 , is calculated by the variational principle. The other deviations depend on χ_{ac} which is known. The integrals in δ_1 and δ_2 are done analytically. The integrals in δ_4 and δ_5 can also be done analytically, but due to the complication a computer program was written to do the calculation. The result of the program is exact, it is not a numerical integration.

The expression for δ_1 , δ_2 , δ_4 and δ_5 are,

$$\delta_1 = \frac{N_2 \sum_{\alpha} Z_{\alpha} e N_{\alpha} \sum_{\beta} \frac{Z_{\alpha}^2 N_{\beta} \sqrt{m_{\beta}}}{(1 + m_{\beta}/m_{\alpha})^{1/2}} \left[\frac{1}{Z_{\alpha} e} - \frac{1}{Z_{\beta} e} \right]}{N^2 \sqrt{m_1} e^2 (Z_2^2 N_2 + Z_3^2 N_3)} \quad (8-15)$$

$$\delta_2 = \frac{N_1 \sum_{\alpha} Z_{\alpha} e N_{\alpha} \sum_{\beta} \frac{Z_{\alpha}^2 N_{\beta} \sqrt{m_{\beta}}}{(1 + m_{\beta}/m_{\alpha})^{1/2}} \left\{ \frac{1}{Z_{\alpha} e} - \frac{1}{Z_{\beta} e} \right\} + \frac{3}{2} \frac{\left(\frac{1 - m_{\beta}}{Z_{\alpha} m_{\alpha} Z_{\beta} e} \right)}{(1 + m_{\beta}/m_{\alpha})}}{N^2 \sqrt{m_1} e^2 \left(\frac{5}{2} - \frac{3Z_{ef}}{2(1 + Z_{ef})} \right) (Z_2^2 N_2 + Z_3^2 N_3)} \quad (8-16)$$

$$\delta_4 = \frac{4N_1 \sum_{\alpha} N_{\alpha} Z_{\alpha} e \sqrt{m_{\alpha}} \sum_{\beta} N_{\beta} V'_{1\alpha\beta}}{3N^2 \sqrt{m_1} e^2 \left(\frac{5}{2} - \frac{3Z_{ef}}{2(1 + Z_{ef})} \right) (Z_2^2 N_2 + Z_3^2 N_3)} \quad (8-17)$$

$$\delta_5 = \frac{4N_1 \sum_{\alpha} N_{\alpha} Z_{\alpha} e \sqrt{m_{\alpha}} \sum_{\beta} N_{\beta} V'_{1\alpha\beta}}{3N^2 \sqrt{m_1} e^2 \left\{ \left(\frac{5}{2} \right)^2 - \frac{30Z_{ef}}{2(1 + Z_{ef})} + \frac{Z_{ef}^2 m_{\alpha} + Z_{ef} \sqrt{2\mu_{ef}}}{(1 + Z_{ef})^2} \right\} (Z_2^2 N_2 + Z_3^2 N_3)} \quad (8-18)$$

Values of $V'_{1\alpha\beta}$ and $V'_{2\alpha\beta}$ are given by the table on page 155. Figure (11) to (20) are plots of the various δ 's.

The δ_3 is less than one. It appears in equations (8-3), (8-6), (8-8) and (8-9), and it indicates the modification of the parallel resistivity of a three species plasma. For a hydrogen deuterium plasma δ_3 is one for any α . δ_3 is a monotonic decreasing function of α . For Z_3 equal to 92 and α equal to 1/2, δ_3 is about .6. For a hydrogen helium plasma of α equal to 1/2, δ_3 is about .96. This small deviation should not necessarily be taken to be indicative of any physical effect resulting from the presence of the third species of particles. It should be remembered that the δ_i 's indicate the deviation from a set of transport coefficients which were obtained from a set of two species transport coefficients by a simple replacement of Z by Z_{eff} and μ by μ_{eff} .

The largest deviation from the Braginskii coefficients occurs for the first particle transport coefficient, (8-1). The δ_1 in this equation indicates an enhancement of the perpendicular resistivity. This is the largest of the δ_i 's. For Z_3 equal to 92 and α equal to about 0.3, δ_1 is about 14. For a hydrogen helium plasma of α equal to 1/2, δ_1 is about 2.

The δ_2 shows about the same dependence as δ_1 , except it is not as large as δ_1 . For Z_3 equal to 92 and α equal to 0.3, δ_2 is about 9. For a hydrogen helium plasma of α equal to 1/2, δ_2 is about 1.8. The δ_4 which appears in the first energy transport coefficient, (8-4), is very

similar to δ_2 . The magnitude of δ_5 is less than δ_1 , δ_2 or δ_4 . It has a minimum of about 0.6 and a maximum of about 1.2.

The physical reason for the large value of δ_1 , is that in the three species plasma the collision of ion with impurities can cause the diffusion of both species, whereas in the two component plasma only ion-electron collision can transport particles across a flux surface. Like particle collision can not transport particles. There is not such a dramatic change in δ_4 since like particle collision do transport energy, and Braginskii transport coefficients include the effect of like particle collisions.

The most striking feature of the δ_i is the rapid rise of δ_1, δ_2 and δ_4 with α for small α . For example: let the impurities be tin, $Z_3=50$, $\alpha=.01$ this corresponds to about .02% by number. In this example δ_1 is about 6. That is the perpendicular resistivity has increased by about a factor six due only to .02% tin.

For small α , $\alpha < .1$, the δ can be approximated by a linear function.

$$\delta_1 = 1.0 + \gamma_1 \alpha \quad (8-12a)$$

$$\delta_2 = 1.0 + \gamma_2 \alpha \quad (8-12b)$$

$$\delta_3 = 1.0 + \gamma_3 \alpha \quad (8-12c)$$

$$\delta_4 = .86 + \gamma_4 \alpha \quad (8-12d)$$

$$\delta_5 = .82 + \gamma_5 \alpha \quad (8-12e)$$

For large $z, z_3 > 5$, the δ_i can be approximated by linear functions of

$$\gamma_1 = 13.6 z_3 \quad (8-20a)$$

$$\gamma_2 = 7.33 z_3 \quad (8-20b)$$

$$\gamma_3 = -.266 z_3 \quad (8-20c)$$

$$\gamma_4 = 8.00 z_3 \quad (8-20d)$$

$$\gamma_5 = .45 z_3 \quad (8-20e)$$

The main enhancement in the D_{n1} , D_{n2} and D_{n3} are due to the δ_2 which multiplies the quantities $\left\langle \frac{|\mathbf{v}|^2}{B^2} \right\rangle$

This quantity is a function of the flux surface function Ψ . As an example, consider the first particle transport coefficient D_{n1} , it can be written as:

$$D_{n1} = c^2 P N \eta_n \delta_2 D^2 \left\langle \frac{1}{B^2} \right\rangle (1 + R_i R_\psi) \quad (8-21)$$

where

$$R_\psi = \frac{\left\langle \frac{|\mathbf{v}|^2}{B^2} \right\rangle}{D^2 \left\langle \frac{1}{B^2} \right\rangle} \quad (8-22)$$

$$R_i = \frac{\delta_i}{a_i \delta_3}$$

(8-23)

The term $|v|$ in R_p is the product of the poloidal magnetic field and v . The poloidal magnetic field is larger on the outer flux surface (small aspect ratio) and zero on the magnetic axis (aspect ratio $\rightarrow \infty$). Figure (21) is a plot of the ratio R_p for magnetic flux surfaces given by Shafranov [55] as a function of the aspect ratio. The parameter which determined these flux surfaces are, total theta current enclosed by the outer flux surface is 1 megamp, the value of the q , magnetic stability factor on the outer surface is 3.

The major axis is 150 cm. The inner radius, the distance from axis of rotational symmetry to the closest flux surface at $z = 0$, is 110 cm. The graph was drawn for three values of the shape parameter S , $S = 1$ corresponds to circular flux surfaces. $S < 1$ corresponds to elongation in the z direction. Figure (22) is a diagram of the three flux surface shapes used in Figure (21). The ratio R_p is insensitive to the shape parameter S .

For a two component plasma R_i varies from 1.96 for $z = 1$ to 3.4 as $z \rightarrow \infty$. R_i can be very much larger

for a three species plasma. For example, $Z_3 = 50$, $\alpha = 0.1$ this is about .30 by number, R_1 is about 6.6.

The maximum for R_1 is about 79 for $Z_3 = 50$ and $\alpha = 0.1$.

Since R_1 is larger on the outer flux surfaces, the enhancement of the particles transport is expected to be greater there.

Finally, Figures (23), (24), (25) are examples of the diffusion coefficients plotted as a function of aspect ratio. The plot were for a magnetic field whose magnetic axis is at $r = 0$ cm. The shear parameter is $S = 1$ (see Figure 12). The total current is enclosed by the outer flux surface is 100 kGauss. The stability factor on the outer flux surface is $q = 2.0$. The quantities plotted are the ratio of the diffusion coefficients to the value of the diffusion coefficients with all δ_i equal to 1, Prager's results. Figure (23) is for a copper plasma, a common impurity in Tokamak, for two different values of α . Figure (24) is for tin, to illustrate the effect of a small quantity of a high Z impurity, and Figure (25) is for a uranium impurity plasma to illustrate a practical upper limit on the enhancement of the transport due to impurities.

V_{100} and V_{200}

$Z_2 A_2$	$Z_3 A_3$	V_{111} V_{111}	V_{112} V_{112}	V_{113} V_{113}	V_{121} V_{121}	V_{122} V_{122}	V_{123} V_{123}	V_{131} V_{131}	V_{132} V_{132}	V_{133} V_{133}
1 2 0	1 1 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-1 115E+04	-2 425E+03	-2 937E+03	-9 167E+03
1 2 0	1 3 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-3 576E+03	-8 085E+02	-7 669E+03	-5 304E+03
1 2 0	2 4 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-4 077E+04	-3 936E+02	-4 769E+01	-8 167E+03
1 2 0	3 7 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-5 757E+04	-1 540E+02	-6 594E+02	-1 042E+04
1 2 0	4 9 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-7 881E+04	-1 701E+01	-1 829E+03	0
1 2 0	5 11 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-7 881E+04	-5 616E+01	-1 829E+03	0
1 2 0	6 12 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-2 899E+05	-7 352E+01	-1 065E+01	-1 385E+04
1 2 0	7 14 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-4 445E+05	-6 179E+01	-1 183E+01	-1 591E+04
1 2 0	8 16 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-5 105E+05	-4 981E+01	-1 108E+03	-1 719E+04
1 2 0	9 18 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-6 026E+05	-4 107E+01	-1 031E+03	-1 837E+04
1 2 0	10 20 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-7 285E+06	-3 033E+01	-9 974E+02	-2 054E+04
1 2 0	12 24 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-8 821E+05	-2 302E+01	-7 695E+02	-2 250E+04
1 2 0	13 27 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-1 472E+06	-2 072E+01	-7 195E+02	-2 298E+04
1 2 0	30 64 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-1 366E+07	-7 971E+00	-3 498E+02	-3 458E+04
1 2 0	20 40 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-1 322E+06	-1 264E+01	-5 407E+02	-3 912E+04
1 2 0	40 80 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-2 056E+07	-5 055E+00	-2 149E+02	-4 874E+04
1 2 0	50 110 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-3 278E+07	-3 769E+00	-1 944E+02	-4 444E+04
1 2 0	60 142 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-4 636E+07	-2 095E+00	-1 631E+02	-4 945E+04
1 2 0	70 172 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-5 245E+07	-2 519E+02	-1 988E+02	-5 104E+04
1 2 0	80 202 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-6 260E+07	-2 127E+00	-1 162E+02	-5 144E+04
1 2 0	90 232 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-7 911E+07	-1 791E+00	-9 917E+01	-5 154E+04
1 2 0	99 262 0	3 936E+05	1 713E+06	1 713E+06	-1 813E+03	-8 496E+03	-1 490E+07	-1 717E+00	-1 010E+02	-5 154E+04

Figure 11 δ_1 vs α

Figure 12 δ_1 vs α

Figure 13 δ_1 vs α

Figure 14 δ_2 vs α

Figure 15 δ_3 vs α

Figure 16

 δ_2 vs α

Figure 17 δ_4 vs α

Figure 18 δ_4 vs α

Figure 19 δ_S vs α

Figure 20

 δ_5 vs α

Figure 21 R_ψ vs Aspect Ratio

Figure 22 Flux Surface

Figure 23 Transport Coefficient for deuterium oxygen plasma for $\alpha = 0.1$ (1.33) and $\alpha = 5.5(119)$

Fig. 25. Transport coefficient ratio α vs. aspect ratio α for S_n^{118} .

APPENDIX A

PFIRSCH SCHLUTER TRANSPORT COEFFICIENTS

In this Appendix the transport coefficient for a resistive plasma will be derived. The approximation of the plasma as a resistive fluid is an appropriate approximation for the collisional or Pfirsch-Schluter regime. The large aspect ratio expansion will be made in the Appendix. The coordinate used in this appendix is shown in Figure (A-1). The toroidal angle is ϕ and the poloidal angle is θ . The inverse aspect ratio is,

$$\epsilon = \frac{a}{R_0}$$

(A-1)

which is assumed to be a small number.

The radial, poloidal and toroidal components of the magnetic field are given respectively by,

$$B_r$$

$$B_\theta \quad B_\phi$$

Figure A-1 ϕ, ϕ_T coordinate system

$$B_T = B_0 (1 + \epsilon \cos \theta_p)$$

(B-4)

The magnitude of the magnetic field is approximately 3.

In order that q ; which is,

$$q = \frac{1}{R_0} \frac{B_T}{B_P}$$

be about 1 to 3, as required for magnetohydrodynamic stability, the toroidal component of the magnetic field should be much larger than the poloidal component.

$$B_T \gg B_P$$

The low β approximation will be valid, that is, the magnetic field pressure is much larger than the gas pressure. In this approximation the largest components of the magnetic field are generated by external currents and are constant in time, thus,

3B
31

(A-7)

$\mathbf{E} = 0$

(A-8)

Equation (A-7) implies that,

$\mathbf{V} \cdot \mathbf{E} = 0$

(A-9)

Thus the electric field can be written in terms of a potential,

$\mathbf{E} = -\nabla \phi$

(A-10)

The trajectory coefficients are calculated by reducing
an expression for the flux of particles crossing a flux

surface per unit area is given by,

$$J = \frac{N}{A} \cdot \vec{v} \cdot \vec{A} \quad (A-11)$$

The particle density is N . The area of the flux surface is A . The area element dA and $v_{\perp}$ is the component of the fluid velocity perpendicular to the flux surface. The perpendicular component of the fluid velocity can be computed from Ohm's law,

$$\vec{v} \times \vec{B} = \eta_{\parallel} J_{\parallel} + \eta_{\perp} J_{\perp} + \nabla \phi \quad (A-12)$$

The parallel and perpendicular resistivity are $\eta_{\parallel}$ and $\eta_{\perp}$ respectively. The current has been divided into a component parallel and perpendicular to $\vec{B}$, $J_{\parallel}$ and $J_{\perp}$ respectively. An equation for $\vec{v}_{\perp}$ can be obtained by taking the cross product of equation (A-12) with $-\vec{B}$.

$$\vec{v}_{\perp} \times \vec{B} = -\eta_{\perp} \vec{J}_{\perp} \times \vec{B} + \eta_{\parallel} \vec{J}_{\parallel} \times \vec{B} - \nabla \phi \times \vec{B} \quad (A-13)$$

The area element, dS , can be written in terms of the pressure gradient ∇P .

$$dS = - \frac{d^2 \nabla P}{|\nabla P|} \quad (A-14)$$

Equation (A-13) and (A-14) are substituted into equation (A-11).

$$\eta = - \frac{N}{A} \frac{d^2}{|\nabla P|} \nabla P \cdot \left\{ c \frac{\nabla \times \nabla \Phi}{R^2} - \frac{\eta c^2}{\theta^2} \nabla P \right\} \quad (A-15)$$

The pressure balance equation,

$$\nabla P = \frac{\rho \times \mathbf{K}}{c} - \frac{\mathbf{B}}{c} \quad (A-16)$$

is substituted into equation (A-15).

$$\frac{N}{A} \frac{d^2}{|\nabla P|} \left\{ \eta_1 J^2 - \frac{(\nabla \Phi) \cdot (\nabla \Phi)}{\theta^2} + \nabla \cdot \nabla \Phi \right\} \quad (A-17)$$

The $\underline{B} \cdot \underline{\nabla \Phi}$ term can be obtained by taking the dot product of Ohm's law, equation (A-12) with $\underline{B}$.

$$\underline{B} \cdot \underline{\nabla \Phi} = -\eta_{\parallel} J_{\parallel} B_0 \quad (\text{A-18})$$

The last term in equation (A-17) can be shown to be zero.

This term can be written as,

$$\int \frac{d^3}{dV} \underline{J} \cdot \underline{\nabla \Phi} = \frac{d}{dP} \int d^3 \underline{J} \cdot \underline{\nabla \Phi} \quad (\text{A-19})$$

The right hand side of equation (A-19) is integrated by parts to yield,

$$\int d^3 \underline{J} \cdot \underline{\nabla \Phi} = - \int d^3 \Phi \underline{\nabla} \cdot \underline{J} + \int dS \cdot \underline{J} \quad (\text{A-20})$$

The first term on the right hand side of equation (A-19) is zero due to the quasineutrality condition.

$$\vec{I} \cdot \vec{J} = 0$$

(A-21)

The second term is zero since $\vec{I}$ is parallel to the constant pressure surfaces.

$$dS \cdot \vec{J} = 0$$

(A-22)

Equation (A-17) can be written as,

$$\alpha^2 = \frac{N}{A} \frac{d^2}{VP} (n_e J_z^2 + n_i J_z^2)$$

(A-23)

The perpendicular component of the current can be calculated from the pressure balance equation (A-15). The parallel component of the current can be computed from the perpendicular component and the quasineutrality condition

(A-21). The current can be written as,

$$\vec{J} = J_z \hat{z} + \frac{c B \times \nabla P}{B^2}$$

(A-24)

Substituting this equation into the quasineutrality condition (A-21) yields,

$$\underline{b} \cdot \nabla J_{||} + c \nabla \cdot \left(\frac{1}{B^2} \right) \underline{B} \times \nabla P + \frac{c}{B^2} \nabla \cdot (\underline{B} \times \nabla P) = 0 \quad (\text{A-23})$$

The last term in this equation is zero due to equation (A-8) and the fact that the curl of a gradient is zero. Equation (A-24) can be written as,

$$\underline{b} \cdot \nabla J_{||} = \frac{2c}{B^3} (\underline{B} \times \nabla P) \cdot \nabla \underline{B} \quad (\text{A-26})$$

Since $\underline{b}$ has no radial component and since $J_{||}$, as any scalar, has no ϕ_r dependence due to the assumption of axisymmetry, the operator $\underline{b} \cdot \nabla$ can be written as,

$$\underline{b} \cdot \nabla = \frac{B_r}{B^2} \frac{\partial}{\partial r} \quad (\text{A-27})$$

The operator on the right hand side of equation (A-26) is,

$$(\underline{B} \times \nabla \rho) \cdot \underline{\nabla} = \frac{B}{r} \frac{\partial \rho}{\partial r} \frac{\partial}{\partial \phi} \quad (\text{A-28})$$

since ρ has only an r dependence. Using equations (A-27) and (A-28) equation (A-26) becomes,

$$B_r \frac{\partial J_z}{\partial \phi} = \frac{2c\epsilon}{B} \frac{\partial \rho}{\partial r} \frac{\partial B}{\partial \phi} \quad (\text{A-29})$$

The derivative of B on the right hand side of equation is obtained by taking the derivative of equation (A-4).

$$\frac{\partial J_z}{\partial \phi} = \frac{2c\epsilon}{B_r} \frac{\partial \rho}{\partial r} \sin \phi \quad (\text{A-30})$$

Thus J_z is given by,

$$J_z = -\frac{2c\epsilon}{B_r} \frac{\partial \rho}{\partial r} \cos \phi \quad (\text{A-31})$$

and J_r is,

$$J_1 = \frac{c}{B} \frac{\partial P}{\partial x} \quad (\text{A-32})$$

Substituting equation (A-31) and (A-32) into equation (A-23) yields,

$$\Theta = -\frac{2\pi R_A}{A \frac{\partial P}{\partial x}} N \int_0^{2\pi} d\phi \left\{ \frac{n_1 c^2}{B^2} + \frac{n_1 + \epsilon^2}{B^2} \cos^2 \phi \right\} \left(\frac{\partial P}{\partial x} \right)^2 \quad (\text{A-33})$$

The area, A , of the torus is,

$$A = 4\pi^2 R_A \quad (\text{A-34})$$

Equation (A-32) becomes,

$$J = -N \frac{c^2 n_1}{B^2} \left(1 + \frac{1}{2} \frac{n_1 + \epsilon^2}{n_1} \frac{1}{K^2 B^2} \right) \frac{\partial P}{\partial x} \quad (\text{A-35})$$

For a hydrogen plasma the ratio of the resistivities is approximately 1.

(A-26)

For a plasma with uniform temperature the particle flux can be written as,

(A-37)

where D is the transport coefficient given by,

(A-38)

If there were no parallel current, as in the case of a straight cylinder with no longitudinal magnetic field, then the transport coefficient would be D_0 .

If ρ is about 3, as in some experiments, the parallel current can cause an order of magnitude increase in the transport coefficients.

APPENDIX B

BANANA TRANSPORT COEFFICIENT

In this appendix the transport coefficient for particles diffusion in the banana regime is derived. This is only an order of magnitude calculation intended to describe the modification of the transport coefficients due to trapped particles. The large aspect ratio expansion is made in this appendix, that is, the inverse of the aspect ratio, ϵ , is small.

$$\epsilon = \frac{a}{R_0} \ll 1 \quad (B-1)$$

The coordinate system has been described in figure (A-1) of Appendix A. The magnetic field used in this appendix is the same as the one used in Appendix A.

As a particle moves along the magnetic field line it experiences a change in the magnitude of the magnetic field. The field is larger on the inside of the torus, near the axis of rotational symmetry, and weaker on the outer edge of the torus. The result of this minimum in the field is that particles with a small velocity parallel to the magnetic field will be trapped on the outer edge of the torus. It will be shown in this appendix

that these trapped particles oscillate in the electric field. In the region of confinement, the electric field is strong enough that the trapped particles oscillate in the electric field between the maxima in the magnetic field before making a collision.

The angle in velocity space, δ , between the velocity of particles, δ , is calculated from the conservation of energy, $\frac{mv^2}{2}$, and magnetic moment, $\frac{mv_{\perp}^2}{2B_0}$.

$$\delta = \sqrt{\epsilon}$$

If the particle distribution function is assumed to be approximately Maxwellian, then the density of trapped particles, n_{tr} , is approximately

$$n_{tr} = \frac{1}{2} n \quad (2-1)$$

where n is the total particle density.

The average velocity of the trapped particles parallel to the magnetic field is small. If it is too small then, they would not be trapped. Therefore, the small velocity is,

$$V_{\parallel} = \sqrt{\epsilon} V_t \quad (2-2)$$

λ_c is the critical wavelength
 of the plasma. The critical
 wavelength is given by

$$\lambda_c = \frac{2\pi c}{\omega_p}$$
 where ω_p is the plasma
 frequency. The plasma
 frequency is given by

$$\omega_p = \sqrt{\frac{n_e e^2}{m_e}}$$
 where n_e is the electron
 density, e is the electron
 charge, and m_e is the
 electron mass.

the maximum value of the function is

The maximum value of the function is

The maximum value of the function is

where Δ is the determinant of the matrix
The determinant of the matrix is
 Δ , is obtained by substituting the
equation (P-7), in the determinant
the result is,

$$(P-8)$$

This is called the banana width, it can be written in terms of the gyroradius, a , as,

$$= \frac{1}{2} \frac{v_{\perp}^2}{\omega_c^2} \frac{1}{a} \quad (B-11)$$

In order to calculate the diffusion coefficient, D , one needs to know the average step size, Δ , and the "step frequency", ν .

$$D = \frac{1}{2} \Delta^2 \nu \quad (B-12)$$

The step size has been calculated, equation (B-11).

The usual classical value of the step size is the gyroradius, a . As can be seen the step size for the trapped particles, equation (B-10), is larger than the gyroradius by a factor of $\sqrt{2}$.

The relevant collision frequency, ν , for this problem is the collision frequency for scattering of trapped particles to untrapped orbits. Since the trapped particles occupy a small angle in velocity space, they can be transferred into untrapped orbits by small angle scattering. If the scattering process is considered to be a diffusion process in velocity space, then the time for diffusing an angle is proportional to the square of the angle. Since the angle occupied by trapped particles in velocity space is $\sqrt{2}$, then the effective collision frequency to be used in equation (B-12) is,

$$\nu_{eff} = \nu_c$$

(B-13)

where ν_c is the classical collision frequency, electron ion collision frequency.

Now the diffusion coefficient can be calculated, equation (B-11) and equation (B-13) are used in equation (B-12). Since all the particles are not trapped particles, equation (B-12) must also be multiplied by the fraction of trapped particles, equation (B-3), the result is,

$$D = \frac{D_0}{\sqrt{\epsilon}} \left(1 + \frac{c^2 \eta_{\perp}}{B^2} \right)^{1/2} \quad (B-14)$$

or,

$$D = D_0 \left(\frac{R}{a} \right)^{3/2} \quad (B-15)$$

where,

$$D_0 = a^2 \nu_c = \frac{c^2 \eta_{\perp}}{B^2} \quad (B-16)$$

The $\eta_{\perp}$ is the resistivity perpendicular to the magnetic field, D_0 is the diffusion coefficient of a straight cylinder plasma.

For a torus of aspect ratio of 3 and q of 3, the diffusion coefficient in the banana regime can be increased by a factor of about 47.

APPENDIX 3

DESCRIPTION OF THE COMPUTER PROGRAM

3.1 INTRODUCTION

There are two computer codes which are involved in the calculation of the diffusion coefficients. The first code, COLINT, computes the velocity integrals which are used in the collisional elements of the matrix of the variational principle. This program does not depend on the magnetic field structure. The input to this program is a description of the trial functions. The second program NEOCO, takes the output of COLINT and the description of the magnetic field, compute the transport coefficients. NEOCO has the ability to select from the output of COLINT the collision integrals it needs. If NEOCO needs an integral that has not been calculated by COLINT, the computation is terminated and COLINT must be run again.

Now a description of the two programs will be given. Some of the equations and methods used by these programs will be given. Section 3.2 describes COLINT. Section 3.2 describes NEOCO, and section 3.3 gives an example of NEOCO's output.

3.2 COLINT

The program COLINT evaluates the velocity integrals needed by NEOCO. These integrals are evaluated analytically. Since there are many terms involved in the calculation of a single integral, a computer code is used to do the calculation. The higher the order of the polynomial used in the trial functions, the larger the number of terms becomes. After the integral is expanded, each term can be written as a product of six simple integrals

all of which can be evaluated analytically.

The equations used by COLINT are given in Appendix

- D. The quantities calculated by COLINT are the $V_{JJ'}^{(a)}$, $V_{JJ'}^{(a)}$, and $V_{JJ'}^{(p)}$, given by equations (D-7), (D-8) and (D-9).

Four subroutines are called which set up various arrays to be used in the calculation. One subroutine calculates the binomial coefficients;

$$C(N, M) = \binom{N-1}{M-1} \quad (C-1)$$

These coefficients are used in the expansion of the F function, (D-23). Three other subroutines generate arrays which contain the values of the following integrals.

$$B_\theta(N, M) = \int_0^\pi \sin^{N-1} \theta \cos^{M-1} \theta d\theta \quad (C-2)$$

$$B_\phi(N, M) = \int_0^{2\pi} \sin^{N-1} \phi \cos^{M-1} \phi d\phi \quad (C-3)$$

$$E(N) = \int_0^\infty s^{N-1} e^{-s} ds \quad (C-4)$$

The values of these integrals stored in the arrays are used to compute the integrals of each of the terms in the expanded form of the V_s , equation (D-7), (D-8) and (D-9).

C.3 NEOCC

The program NEOCC computes the transport coefficients. In this program the Ψ function is given analytically as described in Appendix L. The description of the Ψ , θ , λ coordinate system is set up in this program. It is not necessary that the Ψ function be given by an analytic expression. The part of the code which computes the transport coefficients could work just as well with any flux surfaces described by the arrays $R(J,N)$ and $Z(J,N)$. These are the y and z coordinates of the J^{th} flux surface and the N^{th} λ surface. Also $BP(J,N)$, the poloidal magnetic field, must be given. It is calculated using equation (K-4) given in Appendix K. The calculation of the various matrix elements used in the variational principle involves integrals over a flux surface. There are no integrals in the calculation which are along a λ surface; thus the coordinate system need not be orthogonal in order to compute the transport coefficients.

This program uses the method given in Chapter 6. Equations for the various matrix elements are given in Appendix D.

C.4 EXAMPLE OF COMPUTER PROGRAM OUTPUT

The set of trial functions used in this example is one of the few that gave reasonable results in the banana regime. Unfortunately, this is not a typical case. It was found that if one omits the component of the trial functions which depend on the product of the third order Hermite polynomial and Zeroth order Laguerre, the resulting transport coefficients will show the expected increase in the banana regime. For most sets of trial function the transport coefficients diverge or remain near the collisional value in the banana regime. This particular trial function does not give good results in the collisional regime. The coefficients are low by at most 40% in the collisional regime. The omitted component is needed to give good results in the collisional regime.

Now an example of the graphic output of the program NEDCO will be given. Figure (C-1) gives a plot of the flux surface shape. The parameters that are printed on this page are, NSI, the number of the flux surfaces used, NL, is the number of lambda points on each flux surface, RA is the radius of the magnetic axis in centimeters, RBO is the radius of the intersection of the outer flux surface and the $z=0$ plane, ALPHA is a shape parameter, QI is the magnetohydrodynamic stability factor on the outer flux surface, and CTOTAL is the current enclosed by the outer flux surface in megamperes. The next graph on this figure is of various quantities which are functions of ψ only.

The next figure (C-2) gives the description of the trial function. The order of the various sine, cosine, Hermite, and Laguerre polynomial are given.

Figure (C-3) gives a graph of the nine transport coefficients vs. temperature. The quantities that are plotted are the ratios of the transport coefficients to the collisional transport coefficients of Braginskii. An equation for the collisional transport coefficient is given at the top of the graph. Due to the Onsager relations there is some overlap of these plots.

The next figure (C-4) gives a three dimensional graph of the trial function. The front axis is the parallel velocity. The back axis is the perpendicular velocity. These graphs are plotted at three values of λ . λ equal to zero corresponds to the outside edge of the torus. Also the electric field trial function is plotted vs. λ . These graphs are plotted for only one flux surface.

The next graph page, figure (C-5) is a summary page. It gives the flux surface shape, the toroidal and poloidal magnetic fields as a function of λ for a particular flux surface. Also the transport coefficients are plotted again.

Table (C-1) is a printed summary of the transport coefficients. The values printed are the ratio of the transport coefficient to the Braginskii values.

The program also prints out other numbers which are of a diagnostic nature and are not illustrated in the appendix.

FLOW SURFACES

NS1= 20
 PA= 150.000
 PEO= 150.000
 ALP= 1.000
 TO= 3.000
 COTA= 1.000

PROFILES AT Z=0.0

P=PRESSURE (2.58E+06 DYNES/CM²)
 N=PARTICLE DENSITY
 T=TEMPERATURE
 I=CURRENT (1.00E+06 AMPS)
 G=STABILITY FACTOR (1.00E+00)

Figure C-1

TRIAL FUNCTION DESCRIPTION

TOTAL NUMBER OF PARAMETERS = 36

ELECTRON EVEN

COS SIN HERMITE LAGUERRE

0	1	0	1
0	1	0	2
0	1	2	0
0	1	2	1
0	1	4	0

ION EVEN

COS SIN HERMITE LAGUERRE

0	1	0	1
0	1	0	2
0	1	2	0
0	1	2	1
0	1	4	0

ELECTRON ODD

COS SIN HERMITE LAGUERRE

0	0	1	0
0	0	1	1
0	0	1	2
0	0	3	1
0	0	5	0
0	0	5	1
1	0	1	0
1	0	1	1
1	0	1	2
1	0	3	1
1	0	5	0
1	0	5	1

ION ODD

COS SIN HERMITE LAGUERRE

0	0	1	0
0	0	1	1
0	0	1	2
0	0	3	1
0	0	5	0
0	0	5	1
1	0	1	0
1	0	1	1
1	0	1	2
1	0	3	1
1	0	5	0
1	0	5	1

ELECTRIC FIELD

COS SIN

0	0
1	0

Figure C-2

Figure 2-4

FLUX SURFACES (SHAFRANOV)

FLUX SURFACE SURF
M
MAJOR RADIUS
MINOR RADIUS
ASPECT RATIO
POLARITY
TANGENTIAL
WIND
HEIGHT
APPROX. ECCENTRICITY
TOTAL CURRENT
STABILITY FACTOR

1.0
0.02
0.0

MAGNETIC FIELDS

EXTERNAL
MAGNETIC
FIELD
CALCULATED
LAPLACE

INTERNAL
MAGNETIC
FIELD
CALCULATED
LAPLACE

TRANSPORT COEFFICIENTS

QUANTITIES TRANSPORT COEFFICIENTS ARE THE SAME AS IN THE FIRST RUN. CALCULATED AND TOTAL NUMBER OF PARAMETERS IN THE FIRST RUN IS 10.

EXTERNAL TRANSPORT COEFFICIENTS

EXTERNAL
TRANSPORT
COEFFICIENTS
CALCULATED
LAPLACE

INTERNAL TRANSPORT COEFFICIENTS

INTERNAL
TRANSPORT
COEFFICIENTS
CALCULATED
LAPLACE

TOTAL FLUX TRANSPORT COEFFICIENTS

TOTAL FLUX
TRANSPORT
COEFFICIENTS
CALCULATED
LAPLACE

$\beta = 0.36$ $\gamma = 0.1$ $\alpha = 20$ $K_A = 1.82E+02$ $R_{90} = 1.17E+02$ $A_{PWA} = 1.09E+03$ $G_1 = 3$ $C_L = 0.0$ $C^*_L = 1.00E+00$

[illegible][illegible][illegible]

J	DATE	DW	DW	DW	DW	DW	DW	DW	DW	DW
1	5/1/75	1.191	1.197	1.195	1.199	1.199	1.199	1.197	1.199	1.199
2	5/2/75	1.191	1.197	1.195	1.199	1.199	1.199	1.197	1.199	1.199
3	5/3/75	1.191	1.197	1.195	1.199	1.199	1.199	1.197	1.199	1.199
4	5/4/75	1.191	1.197	1.195	1.199	1.199	1.199	1.197	1.199	1.199
5	5/5/75	1.191	1.197	1.195	1.199	1.199	1.199	1.197	1.199	1.199
6	5/6/75	1.191	1.197	1.195	1.199	1.199	1.199	1.197	1.199	1.199
7	5/7/75	1.191	1.197	1.195	1.199	1.199	1.199	1.197	1.199	1.199
8	5/8/75	1.191	1.197	1.195	1.199	1.199	1.199	1.197	1.199	1.199
9	5/9/75	1.191	1.197	1.195	1.199	1.199	1.199	1.197	1.199	1.199
10	5/10/75	1.191	1.197	1.195	1.199	1.199	1.199	1.197	1.199	1.199
11	5/11/75	1.191	1.197	1.195	1.199	1.199	1.199	1.197	1.199	1.199
12	5/12/75	1.191	1.197	1.195	1.199	1.199	1.199	1.197	1.199	1.199

[illegible]

Table C-1

EVALUATION OF MATRIX ELEMENTS

For the purpose of simplification in this appendix, the trial functions are written as,

$$\psi_{\alpha} = \sum_{j=1}^{N_{\alpha}} A_{\alpha j}(\varphi) Y_j^{\alpha}(\lambda, u, w) \quad (2-1)$$

$$\varepsilon = \sum_{j=1}^{N_{\varepsilon}} A_{\varepsilon j}(\varphi) Y_j^{\varepsilon}(\lambda) \quad (2-2)$$

where,

$$Y_j^{\alpha} = S_j^{\alpha}(\lambda) \mathcal{H}_{K\alpha}(\frac{u}{\sqrt{\epsilon_{\alpha}}}) \mathcal{L}_{L_{\alpha}}(\frac{w^2}{\sqrt{\epsilon_{\alpha}}}) \quad (2-3)$$

$$Y_j^{\varepsilon} = S_j^{\varepsilon}(\lambda) \quad (2-4)$$

The A_j^{α} are the Fourier coefficients which are computed by the variational principle. The S_j^{α} are sine or cosine functions, $\mathcal{H}_K$ is a Hermite polynomial and $\mathcal{L}_{L_{\alpha}}$ is a Laguerre polynomial. The result of the calculation of the various elements of the matrix $\underline{\underline{C}}$, equation (6-18), and the elements of the vectors h_1 , h_2 and h_3 , equations (6-27), (6-28) and (6-29), are given in this appendix. The elements of the matrix $\underline{\underline{C}}$ consist of three types, collisional operator, orbit operator, and electric field. The matrix

$$\langle Y_J^\alpha \rangle \langle Y_J^\alpha \rangle = \langle S_J^\alpha S_J^\alpha \rangle \frac{F_a N_a}{m_a \pi^3}.$$

$$\left\{ Z_a^2 e^2 N_a \left(\frac{m_a}{2T} \right)^{3/2} V_{JJ}^{\alpha\alpha 1} + Z_p^2 e^2 N_p \left(\frac{m_p}{2T} \right)^{3/2} V_{JJ}^{\alpha\alpha 2} \right\}$$

$$\langle Y_J^\alpha \rangle \langle Y_J^\beta \rangle = \langle S_J^\alpha S_J^\beta \rangle \frac{F_a N_a N_p}{m_a \pi^3} \frac{e^2}{(2T)} \left(\frac{m_p}{2T} \right)^{3/2} V_{JJ}^{\alpha\beta}$$

The $V_{JJ}^{\alpha\alpha 1}$, $V_{JJ}^{\alpha\alpha 2}$, and $V_{JJ}^{\alpha\beta}$ are computed by the program LIGHT. They are given by;

$$V_{JJ}^{\alpha\alpha 1} = \frac{1}{2} \sum_{k_1 k_2} \vec{k}_1 \cdot \vec{k}_2 \cdot \frac{1}{k_1 k_2} e^{-i \vec{k}_1 \cdot \vec{r}_{JJ}} e^{-i \vec{k}_2 \cdot \vec{r}_{JJ}}.$$

$$\left\{ \sum_{k_1 k_2} \vec{k}_1 \cdot \vec{k}_2 \cdot \frac{1}{k_1 k_2} e^{-i \vec{k}_1 \cdot \vec{r}_{JJ}} e^{-i \vec{k}_2 \cdot \vec{r}_{JJ}} \right\}^T$$

$$V_{JJ}^{\alpha\beta} = \frac{1}{2} \sum_{k_1 k_2} \vec{k}_1 \cdot \vec{k}_2 \cdot \frac{1}{k_1 k_2} e^{-i \vec{k}_1 \cdot \vec{r}_{JJ}} e^{-i \vec{k}_2 \cdot \vec{r}_{JJ}}.$$

$$\sum_{k_1 k_2} \vec{k}_1 \cdot \vec{k}_2 \cdot \frac{1}{k_1 k_2} e^{-i \vec{k}_1 \cdot \vec{r}_{JJ}} e^{-i \vec{k}_2 \cdot \vec{r}_{JJ}}.$$

$$\underline{v}' = \frac{\underline{v}}{v_{\text{ex}}}$$

(D-13)

The second change of variables is,

$$\underline{s} = \underline{v} + u \underline{v}'$$

(D-14)

$$\underline{s}' = \underline{v} - \underline{v}'$$

(D-15)

This change simplifies the $|\underline{v} - \underline{v}'|$ term. It is reduced to

$|\underline{s}'|$ which in spherical coordinates is one of the independent variables. After these changes and much algebra the integrals, (D-7), (D-8), and (D-9) can be written as,

$$V_{JJ'}^{(000)} = \sum_a \sum_{a'} \gamma^{M_{aJ}^a + M_{a'J'}^{a'} + 2N_{aJ}^a + 2N_{a'J'}^{a'} + 1} \mathcal{L}_{aJ}^a \mathcal{L}_{a'J'}^{a'}$$

$$\int d^3s d^3s' \left\{ G_{aaJ1}^{aa'J'} - G_{aaJ2}^{aa'J'} \right\} e^{-s^2} e^{-s'^2} \quad (D-16)$$

$$V_{JJ'}^{(000)} = \sum_a \sum_{a'} \gamma^{M_{aJ}^a + M_{a'J'}^{a'} + 2N_{aJ}^a + 2N_{a'J'}^{a'} + 1} \mathcal{L}_{aJ}^a \mathcal{L}_{a'J'}^{a'}$$

$$\int d^3s d^3s' G_{aaJ1}^{aa'J'} e^{-s^2} e^{-us'^2} \quad (D-17)$$

$$V_{JJ'}^{a\beta} = \sum_a \sum_{a'} V_{aa'}^{M_{a\beta}^a + M_{a\beta}^{a'} + N_{a\beta}^a + N_{a\beta}^{a'}} \quad (2-17)$$

$$F(s_1, s_2) G_{aa'j_2}^{a\beta} e^{-s_1^2} e^{-s_2^2} \quad (2-18)$$

where,

$$u = \frac{m_\beta}{m_a} \quad (2-19)$$

$$v = \frac{1}{1 + m_\beta/m_a} \quad (2-20)$$

The equation for G is,

$$\begin{aligned} G_{aa'j_2}^{a\beta} = \frac{1}{16\pi^3} \{ & 4N_{a\beta}^a N_{a\beta}^{a'} [s_1^2 + s_2^2] F(1, 0, M_{a\beta}^a, N_{a\beta}^a - 1, 1) F(1, 0, M_{a\beta}^{a'}, N_{a\beta}^{a'} - 1, 1) + \\ & + 4N_{a\beta}^a N_{a\beta}^{a'} s_1 s_2 F(1, 0, M_{a\beta}^a, N_{a\beta}^a - 1, 1) F(0, 1, M_{a\beta}^{a'}, N_{a\beta}^{a'} - 1, 1) \\ & - 2N_{a\beta}^a M_{a\beta}^{a'} s_1 s_2 F(1, 0, M_{a\beta}^a, N_{a\beta}^a - 1, 1) F(0, 0, M_{a\beta}^{a'}, N_{a\beta}^{a'} - 1, 1) \\ & + 4N_{a\beta}^a N_{a\beta}^{a'} [s_1^2 + s_2^2] F(0, 1, M_{a\beta}^a, N_{a\beta}^a - 1, 1) F(0, 1, M_{a\beta}^{a'}, N_{a\beta}^{a'} - 1, 1) \\ & - 4N_{a\beta}^a N_{a\beta}^{a'} s_1 s_2 F(0, 1, M_{a\beta}^a, N_{a\beta}^a - 1, 1) F(0, 0, M_{a\beta}^{a'}, N_{a\beta}^{a'} - 1, 1) \\ & - 2N_{a\beta}^a M_{a\beta}^{a'} s_1 s_2 F(0, 1, M_{a\beta}^a, N_{a\beta}^a - 1, 1) F(0, 0, M_{a\beta}^{a'}, N_{a\beta}^{a'} - 1, 1) \\ & + M_{a\beta}^a M_{a\beta}^{a'} [s_1^2 + s_2^2] F(0, 0, M_{a\beta}^a - 1, N_{a\beta}^a, 1) F(0, 0, M_{a\beta}^{a'} - 1, N_{a\beta}^{a'}, 1) \\ & - 2M_{a\beta}^a N_{a\beta}^{a'} s_1 s_2 F(0, 0, M_{a\beta}^a - 1, N_{a\beta}^a, 1) F(0, 1, M_{a\beta}^{a'}, N_{a\beta}^{a'} - 1, 1) \\ & - 2M_{a\beta}^a N_{a\beta}^{a'} s_1 s_2 F(0, 0, M_{a\beta}^a - 1, N_{a\beta}^a, 1) F(0, 1, M_{a\beta}^{a'}, N_{a\beta}^{a'} - 1, 1) \} \end{aligned}$$

(2-21)

The S_1, S_2, S_3 are the components of the vector S , in spherical coordinates,

$$S_1 = S \sin \theta \cos \phi \quad (D-22a)$$

$$S_2 = S \sin \theta \sin \phi \quad (D-22b)$$

$$S_3 = S \cos \theta \quad (D-22c)$$

$$S'_1 = S' \sin \theta' \cos \phi' \quad (D-22d)$$

$$S'_2 = S' \sin \theta' \sin \phi' \quad (D-22e)$$

$$S'_3 = S' \cos \theta' \quad (D-22f)$$

The function F is a product defined by,

$$F(K, L, m, n, j) = [t_{j1} + t_{j2}]^K [t_{j3} + t_{j4}]^L [t_{j5} + t_{j6}]^M [t_{j7} + t_{j8} + t_{j9} + t_{j10}]^N \quad (D-23)$$

The terms in this equation are given by,

$$t_{jnm} = C_{jnm} \sum_{i=1}^{10} f_i^{I_{inm}} \quad (D-24)$$

$$\begin{aligned}
 f_3 &= \sin \theta \\
 f_4 &= \cos \theta \\
 f_5 &= \sin \phi \\
 f_6 &= \cos \phi
 \end{aligned}$$

$$\begin{aligned}
 f_7 &= \sin^2 \theta \\
 f_8 &= \cos^2 \theta \\
 f_9 &= \sin^2 \phi \\
 f_{10} &= \cos^2 \phi
 \end{aligned}$$

The coefficients C_{jnm} in the powers $-2nm$ are given in table B-1.

As one can see the substitution of the V function into the equation for χ will lead to many terms, especially for a radial function with high order Hermite and Laguerre polynomials. This is the reason why a program was written to do the calculation of the V 's.

Next the calculation of the dipole operator matrix element will be given. It can be written as,

$$\begin{aligned}
 \langle Y_J^\alpha, H_\alpha Y_J^\alpha \rangle &= \langle S_J^\alpha b \cdot V S_J^\alpha \rangle \int d^3v \chi_{K\alpha J} L_{L\alpha J} \chi_{K\alpha J}^* L_{L\alpha J}^* f_a^{(0)} \\
 &+ \langle S_J^\alpha V \cdot b S_J^\alpha \rangle \int d^3v \chi_{K\alpha J} f_a^{(0)} \left\{ \frac{v_x^2 - v_y^2}{2} L_{L\alpha J}^* L_{L\alpha J} + \frac{v_x^2 + v_y^2}{2} L_{L\alpha J}^* L_{L\alpha J} \right\} f_a^{(0)} \quad (C-1)
 \end{aligned}$$

In the velocity integral the following properties of the orthogonal polynomials are used.

TABLE A-1

n	$\cos n$	$\sin n$	$\cos 2n$	$\sin 2n$	$\cos 3n$	$\sin 3n$	$\cos 4n$	$\sin 4n$	$\cos 5n$	$\sin 5n$	$\cos 6n$	$\sin 6n$	$\cos 7n$	$\sin 7n$	$\cos 8n$	$\sin 8n$	$\cos 9n$	$\sin 9n$	$\cos 10n$	$\sin 10n$
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
2	0	1	-1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	1	0
3	-1	0	1	0	-1	0	1	0	-1	0	1	0	-1	0	1	0	-1	0	1	0
4	0	1	-1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	1	0
5	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
6	0	1	-1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	1	0
7	-1	0	1	0	-1	0	1	0	-1	0	1	0	-1	0	1	0	-1	0	1	0
8	0	1	-1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	1	0
9	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
10	0	1	-1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	1	0

$$u \mathcal{P}_k = \frac{1}{2} \left\{ \mathcal{P}_{k+1} + 2K \mathcal{P}_{k-1} \right\} \quad (D-27)$$

$$\omega^2 \mathcal{L}_L = - (L+1) \mathcal{L}_{L-1} + (2L+1) \mathcal{L}_L - L \mathcal{L}_{L+1} \quad (D-28)$$

$$\frac{d\mathcal{P}_k}{du} = 2K \mathcal{P}_{k-1} \quad (D-29)$$

$$\omega^2 \frac{d\mathcal{L}_L}{d(\omega^2)} = L \left\{ \mathcal{L}_L - \mathcal{L}_{L-1} \right\} \quad (D-30)$$

$$\int_{-\infty}^{\infty} du e^{-u^2} \mathcal{P}_k(u) \mathcal{P}_k(u) = \sqrt{\pi} 2^k k! \delta_{kk'} \quad (D-31)$$

$$\int_{-\infty}^{\infty} d(\omega^2) e^{-\omega^2} \mathcal{L}_L(\omega^2) \mathcal{L}_L(\omega^2) = \delta_{LL'} \quad (D-32)$$

After using the equation given in Appendix K for $\underline{b} \cdot \underline{\nabla}$ and $\underline{\nabla} \cdot \underline{b}$, (K-10) and (K-11), this matrix element can be written as,

$$(\Psi_J^\alpha H_a \Psi_J^\alpha) = S_1 V_1 + S_2 V_2 \quad (D-33)$$

where,

$$S_1 = 2\pi \int_0^1 d\lambda \frac{S_J^\alpha}{B} \frac{dS_J^\alpha}{d\lambda} \quad (D-34)$$

$$V_1 = N_A \nu_a \{ 2^{k_{aJ}} k_{aJ}! \} \left\{ \frac{\delta_{k_{aJ}, k_{aJ}+1}}{2} + k_{aJ} \delta_{k_{aJ}, k_{aJ}-1} \right\} L_{k_{aJ}} \quad (1-37)$$

$$V_2 = N_A \nu_a \{ k_{aJ}! \} \left\{ \frac{\delta_{k_{aJ}, k_{aJ}+1}}{2} + k_{aJ} \delta_{k_{aJ}, k_{aJ}-1} \right\} L_{k_{aJ}} \quad (1-38)$$

$$- L_{k_{aJ}+1} L_{k_{aJ}} \} - 2^{k_{aJ}} k_{aJ}! \left\{ \frac{\delta_{k_{aJ}, k_{aJ}+1}}{2} + k_{aJ} \delta_{k_{aJ}, k_{aJ}-1} \right\} L_{k_{aJ}} \quad (1-39)$$

$$L_{k_{aJ}} \left\{ \delta_{k_{aJ}, k_{aJ}+1} - \delta_{k_{aJ}, k_{aJ}-1} \right\} \quad (1-40)$$

The δ_{ij} is the Kronecker delta function.

The electric field matrix element is,

$$\left(\frac{1}{T} f_a^{(0)} Y_J^a Y_J^E \right) = \langle S_J^a S_J^E \rangle \frac{2e}{T} \left(d^3 u f_a^{(0)} L_{k_{aJ}} \right) \quad (1-41)$$

After using the orthonormality relation, it can be

written as,

$$\left(\frac{1}{T} f_a^{(0)} Y_J^a Y_J^E \right) = \frac{2e}{T} \left(d^3 u f_a^{(0)} L_{k_{aJ}} \right) \quad (1-42)$$

The components of the vector $\underline{h}_1$, $\underline{h}_2$, and $\underline{h}_3$ will now be given. Components of the vector $\underline{h}_1$ defined by equations (4-30) and (4-31) are,

$$(Y_J^A, h_{1J}) = - \left\langle \frac{\nabla \Psi \cdot \underline{b} \times \nabla \underline{b}}{\mathcal{L}_A} S_J^A \right\rangle \int d^3v \left[\frac{w^2}{2} + u^2 \right] f_a^{(0)} \mathcal{L}_{L_{AJ}} S_{KAJ} \quad (4-32)$$

After doing the velocity integral this becomes,

$$(Y_J^A, h_{1J}) = -2\pi D \int_0^1 d\lambda \frac{S_J^A}{B^3} \frac{\partial B}{\partial \lambda} \frac{m_a c N_a}{\mathcal{L}_A e} \left\{ \frac{1}{2} \delta_{2,KAJ} \delta_{0,L_{AJ}} - \delta_{0,KAJ} \delta_{1,L_{AJ}} + 4 \delta_{2,KAJ} \delta_{0,L_{AJ}} \right\} \quad (4-33)$$

Components of the vector $\underline{h}_2$ are given by equations (4-32) and (4-33).

$$(Y_J^A, h_{2J}) = - \left\langle \frac{\nabla \Psi \cdot \underline{b} \times \nabla \underline{b}}{\mathcal{L}_A} S_J^A \right\rangle \int d^3v \left[\frac{w^2}{2} + u^2 \right] \frac{m_a}{2T} v^2 f_a^{(0)} \mathcal{L}_{L_{AJ}} S_{KAJ} \quad (4-34)$$

After doing the velocity integral this becomes,

$$(Y_J^A, h_{2J}) = -2\pi D \int_0^1 d\lambda \frac{S_J^A}{B^3} \frac{\partial B}{\partial \lambda} \frac{2N_a T c}{\mathcal{L}_A e} \left\{ \frac{5}{2} \delta_{2,KAJ} \delta_{0,L_{AJ}} - \frac{11}{4} \delta_{0,KAJ} \delta_{1,L_{AJ}} + 9 \delta_{2,KAJ} \delta_{0,L_{AJ}} - 3 \delta_{2,KAJ} \delta_{1,L_{AJ}} + 24 \delta_{2,KAJ} \delta_{0,L_{AJ}} \right\} \quad (4-35)$$

The other components of the vectors $\underline{h}_1$ and $\underline{h}_2$, equation (6-34), which do not have a velocity integral are,

$$\left\langle Y_J^E \frac{h_2}{T} \right\rangle = \left\langle S_J^E \frac{cDP}{B} \right\rangle = \frac{2\pi DcP}{T} \int_0^1 d\lambda \int \frac{S_J^E}{B} \quad (6-35)$$

The components of the vector $\underline{h}_1$, equation (6-35) also does not have a velocity integral, it is,

$$\left\langle Y_J^E \frac{h_1}{T} \right\rangle = \left\langle S_J^E \frac{cDB}{4\pi T} \right\rangle = \frac{Dc}{2T} \int_0^1 d\lambda \int B S_J^E \quad (6-36)$$

This completes the set of matrix and vector elements needed in the variational principle.

APPENDIX B

The collision operator that is used in the Landau collision operator. For the α species it can be written as,

$$\frac{\partial f_{\alpha}}{\partial t} = \dots \quad (E-1)$$

where,

$$j_{\alpha\alpha} = \frac{2\pi \ln \Lambda e^4 Z_{\alpha}^4}{m_{\alpha}} \int d^3v' \nabla_v \nabla_{v'} |v-v'| \cdot \left\{ \frac{f_{\alpha}}{m_{\alpha}} \nabla_{v'} f_{\alpha}' - \frac{f_{\alpha}'}{m_{\alpha}} \nabla_v f_{\alpha} \right\} \quad (E-2)$$

$$j_{\alpha\beta} = \frac{2\pi \ln \Lambda e^4 Z_{\alpha} Z_{\beta}^2}{m_{\alpha}} \int d^3v' \nabla_v \nabla_{v'} |v-v'| \cdot \left\{ \frac{f_{\beta}}{m_{\alpha}} \nabla_{v'} f_{\beta}' - \frac{f_{\beta}'}{m_{\alpha}} \nabla_v f_{\alpha} \right\} \quad (E-3)$$

$$\Lambda = \left(\frac{3}{2 Z_{\alpha}^2 e^8} \left(\frac{T^3}{\pi N_e} \right)^{1/2} \right) \quad (E-4)$$

the linear part of the distribution function is,

$$f_{\alpha} = f_{\alpha}^{(0)} [1 + y_{\alpha}] \quad (E-5)$$

$$f_{\beta} = f_{\beta}^{(0)} [1 + y_{\beta}] \quad (E-6)$$

where $f_{\alpha}^{(0)}$ and $f_{\beta}^{(0)}$ are the Maxwellians. y_{α} and y_{β} are the first-order part of the distribution function.

Introducing,

$$\nabla_{\alpha} f_{\alpha}^{(0)} = - \frac{m_{\alpha}}{T} f_{\alpha}^{(0)} \mathbf{v} \quad (E-7)$$

and,

$$(\mathbf{v} - \mathbf{v}') \cdot \nabla_{\alpha} \nabla_{\alpha}' |\mathbf{v} - \mathbf{v}'| = 0 \quad (E-8)$$

and keeping only lowest order terms, equation (H-3) and (H-4) become,

$$\mathbf{j}_{\alpha\alpha} = \frac{2\pi i n e^2 Z_{\alpha}^2}{m_{\alpha}} \int d^3 v' \nabla_{\alpha}' \nabla_{\alpha} |\mathbf{v} - \mathbf{v}'| \cdot [\nabla_{\alpha}' y_{\alpha}' - \nabla_{\alpha} y_{\alpha}] \frac{f_{\alpha}^{(0)} f_{\alpha}'^{(0)}}{m_{\alpha}} \quad (E-9)$$

$$L_{\text{eff}} = \frac{2\pi \hbar c^2}{m_0} \left(\frac{1}{2} \left(\frac{d\mathbf{r}}{dt} \right)^2 - \frac{1}{2} \left(\frac{d\mathbf{r}}{dt} \right)^2 \right)$$

This is the linearized form of the collision operator used in this work.

APPENDIX F

COLLISION OPERATOR OF χ_{ac}

In this Appendix it will be shown that,

$$K_{aa} \chi_{ac} + K_{ab} \chi_{bc} = 0 \quad (F-2)$$

The term in this equation can be written as,

$$K_{aa} \chi_{ac} = \frac{\Gamma_a}{m_a 2\pi} \left[\frac{z_a^2 e^2}{v_{ca}^3 v_{cp}^3} \frac{N_a N_p}{\pi} v_{ta} (I_1 - I_2) - \frac{z_a^2 e^2}{v_{ca}^3 v_{cp}^3} \frac{N_a N_p}{\pi} v_{ta} I_3 \right] \quad (F-3)$$

$$K_{ab} \chi_{bc} = \frac{\Gamma_a z_b^2 e^2}{m_p 2\pi} \frac{N_a N_p}{v_{ca}^3 v_{cp}^3} v_{ta} I_4 \quad (F-4)$$

where,
$$I_1 = \int d\phi \nabla_{\nu} \cdot \int d^3 \nu' e^{-\nu^2} e^{-\nu'^2} \nabla_{\nu} \nabla_{\nu'} |\nu - \nu'| \cdot \nabla_{\nu'} \chi'_{ac} \quad (F-4)$$

$$I_2 = \int d\phi \nabla_{\nu} \cdot \int d^3 \nu' e^{-\nu^2} e^{-\nu'^2} \nabla_{\nu} \nabla_{\nu'} |\nu - \nu'| \cdot \nabla_{\nu'} \chi'_{ac} \quad (F-5)$$

$$I_3 = \int d\phi \nabla_{\nu} \cdot \int d^3 \nu' e^{-\nu^2} e^{-\nu'^2} \nabla_{\nu} \nabla_{\nu'} |\nu - \nu'| \cdot \nabla_{\nu'} \chi'_{bc} \quad (F-6)$$

$$I_4 = \int d\phi \nabla_{\nu} \cdot \int d^3 \nu' e^{-\nu^2} e^{-\nu'^2} \nabla_{\nu} \nabla_{\nu'} |\nu - \nu'| \cdot \nabla_{\nu'} \chi'_{pc} \quad (F-7)$$

Details will be given of the proof that I_1 is zero. The proof that the other I_i are zero will not be given since these proofs are done in the same manner.

From equation (5-7), χ_{ac} can be written as,

$$\chi_{ac} = A_2 v_2 + B_2 v_2^2 \quad (E-2)$$

where,

$$A_2 = \frac{1}{\lambda_a} \left[G + \frac{ze}{T} \frac{dG}{d\psi} \right] \quad (E-3)$$

$$B_2 = \frac{1}{\lambda_a} \frac{m_a v^2}{2T} G_2 \quad (E-4)$$

The gradient in velocity of χ_{ac} is,

$$\nabla_\nu \chi_{ac} = \hat{e}_2 [A_2 + 2B_2 v_2] + B_2 v_2 \quad (E-5)$$

where the unit vector, $\hat{e}_2$ is given by equation (5-5) and v_2 is the component of the velocity in the direction of $\hat{e}_2$. Starting with equation (E-4) for I_1 , and using equation (E-5), I_1 becomes,

$$I_1 = \int d\phi \int d^3 v' e^{-v'^2} e^{-v'^2} [-2v \cdot \nabla_\nu \nabla_\nu |v-v'| + \nabla_\nu \cdot \nabla_\nu \nabla_\nu |v-v'|] \cdot [\hat{e}_2 (A_2 + v^2 B_2) + B_2 v v_2] \quad (E-6)$$

$$I_1 = -2 \int d\phi \int d^3 v' e^{-v'^2} e^{-v'^2} v \cdot \nabla_\nu \nabla_\nu |v-v'| \cdot [\hat{e}_2 (A_2 + v^2 B_2) + B_2 v v_2] \\ - \int d\phi \int d^3 v' e^{-v'^2} e^{-v'^2} \nabla_\nu \cdot \nabla_\nu \nabla_\nu |v-v'| \cdot [\hat{e}_2 (A_2 + v^2 B_2) + B_2 v v_2] \quad (E-7)$$

The following results have been used in the second integral,

$$\nabla_0 \cdot \nabla_0 \left(\frac{1}{r} \right) = -\nabla_0^2 \left(\frac{1}{r} \right) = -\delta(\mathbf{r}) \quad (F-12)$$

The second integral in equation (F-13) is now integrated by parts with respect to $\mathbf{r}'$.

$$\begin{aligned} L_2 = -\frac{1}{2} \int d^3r' \left[\frac{1}{r} \nabla_0 \cdot \nabla_0' \left(\frac{1}{r'} \right) \right] \left[\frac{1}{r'} \nabla_0' \cdot \nabla_0 \left(\frac{1}{r} \right) \right] \\ + \frac{1}{2} \int d^3r' \left[\frac{1}{r} \nabla_0 \cdot \nabla_0' \left(\frac{1}{r'} \right) \right] \left[\frac{1}{r'} \nabla_0' \cdot \nabla_0 \left(\frac{1}{r} \right) \right] \\ + 3 \frac{1}{r} \nabla_0 \cdot \nabla_0' \left(\frac{1}{r'} \right) + \frac{1}{r'} \nabla_0' \cdot \nabla_0 \left(\frac{1}{r} \right) \end{aligned} \quad (F-13)$$

In spherical coordinates the velocity components are,

$$v_1 = v \sin \theta \cos \phi \quad (F-14)$$

$$v_2 = v \sin \theta \sin \phi \quad (F-15)$$

$$v_3 = v \cos \theta \quad (F-16)$$

The dot product of $\mathbf{v}$ and $\mathbf{v}'$ is,

$$\mathbf{v} \cdot \mathbf{v}' = vv' \left[\sin \theta \sin \theta' \cos(\phi - \phi') + \cos \theta \cos \theta' \right] \quad (F-17)$$

If the following change of variables is made in the integrals of equation (F-13),

$$\beta = \phi - \phi' \quad (F-18a)$$

$$\beta' = \phi + \phi' \quad (F-18b)$$

then the dot product is only a function of β .

$$\underline{u} \cdot \underline{u}' = \mu \mu' \left\{ \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \beta \right\} \quad (E-14)$$

There are five integrands in equation (E-15). They are,

$$i_1 = \underline{u} \cdot \nabla_{\underline{u}} \nabla_{\underline{u}'} |\underline{u} - \underline{u}'| \cdot \hat{e}_2 \quad (E-15a)$$

$$i_2 = \underline{u} \cdot \nabla_{\underline{u}} \nabla_{\underline{u}'} |\underline{u} - \underline{u}'| \cdot \underline{u}' \cdot \underline{u}_2 \quad (E-15b)$$

$$i_3 = \underline{u}' \cdot \nabla_{\underline{u}} \nabla_{\underline{u}'} |\underline{u} - \underline{u}'| \cdot \underline{u}' \cdot \underline{u}_2' \quad (E-15c)$$

$$i_4 = \hat{e}_2 \cdot \nabla_{\underline{u}} \nabla_{\underline{u}'} |\underline{u} - \underline{u}'| \cdot \underline{u}' \quad (E-15d)$$

$$i_5 = \underline{I} \cdot \nabla_{\underline{u}} \nabla_{\underline{u}'} |\underline{u} - \underline{u}'| \quad (E-15e)$$

Using the equation for $\nabla_{\underline{u}} \nabla_{\underline{u}'} |\underline{u} - \underline{u}'|$,

$$\nabla_{\underline{u}} \nabla_{\underline{u}'} |\underline{u} - \underline{u}'| = \frac{(\underline{u} - \underline{u}')^2 \underline{I}}{|\underline{u} - \underline{u}'|^3} - \frac{(\underline{u} - \underline{u}')(\underline{u} - \underline{u}')}{|\underline{u} - \underline{u}'|^3} \quad (E-16)$$

the i_5 become,

$$i_5 = \frac{(\mu^2 + \mu'^2 - 2\mu\mu')\mu_2 - (\mu^2 - \mu\mu')(\mu_2 - \mu_2')}{[\mu^2 + \mu'^2 - 2\mu\mu']^{3/2}} \quad (E-17a)$$

$$i_2 = \frac{(\nu^2 + \nu'^2 - 2\nu\nu')\nu\nu'\nu_2' - (\nu^2 - \nu\nu')(\nu\nu' - \nu'^2)\nu_2}{[\nu^2 + \nu'^2 - 2\nu\nu']^{3/2}} \quad (\text{F-22f})$$

$$i_3 = \frac{(\nu^2 + \nu'^2 - 2\nu\nu')\nu^2\nu_2' - (\nu\nu' - \nu'^2)(\nu\nu' - \nu'^2)\nu_2'}{[\nu^2 + \nu'^2 - 2\nu\nu']^{3/2}} \quad (\text{F-22c})$$

$$i_4 = \frac{(\nu^2 + \nu'^2 - 2\nu\nu')\nu_2' - (\nu_2' - \nu_2')(\nu\nu' - \nu'^2)}{[\nu^2 + \nu'^2 - 2\nu\nu']^{3/2}} \quad (\text{F-22d})$$

$$i_5 = \frac{(\nu^2 + \nu'^2 - 2\nu\nu') - (\nu^2 + \nu'^2 - 2\nu\nu')}{[\nu^2 + \nu'^2 - 2\nu\nu']^{3/2}} = 0 \quad (\text{F-22e})$$

Recall that $\nu\nu'$ depends only on β , also each of the above integrand contains a factor of ν_2 or ν_2' , no product of these terms and none to a power greater than one, except i_5 which is zero

for any velocity. The term with ν_2 or ν_2' depend on $\sin(\frac{\theta+\theta'}{2})$ or $\sin(\frac{\theta-\theta'}{2})$ respectively. The integration over β is from

0 to 2π . Thus the integral over β' of all the integrands of equation (F-15) are zero due to the integration over β , thus,

$$I_1 = 0 \quad (\text{F-23})$$

The other I_2 can be expanded in the same manner to yield integrands similar to equations (F-22a) to (F-22e) all of which are zero, thus yielding equation (F-1).

APPENDIX C

EVALUATION OF $\underline{u}_\alpha$ AND $\underline{q}_\alpha$

In this appendix an expression is given for the fluid velocity $\underline{u}_\alpha$ and the heat flow vector $\underline{q}_\alpha$. The definition of $\underline{u}_\alpha$ is,

$$N_\alpha \underline{u}_\alpha = \int d^3v f_\alpha \underline{v} \quad (C-1)$$

using the expression for f_α and $\underline{v}$

$$f_\alpha = f_\alpha^{(0)} (1 + \chi_\alpha + \underline{w} \cdot \underline{d}_\alpha) \quad (C-2)$$

$$\underline{v} = \underline{u}_b + \underline{w} \quad (C-3)$$

The Maxwellian is even in $\underline{v}$ and thus makes no contribution to $\underline{u}_\alpha$. $\underline{d}_\alpha$ is defined in equation (3-78). Equation (C-1) now becomes,

$$N_\alpha \underline{u}_\alpha = \int d^3v f_\alpha^{(0)} \chi_\alpha (\underline{u}_b + \underline{w}) + \int d^3v f_\alpha^{(0)} \chi_{\alpha c} (\underline{u}_b + \underline{w}) \quad (C-4)$$

Since χ_α is independent of ϕ and $\underline{w}$ contains $\sin\phi$ and $\cos\phi$, the second term in the first integral is zero. Also since $\chi_{\alpha c}$ contains $\sin\phi$, the first term in the second integral is also zero. Now using,

$$\underline{w} = \hat{e}_1 w \cos\phi + \hat{e}_2 w \sin\phi \quad (C-5)$$

equation (C-4) becomes,

$$N_\alpha \underline{u}_\alpha = b \int d^3v f_\alpha^{(0)} \chi_\alpha \underline{u} + \hat{e}_2 \frac{|v\phi|}{\Lambda_\alpha} \int w dw d\phi w^2 \sin^2\phi \{A_\alpha + v^2 C_\alpha\} \hat{e}_2 \quad (C-6)$$

where A_α and C_α are terms from $\underline{d}_\alpha$,

$$A_\alpha = \frac{2kN}{\partial\varphi} - \frac{3}{2} \frac{\partial kT}{\partial\varphi} + \frac{2k_e}{T} \frac{\partial \bar{\epsilon}}{\partial\varphi} \quad (C-7a)$$

$$C_\alpha = \frac{\pi a}{2T} \frac{\partial kT}{\partial\varphi} \quad (C-7b)$$

The second integral can be evaluated. The result is,

$$N_a u_a = b \left(d^3 v f_a^{(0)} \chi_a u + \frac{b \times \nabla \psi}{m_a n_a} \left\{ \frac{\partial \rho_a}{\partial \psi} + z_a e N_a \frac{\partial \Phi}{\partial \psi} \right\} \right) \quad (1-9)$$

Similarly, q_a is defined by,

$$q_a = \int d^3 v \frac{m_a v^2}{2} v f_a \quad (6-9)$$

$$q_a = b \left(d^3 v \frac{m_a v^2}{2} v \chi_a f_a^{(0)} + \frac{5}{2} \frac{b \times \nabla \psi}{m_a n_a} \left\{ \frac{\partial (N_a T^2)}{\partial \psi} + N_a T z_a e \frac{\partial \Phi}{\partial \psi} \right\} \right) \quad (C-10)$$

APPENDIX B

A VECTOR IDENTITY

In this appendix a vector identity which is used in the reduction of the moment equations, is derived. This identity was given by Bernstein [40]. Starting with,

$$\nabla \times \frac{\mathbf{b}}{b} = \nabla \times \frac{\mathbf{B}}{B^2} \quad (B-1)$$

This can be written as,

$$\nabla \times \frac{\mathbf{b}}{b} = \frac{\nabla \times \mathbf{B}}{B^2} + \frac{\mathbf{b}}{B} \times \nabla \ln B + \frac{\mathbf{b}}{B^2} \times \nabla \left(\frac{\mathbf{B} \cdot \mathbf{B}}{2} \right)$$

Now after using,

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = 2(\mathbf{B} \cdot \nabla) \mathbf{A} + 2\mathbf{B} \times (\nabla \times \mathbf{A}) \quad (B-2)$$

this becomes,

$$\nabla \times \frac{\mathbf{b}}{b} = \frac{\nabla \times \mathbf{B}}{B^2} + \frac{\mathbf{b}}{B} \times \nabla \ln B + \frac{\mathbf{b}}{B^2} \times \mathbf{b} \cdot \nabla \ln B + \frac{\mathbf{b}}{B^2} \times (\mathbf{b} \times (\nabla \times \mathbf{B})) \quad (B-3)$$

After expanding the triple cross product, the first term on the right side is cancelled by one term of the triple cross product. The result is,

$$\nabla \times \frac{\mathbf{b}}{b} = \frac{\mathbf{b} \cdot \nabla \times \mathbf{B}}{B^2} + \frac{\mathbf{b}}{B} (\nabla \ln B + \mathbf{b} \cdot \nabla \ln b) \quad (B-4)$$

APPENDIX I

EVALUATION OF COLLISION INTEGRAL IN

χ_{ac}

In this appendix the collision integral in the transport equation (4-47) and (4-48) will be evaluated. The space part of these integrals are not evaluated here. The integral in equation (4-43) is,

$$I_1 = \nabla \psi \cdot \mathbf{b} \times \int d^3 v \, m v \frac{\partial f_a}{\partial t} \quad (I-1)$$

This integral after two dimensional variables can be evaluated analytically. The integral arising in equation (4-46) is,

$$I_2 = \nabla \psi \cdot \mathbf{b} \times \int d^3 v \, \frac{m v^2}{2} \frac{\partial f_a}{\partial t} \quad (I-2)$$

After the same two dimensional variables, there are now terms in the integrand due to the three dimensional terms which must be evaluated and summed. In appendix II we will evaluate this integral.

The value of the integral obtained here is not exact. The value evaluated the integral by considering each term exactly, not by numerical integration.

The simplified collisional integral which is given in appendix II

$$\frac{\partial f_a}{\partial t} = -\nabla_v \cdot (j_{av} + j_{avp}) \quad (I-3)$$

$$j_{av} = \Gamma_a Z_a^2 \int d^3 v' \nabla_v \cdot \nabla_{v'} |v - v'| \left\{ \nabla_{v'} y_a' - \nabla_v y_a \right\} \frac{f_a^{(a)} f_a^{(a')}}{m_a} \quad (I-4)$$

$$j_{avp} = \Gamma_a Z_a^2 \int d^3 v' \nabla_v \cdot \nabla_{v'} |v - v'| \left\{ \frac{\nabla_{v'} y_a'}{m_p} - \frac{\nabla_v y_a}{m_a} \right\} f_a^{(a)} f_p^{(a')} \quad (I-5)$$

The integral I_1 is due to the j_{av} term. The part of I_1 due to j_{av} is, j_{av} is due to the collisional momentum of the collisional integral. The part of I_1 due to j_{avp} is due to χ_a is due to χ_a being independent of ϕ . The only

contribution to the integral comes from the cross collision term due to χ_{ac} . The integral can be written as,

$$I_1 = -C |\nabla \psi| \int d^3 v_1 d^3 v_2 \cdot \nabla_v \nabla_{v'} |v - v'| \cdot \left\{ \frac{\nabla_{v'} \chi_{ac}}{m_p} - \frac{\nabla_v \chi_{ac}}{m_a} \right\} e^{-\frac{v^2}{2T_a}} e^{-\frac{v'^2}{2T_p}} \quad (I-1)$$

where,

$$\hat{e}_2 = \frac{\mathbf{b} \times \nabla \psi}{|\nabla \psi|} \quad (I-2)$$

$$C = \Gamma_a Z_p^2 m_a N_a \left(\frac{m_a}{2\pi T} \right)^{3/2} N_p \left(\frac{m_p}{2\pi T} \right)^{3/2} \quad (I-3)$$

The first change of variables is to a dimensionless velocity,

$$\underline{v} = \left(\frac{m_a}{2T} \right)^{1/2} \underline{v} \quad (I-4)$$

$$\underline{v}' = \left(\frac{m_a}{2T} \right)^{1/2} \underline{v}' \quad (I-5)$$

The second change of variables is,

$$\underline{s} = \underline{v} - \mu \underline{v}' \quad (I-6)$$

$$\underline{s}' = \underline{v} - \underline{v}' \quad (I-7)$$

where,

$$\mu = \frac{m_p}{m_a} \quad (I-8)$$

$$\gamma = \frac{1}{(1+\mu)} \quad (I-9)$$

$$\nabla_v \nabla_{v'} |v - v'| = \nabla_s \cdot \nabla_{s'} |s| = \frac{s'^2 \underline{s} - s' s'}{|s'|^3} \quad (I-10)$$

the integral becomes,

$$I_1 = C |\nabla \psi| \left(\frac{2T}{m_a} \right)^2 \gamma^3 \int d^3 s d^3 s' \left\{ \frac{s'^2 \underline{s} - s' s'}{|s'|^3} \right\} \cdot \left\{ \frac{\nabla_{s'} \chi_{ac}}{m_p} - \frac{\nabla_s \chi_{ac}}{m_a} \right\} e^{-\gamma(s^2 + \mu s'^2)} \quad (I-11)$$

$$I_1 = C(\sqrt{\psi})^2 \left(\frac{2T}{m_a} \right)^{5/2} \left\{ \left[\frac{A_a}{N_{a m_a}} - \frac{A_p}{N_{p m_p}} \right] t_1 + \left[\frac{1}{N_{a m_a}} - \frac{\mu}{N_{p m_p}} \right] \gamma^2 \frac{\partial \ln T}{\partial \psi} t_2 \right. \\ \left. + \left[\frac{\mu^2}{N_{a m_a}} - \frac{\mu}{N_{p m_p}} \right] \gamma^2 \frac{\partial \ln T}{\partial \psi} t_3 + \left[\frac{1}{N_{a m_a}} - \frac{\mu}{N_{p m_p}} \right] \gamma^2 \frac{\partial \ln T}{\partial \psi} t_4 \right\}$$

t_i are integrals which can be evaluated.

$$t_1 = \int d^3 s d^3 s' \frac{(s_1'^2 - s_1^2)}{s_1^3} e^{-\gamma(s^2 + \mu s'^2)} = \frac{4}{3} \frac{\pi^{5/2}}{\mu \gamma^{5/2}}$$

$$t_2 = \int d^3 s d^3 s' (s_1'^2 + s_1^2) \frac{s_1^2}{s_1^3} e^{-\gamma(s^2 + \mu s'^2)} = 2 \frac{\pi^{5/2}}{\mu \gamma^{5/2}}$$

$$t_3 = \int d^3 s d^3 s' (s_1'^2 + s_1^2) \frac{s_1'^2}{s_1^3} e^{-\gamma(s^2 + \mu s'^2)} = \frac{4}{3} \frac{\pi^{5/2}}{\mu^2 \gamma^{5/2}}$$

$$t_4 = \int d^3 s d^3 s' (s_1'^2 + s_1^2) \frac{s_1^2}{s_1^3} e^{-\gamma(s^2 + \mu s'^2)} = \frac{2}{3} \frac{\pi^{5/2}}{\mu \gamma^{5/2}}$$

$$A_a = \frac{\partial \ln N}{\partial \psi} - \frac{3}{2} \frac{\partial \ln T}{\partial \psi} + \frac{z_a e}{T} \frac{\partial \Phi}{\partial \psi}$$

For the equal temperature case the first term in the first and the corresponding (-1) in the second of Φ are the same.

$$I_1 = 4\sqrt{\pi} \frac{c(\sqrt{\psi})^2}{B} z_a^2 z_p^2 e^{\psi} \ln \Lambda N_a N_p \left(\frac{m_p}{2T} \right)^{1/2} \left(1 + \frac{m_p}{m_a} \right)^{-1/2} \\ \left\{ \left[\frac{1}{z_a e} - \frac{1}{z_p e} \right] \cdot \left[\frac{\partial \ln N}{\partial \psi} - \frac{3}{2} \frac{\partial \ln T}{\partial \psi} \right] \frac{2}{3} + \left[\frac{z_a e}{1 + m_p/m_a} - \frac{z_p e}{z_p e} \right] \frac{\partial \ln T}{\partial \psi} \right\}$$

where the first term I_2 is the collision term for the α particles. The second term $I_{\alpha\beta}$ is the collision term for the β particles. The third term $I_{\beta\alpha}$ is the collision term for the α particles. The fourth term $I_{\beta\beta}$ is the collision term for the β particles. The fifth term $I_{\alpha\alpha}$ is the collision term for the α particles. The sixth term $I_{\beta\beta}$ is the collision term for the β particles. The seventh term $I_{\alpha\alpha}$ is the collision term for the α particles. The eighth term $I_{\beta\beta}$ is the collision term for the β particles. The ninth term $I_{\alpha\alpha}$ is the collision term for the α particles. The tenth term $I_{\beta\beta}$ is the collision term for the β particles. The eleventh term $I_{\alpha\alpha}$ is the collision term for the α particles. The twelfth term $I_{\beta\beta}$ is the collision term for the β particles. The thirteenth term $I_{\alpha\alpha}$ is the collision term for the α particles. 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The eighty-fifth term $I_{\alpha\alpha}$ is the collision term for the α particles. The eighty-sixth term $I_{\beta\beta}$ is the collision term for the β particles. The eighty-seventh term $I_{\alpha\alpha}$ is the collision term for the α particles. The eighty-eighth term $I_{\beta\beta}$ is the collision term for the β particles. The eighty-ninth term $I_{\alpha\alpha}$ is the collision term for the α particles. The ninetieth term $I_{\beta\beta}$ is the collision term for the β particles. The ninety-first term $I_{\alpha\alpha}$ is the collision term for the α particles. The ninety-second term $I_{\beta\beta}$ is the collision term for the β particles. The ninety-third term $I_{\alpha\alpha}$ is the collision term for the α particles. The ninety-fourth term $I_{\beta\beta}$ is the collision term for the β particles. The ninety-fifth term $I_{\alpha\alpha}$ is the collision term for the α particles. The ninety-sixth term $I_{\beta\beta}$ is the collision term for the β particles. The ninety-seventh term $I_{\alpha\alpha}$ is the collision term for the α particles. The ninety-eighth term $I_{\beta\beta}$ is the collision term for the β particles. The ninety-ninth term $I_{\alpha\alpha}$ is the collision term for the α particles. The hundredth term $I_{\beta\beta}$ is the collision term for the β particles.

$$I_2 = \frac{m_\alpha}{2} |\nabla \Psi| \Gamma_\alpha \left\{ Z_\alpha^2 I_{\alpha\alpha} + Z_\beta^2 I_{\alpha\beta} \right\}$$

where,

$$I_{\alpha\alpha} = \int d^3v \frac{\nabla \Psi \cdot \nabla v}{|\nabla \Psi|} v^2 \nabla_v \cdot \int d^3v' \nabla_v \nabla_{v'} |v-v'| \cdot \left\{ \frac{\nabla_v \chi'_{\alpha\alpha}}{m_\alpha} - \frac{\nabla_v \chi_{\alpha\alpha}}{m_\alpha} \right\} f_{\alpha}^{(0)} f_{\alpha}^{(0)}$$

$$I_{\alpha\beta} = \int d^3v \frac{\nabla \Psi \cdot \nabla v}{|\nabla \Psi|} v^2 \nabla_v \cdot \int d^3v' \nabla_v \nabla_{v'} |v-v'| \cdot \left\{ \frac{\nabla_v \chi'_{\alpha\beta}}{m_\beta} - \frac{\nabla_v \chi_{\alpha\beta}}{m_\alpha} \right\} f_{\alpha}^{(0)} f_{\beta}^{(0)}$$

The above two integrals differ only in that the first contains $\chi'_{\alpha\alpha}$ and the other contains $\chi'_{\alpha\beta}$. An expression for $I_{\alpha\beta}$ will be obtained. Then $I_{\alpha\alpha}$ will be obtained by setting $\beta = \alpha$ in the expression.

After an integration by parts and a change to the relative velocity the integral becomes,

$$I_{\alpha\beta} = \frac{N_\alpha N_\beta}{\pi^3} \mu^{3/2} \int d^3v d^3v' A \cdot B \cdot C e^{-v^2} e^{-\mu v'^2}$$

where,

$$A = 2v_2 v + v^2 e_2$$

$$B = \nabla_v \nabla_{v'} |v-v'|$$

$$C = \left\{ \frac{\nabla_v \chi'_{\alpha\beta}}{m_\beta} - \frac{\nabla_v \chi_{\alpha\beta}}{m_\alpha} \right\}$$

Making another change of variables,

$$s = v - \mu v'$$

$$s' = v - v'$$

the integral can be written as,

$$I_{\alpha\beta} = \frac{N_\alpha N_\beta}{\pi^3} \mu^{3/2} \int d^3s d^3s' A \cdot B \cdot C e^{-s^2} e^{-\mu s'^2}$$

where,

$$\underline{A} = \gamma^2 \{ a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3 \} \quad (\text{I-31a})$$

$$a_1 = s_2 s_1 + \mu s_1' s_1 + \mu s_2 s_1' + \mu^2 s_1' s_1' \quad (\text{I-31b})$$

$$a_2 = s^2 + \mu^2 s_1'^2 + s_2^2 + \mu^2 s_2'^2 + 2\mu s_1 s_1' + 4\mu s_2 s_2' + 2\mu s_3 s_3' \quad (\text{I-31c})$$

$$a_3 = s_2 s_3 + \mu s_1' s_3 + \mu s_2 s_3' + \mu^2 s_1' s_3' \quad (\text{I-31d})$$

$$\underline{B} = \frac{1}{|s'|^3} \begin{Bmatrix} s_1'^2 s_1' s_1' & -s_1' s_2' & -s_1' s_3' \\ -s_2' s_1' & s_1'^2 s_2' s_2' & -s_2' s_3' \\ -s_3' s_1' & -s_3' s_2' & s_1'^2 s_3' s_3' \end{Bmatrix} \quad (\text{I-32})$$

$$\underline{C} = G \frac{\partial \ln T}{\partial \psi} + H \left\{ \frac{\partial \ln N}{\partial \psi} - \frac{3}{2} \frac{\partial \ln T}{\partial \psi} \right\} \quad (\text{I-33})$$

$$\underline{G} = \frac{c(174)}{B} \gamma^2 \left(\frac{2T}{m_u} \right)^{1/2} \{ g_1 \hat{e}_1 + g_2 \hat{e}_2 + g_3 \hat{e}_3 \} \quad (\text{I-34a})$$

$$g_1 = 2 \left\{ s_1 s_2 \left[\frac{\mu}{z_{pe}} - \frac{1}{z_{ce}} \right] - [s_1' s_2 + s_1 s_2'] \left[\frac{1}{z_{pe}} + \frac{1}{z_{ce}} \right] \mu + s_1' s_2' \mu \left[\frac{1}{z_{pe}} - \frac{\mu}{z_{ce}} \right] \right\} \quad (\text{I-34b})$$

$$g_2 = s_1' \left[\frac{\mu}{z_{pe}} - \frac{1}{z_{ce}} \right] + s_1'^2 \left[\frac{\mu}{z_{pe}} - \frac{\mu^2}{z_{ce}} \right] - 2s_1 s_1' \left[\frac{\mu}{z_{pe}} + \frac{\mu}{z_{ce}} \right] \\ - 6s_2 s_2' \left[\frac{\mu}{z_{pe}} + \frac{\mu}{z_{ce}} \right] - 2s_2 s_2' \left[\frac{\mu}{z_{pe}} - \frac{\mu^2}{z_{ce}} \right] + 2s_2^2 \left[\frac{\mu}{z_{pe}} - \frac{1}{z_{ce}} \right] + 2s_2 s_1' \left[\frac{\mu}{z_{pe}} - \frac{\mu^2}{z_{ce}} \right] \quad (\text{I-34c})$$

$$g_3 = s_2 s_3 \left[\frac{\mu}{z_{pe}} - \frac{1}{z_{ce}} \right] - s_2 s_3' \left[\frac{\mu}{z_{pe}} + \frac{\mu}{z_{ce}} \right] - s_2' s_3 \left[\frac{\mu}{z_{pe}} + \frac{\mu}{z_{ce}} \right] + s_2' s_3' \left[\frac{\mu}{z_{pe}} - \frac{\mu^2}{z_{ce}} \right] \quad (\text{I-34d})$$

$$\underline{H} = \frac{c(174)}{B} \left(\frac{2T}{m_u} \right)^{1/2} \{ h_1 \hat{e}_1 + h_2 \hat{e}_2 + h_3 \hat{e}_3 \} \quad (\text{I-35a})$$

$$h_1 = 0$$

$$h_2 = \left\{ \frac{1}{2\rho e} - \frac{1}{2\kappa e} \right\}$$

$$h_3 = 0$$

The integral can be written as,

$$\begin{aligned} I_{q0} = N_A N_p \frac{1}{\pi^3} \frac{e^{3/2} s}{B} \left(\frac{2T}{m_A} \right)^{1/2} & \left\{ \frac{\partial h T}{\partial \psi} \frac{1}{s} \sum_{i,j} a_i b_{ij} g_j e^{-\gamma(s^2 + \mu s^2)} \right. \\ & \left. + \left[\frac{\partial h N}{\partial \psi} - \frac{3}{2} \frac{\partial h T}{\partial \psi} \right] \left(\frac{d^3 s d^3 s'}{s^3} \sum_{i,j} a_i b_{ij} h_j e^{-\gamma(s^2 + \mu s^2)} \right) \right\} \end{aligned} \quad (I-3-3)$$

The integral I_2 can be written as,

$$\begin{aligned} I_2 = -h_A h_Z N_A \frac{1}{B \pi^2} \left(\frac{2T}{m_A} \right)^{1/2} & \left\{ \frac{N_A Z_A^2}{\lambda^5} \left[\frac{\partial h T}{\partial \psi} \frac{V_{1A\kappa}}{4} + \left(\frac{\partial h N}{\partial \psi} - \frac{3}{2} \frac{\partial h T}{\partial \psi} \right) V_{2A\kappa} \right] \right. \\ & \left. + \left(\frac{m_p}{m_A} \right)^{3/2} \frac{N_A Z_p^2}{[1 + m_p/m_A]^3} \left[\frac{\partial h T}{\partial \psi} \frac{V_{1A\beta}}{(1 + m_p/m_A)^2} + \left(\frac{\partial h N}{\partial \psi} - \frac{3}{2} \frac{\partial h T}{\partial \psi} \right) V_{2A\beta} \right] \right\} \end{aligned} \quad (I-3-4)$$

where,

$$V_{1A\beta} = \int \frac{d^3 s d^3 s'}{s^3} \sum_{i,j} a_i b_{ij} g_j e^{-\gamma(s^2 + \mu s^2)} \quad (I-3-5)$$

$$V_{2A\beta} = \int \frac{d^3 s d^3 s'}{s^3} \sum_{i,j} a_i b_{ij} h_j e^{-\gamma(s^2 + \mu s^2)} \quad (I-3-6)$$

For a deuterium plasma the above quantities obtained from the computer are,

$$V_{1ei} = 9.07 \times 10^{30} \quad (I-3-7)$$

$$V_{2ei} = 2.92 \times 10^{23} \quad (I-3-8)$$

$$V_{1ee} = 6.59 \times 10^{12} \quad (I-3-9)$$

$$V_{2ee} = 0 \quad (I-3-10)$$

$$V_{1ie} = -2.14 \times 10^{15} \quad (I-39e)$$

$$V_{2ie} = -7.14 \times 10^{14} \quad (I-39f)$$

$$V_{1ii} = -6.59 \times 10^{12} \quad (I-39g)$$

$$V_{2ii} = 0 \quad (I-39h)$$

DIFFERENTIAL EQUATION FOR E_θ

In this Appendix a differential equation for E_θ is obtained. the equation for ψ is,

$$\nabla \cdot \left(\frac{\nabla \psi}{r^2} \right) = -\frac{1}{2r^2} \frac{\partial D^2}{\partial \psi} \quad (1-1)$$

The theta component of the electric field, E_θ , the flux surface velocity V_θ , and ρ are related by,

$$\frac{\partial \psi}{\partial t} = -crE_\theta = -V_\theta \frac{\partial \psi}{\partial r} \quad (1-2)$$

An equation for $\frac{\partial \psi}{\partial t}$ is obtained by taking the time derivative of equation (J-1) with respect to time with r and z held constant.

$$\nabla \cdot \left(\frac{\nabla \frac{\partial \psi}{\partial t}}{r^2} \right) = -4\pi \frac{\partial \rho}{\partial t} \frac{\partial \psi}{\partial \psi} - \frac{1}{2r^2} \frac{\partial}{\partial t} \frac{\partial D^2}{\partial \psi} \quad (1-3)$$

For any function $F(\psi, t)$, the partial derivative with respect to time with r and z held constant, $\frac{\partial F}{\partial t}$, the derivative with respect to time with ψ held constant $\frac{\partial F}{\partial \psi}$ are related by,

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial t} + \frac{\partial \psi}{\partial t} \frac{\partial F}{\partial \psi} \quad (1-4)$$

Equation can be written as,

$$\nabla \cdot \left(\frac{\nabla \frac{\partial \psi}{\partial t}}{r^2} \right) = -4\pi \frac{d}{d\psi} \frac{dP}{dt} - 4\pi \frac{\partial \psi}{\partial t} \frac{d^2 P}{d\psi^2} - \frac{1}{2r^2} \frac{d}{d\psi} \frac{dD^2}{dt} - \frac{1}{2r^2} \frac{\partial \psi}{\partial t} \frac{d^2 D^2}{d\psi^2} \quad (J-5)$$

The first term in equation (J-5) can be obtained from equation (4-54).

$$\frac{d}{d\psi} \frac{dP}{dt} = \frac{d}{d\psi} \left\{ \frac{5}{3} \frac{P_0}{\langle \eta \rangle} \frac{d}{d\psi} \langle V \cdot \nabla \psi \rangle + \frac{2}{3 \langle \eta \rangle} \frac{d \mathcal{Z}_E}{d\psi} + \frac{D G_3}{6\pi \langle \eta \rangle} \mathcal{Z}_D \right\} \quad (J-6)$$

where the fluxes $\mathcal{Z}_E$ and $\mathcal{Z}_D$ are given by,

$$\mathcal{Z}_E = D_{E1} G_1 + D_{E2} G_2 + D_{E3} G_3 \quad (J-7)$$

$$\mathcal{Z}_D = D_{D1} G_1 + D_{D2} G_2 + D_{D3} G_3 \quad (J-8)$$

An equation for the third term on the right hand side of equation can be obtained from equation (5-80)

$$\frac{d}{dt} \left\{ \int d^3 r \frac{D}{r^2} \right\} = -\mathcal{Z}_D \quad (J-9)$$

After taking a derivative with respect to ψ using the following equation,

$$\frac{d}{dt} \left\langle \frac{1}{r^2} \right\rangle = \frac{d}{d\psi} \left\langle \frac{\partial \psi}{\partial t} \right\rangle \quad (J-10)$$

The equation for $\frac{dD}{dt}$ is given by,

$$\frac{dD}{dt} = \frac{1}{\left\langle \frac{1}{r^2} \right\rangle} \left\{ cD \frac{\partial}{\partial \psi} \left\langle \frac{\mathcal{E}_\theta}{r^2} \right\rangle + \frac{d\mathcal{Z}_0}{d\psi} \right\} \quad (J-11)$$

where,

$$\mathcal{E}_\theta = rE_\theta \quad (J-12)$$

thus,

$$\frac{d}{d\psi} \frac{dD^2}{dt} = \frac{d}{d\psi} \left\{ \frac{2D}{\left\langle \frac{1}{r^2} \right\rangle} \left[cD \frac{d}{d\psi} \left\langle \frac{\mathcal{E}_\theta}{r^2} \right\rangle + \frac{d\mathcal{Z}_0}{d\psi} \right] \right\} \quad (J-13)$$

The equation for $\mathcal{E}_\theta$ is obtained by substituting equations (J-6) and (J-13) into (J-5).

$$\begin{aligned} \nabla \cdot \left(\frac{\nabla \mathcal{E}_\theta}{r^2} \right) - 4\pi \frac{d}{d\psi} \left\{ \frac{5}{3} \frac{P}{\langle 1 \rangle} \frac{d}{d\psi} \langle \mathcal{E}_\theta \rangle \right\} - \frac{1}{r^2} \frac{d}{d\psi} \left\{ \frac{D^2}{\left\langle \frac{1}{r^2} \right\rangle} \frac{d}{d\psi} \left\langle \frac{\mathcal{E}_\theta}{r^2} \right\rangle \right\} + \left\{ 4\pi \frac{d^2 P}{d\psi^2} + \frac{1}{2r^2} \frac{d^2 D^2}{d\psi^2} \right\} = \\ \frac{4\pi}{c} \frac{d}{d\psi} \left\{ \frac{2}{3\langle 1 \rangle} \frac{d\mathcal{Z}_0}{d\psi} + \frac{DG_3 \mathcal{Z}_0}{6\pi \langle 1 \rangle} \right\} + \frac{1}{r^2} \frac{d}{d\psi} \left\{ \frac{D}{c \left\langle \frac{1}{r^2} \right\rangle} \frac{d\mathcal{Z}_0}{d\psi} \right\} \quad (J-14) \end{aligned}$$

This is the partial differential integral equation for ∇E_0 .

The boundary condition on ∇E_0 is that it be given on the outer flux surface.

APPENDIX K

Ψ SURFACE and Ψ, θ, λ COORDINATE SYSTEM

In this work the Ψ function that is used is the one given by Shafranov [55]. In the equation for Ψ (2-17) one can choose $\frac{dD}{d\Psi}$ and $\frac{dP}{d\Psi}$, the choice made by Shafranov which can yield a simple analytical solution to the Ψ equation are,

$$\frac{dP}{d\Psi} = \text{constant} \quad (K-1)$$

$$\frac{dD}{d\Psi} = 0 \quad (K-2)$$

The solution to equation (2-17) for this choice is

$$\Psi = \frac{\bar{J}_0 \pi}{2c(1+\alpha^2)} \frac{r^2}{R} [2R^2 - r^2 - 4\alpha^2 z^2] - \Psi_c \quad (K-3)$$

Surface of Constant λ

Surface of
Constant θ

Figure L-1 ψ, θ, λ Coordinate System

$$d\Omega^2 = h_\psi^2 d\psi^2 + h_\theta^2 d\theta^2 + h_\lambda^2 d\lambda^2 \quad (K-7)$$

where,

$$h_\psi = \frac{1}{r B_r} \quad (K-8)$$

$$h_\theta = r \quad (K-9)$$

$$h_\lambda = \int B_r \quad (K-10)$$

The gradient of a function f and the divergence of a vector A are,

$$\nabla f = \hat{e}_\psi r B_r \frac{\partial f}{\partial \psi} + \hat{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} + \hat{e}_\lambda \frac{1}{\int B_r} \frac{\partial f}{\partial \lambda} \quad (K-11)$$

$$\nabla \cdot A = \frac{1}{\int B_r} \left\{ \frac{\partial}{\partial \psi} [r \int B_r A_\psi] + \frac{\partial}{\partial \theta} \left[\frac{\int B_r}{r} A_\theta \right] + \frac{\partial}{\partial \lambda} \left[\frac{A_\lambda}{B_r} \right] \right\} \quad (K-12)$$

Two quantities which are used in the orbit operator are,

$$b \cdot \nabla f = \frac{1}{\int B_r} \frac{\partial f}{\partial \lambda} \quad (K-13)$$

$$\nabla \cdot b = - \frac{1}{\int B_r} \frac{\partial \ln B}{\partial \lambda} \quad (K-14)$$

where f is any function.

The area of the surface element on the flux surface is,

$$d^2 = h_2 d\theta h_2 d\lambda \quad (4-14)$$

After using,

$$|\nabla\psi|^2 = r B_p \quad (4-15)$$

the surface average of any function can be written as,

$$\langle F \rangle = \frac{\int d^2 F}{|\nabla\psi|^2} = 2\pi \int d\lambda \delta F \quad (4-17)$$

The coordinate system defined in equation (4-7) is orthogonal. However, the calculation of the transport coefficients is done separately on each flux surface and does not require that the coordinate system be orthogonal. The λ points can be spaced on the flux surface any way that is convenient. In this calculation $\lambda = 0$ is placed on the inside of the flux surface.

The λ points are placed equal distances apart on each flux surface. In a true orthogonal coordinate system this could only be done on one flux surface. If $d\lambda$ is the incremental difference in λ between two adjacent points on a flux surface, and dL is the length between these two points, the δ is,

$$\bar{f} = \frac{\Delta L}{\beta_p \Delta \lambda}$$

(K-18)

This quantity is needed in the calculation of various flux surface averages defined by equation (K-17).

APPENDIX L

RELATIONSHIPS USED IN THE ELIMINATION OF ELECTROSTATIC
POTENTIAL

In this appendix the following three relationships are proved.

$$K_{ap}(z_p, m_p u) + K_{am}(z_m, m_m u) = 0 \quad (L-1)$$

$$H_a\left(\frac{uD}{\mathcal{L}_a}\right) = h_{aT} \quad (L-2)$$

and,

$$H_a\left(\frac{m_a v^2}{T} \frac{uD}{\mathcal{L}_a}\right) = h_{aT} \quad (L-3)$$

Putting the expression for the collision operator into (L-1) and using

$$\nabla_a u = \underline{b} \quad (L-4)$$

$$\nabla_a u' = \underline{b} \quad (L-5)$$

the equation becomes,

$$\begin{aligned} K_{ap}(z_p, m_p u) + K_{am}(z_m, m_m u) = & \Gamma_a \int d\phi \nabla_a \cdot \left\{ z_a^2 z_a^2 \int d^3 v_a' f_a^{(m)} f_a^{(m)'} \nabla_a \nabla_a |v-v'| \cdot \underline{b} c \right. \\ & - z_a^2 z_a^2 \int d^3 v_a' f_a^{(m)} f_a^{(m)'} \nabla_a \nabla_a |v-v'| \cdot \underline{b} c - z_a^2 z_a^2 \int d^3 v_a' f_a^{(m)} f_a^{(m)'} \nabla_a \nabla_a |v-v'| \cdot \underline{b} c \left. \right\} \\ & + \Gamma_a z_p^2 \int d\phi \nabla_a \cdot \left\{ z_p^2 z_p^2 \int d^3 v_p' f_p^{(m)} f_p^{(m)'} \nabla_p \nabla_p |v-v'| \cdot \underline{b} c \right. \end{aligned} \quad (L-6)$$

The first and second term cancel as well as the third and fourth, thus the left hand side is zero which gives equation (L-1).

The orbit operator, H_a , is defined by equation (3-49). Any function of Ψ only, like D will commute with the orbit operator.

It can be shown that,

$$H_a\left(\frac{uD}{\mathcal{L}_a}\right) = \frac{f_a^{(m)} D}{\mathcal{L}_a} \left\{ \frac{u^2}{2} \nabla_a \cdot \underline{b} - u^2 \underline{b} \cdot \nabla_a \ln \beta \right\} \quad (L-7)$$

The unit vector, $\hat{e}_2$, can be written as,

$$\hat{e}_2 = \nabla \phi \frac{r^2 B}{|\nabla \phi|} - \frac{D}{|\nabla \phi|} \underline{b} \quad (L-2)$$

Since $\nabla \phi \cdot \nabla \ln B = 0$, due to axisymmetry, it can be shown that,

$$\hat{e}_2 \cdot \nabla \ln B = - \frac{D}{|\nabla \phi|} \underline{b} \cdot \nabla \ln B \quad (L-3)$$

Using $\nabla \phi = 0$ it can be shown that,

$$\nabla \cdot \underline{b} = \frac{|\nabla \phi|}{D} \hat{e}_2 \cdot \nabla \ln B \quad (L-10)$$

The right hand side of the above equation can be shown to be equal to $\frac{|\nabla \phi|}{B} \hat{e}_2 \cdot (\underline{b} \cdot \nabla \underline{b})$. Starting with,

$$\hat{e}_2 \cdot (\underline{b} \cdot \nabla \underline{b}) = - [\underline{b} \times (\nabla \times \underline{b})] \cdot \hat{e}_2 \quad (L-11)$$

or,

$$\hat{e}_2 \cdot (\underline{b} \cdot \nabla \underline{b}) = - \hat{e}_2 \cdot \nabla \times \underline{b} \quad (L-12)$$

Equation (L-12) becomes,

$$\hat{e}_2 \cdot (\underline{b} \cdot \nabla \underline{b}) = - \nabla \cdot \hat{e}_2 - \hat{e}_2 \cdot \nabla \ln |\nabla \phi| \quad (L-13)$$

After using,

$$\nabla \cdot \hat{e}_2 = - \hat{e}_2 \cdot \nabla \ln |\nabla \phi| - \hat{e}_2 \cdot \nabla \ln B \quad (L-14)$$

equation (L-13) becomes,

$$\hat{e}_2 \cdot (\underline{b} \cdot \nabla \underline{b}) = \hat{e}_2 \cdot \nabla \ln B \quad (L-15)$$

Equation (L-10) now becomes,

$$\nabla \cdot \underline{b} = \frac{|\nabla \phi|}{D} \hat{e}_2 \cdot (\underline{b} \cdot \nabla \underline{b}) \quad (L-16)$$

Substitution of equations (L-2) and (L-16) into (L-7) yields,

$$H_A \left(\frac{u D}{\mu_0} \right) = f_A' \frac{|\nabla \phi|}{\mu_0} \hat{e}_2 \cdot \left[\frac{u^2}{2} \nabla \ln B + u^2 \underline{b} \cdot \nabla \underline{b} \right] \quad (L-17)$$

After using equation (3-101) for $h_{\alpha N}$, this yields equation (L-2).

The derivation of equation (L-3) is similar to the above derivation. Starting with,

$$H_{\alpha} \left(\frac{m_{\alpha} v^2}{2} u \frac{D}{N_{\alpha}} \right) = \frac{f_{\alpha}^{(0)}}{N_{\alpha}} \left\{ \frac{m_{\alpha} v^2}{2} \frac{w^2}{2} \nabla \cdot \underline{b} - \frac{m_{\alpha} v^2}{2} u^2 \underline{b} \cdot \nabla \ln B \right\} \quad (L-18)$$

after using the equation (L-9) and (L-16) this becomes,

$$H_{\alpha} \left(\frac{m_{\alpha} v^2}{2} u \frac{D}{N_{\alpha}} \right) = \frac{f_{\alpha}^{(0)} \langle \nabla \psi \rangle}{N_{\alpha}} \frac{m_{\alpha} v^2}{2} \underline{e}_2 \cdot \left\{ \frac{w^2}{2} \nabla \ln B + u^2 \underline{b} \cdot \nabla \underline{b} \right\} \quad (L-19)$$

After using equation (3-102) for $h_{\alpha T}$ this yields equation (L-3).

APPENDIX M

SYMMETRY PROPERTIES OF COLLISION AND ORBIT OPERATOR

In this appendix some of the properties of the operators used in the variational principle are given. The inner product of the self-collision operator is,

$$\begin{aligned}
 (\chi K_{\alpha\alpha} \chi) = & \frac{f_{\alpha}}{2\pi m_{\alpha}} \left(\frac{d^2}{dv^2} \right) \left[\frac{d^2}{d\alpha^2} \int d^3 \chi \nabla_{\alpha} \cdot \left\{ d^3 v' f_{\alpha}^{(0)} f_{\alpha}^{(0')} \nabla_{\alpha} \nabla_{\alpha} |v - v'| \cdot \left\{ \nabla_{\alpha} \chi' - \nabla_{\alpha} \chi \right\} \right. \right. \\
 & \left. \left. + \frac{d^2}{d\beta^2} \int d^3 \chi \nabla_{\alpha} \cdot \left\{ d^3 v' f_{\alpha}^{(0)} f_{\beta}^{(0')} \nabla_{\alpha} \nabla_{\alpha} |v - v'| \cdot \nabla_{\alpha} \chi \right\} \right] \quad (M-1)
 \end{aligned}$$

Both terms are integrated by parts then in the first term v and v' are interchanged. Half of the result is then subtracted from half of the previous results. Also using the following equation,

$$\nabla_{\alpha} \nabla_{\alpha} |v - v'| = \nabla_{\alpha} \nabla_{\alpha} |v - v'| \quad (M-2)$$

equation (M-1) can now be written as,

$$\begin{aligned}
 (\chi K_{\alpha\alpha} \chi) = & \frac{f_{\alpha}}{2\pi m_{\alpha}} \left(\frac{d^2}{dv^2} \right) \left[\frac{d^2}{d\alpha^2} \int d^3 \chi \left\{ d^3 v' f_{\alpha}^{(0)} f_{\alpha}^{(0')} [\nabla_{\alpha} \chi' - \nabla_{\alpha} \chi] \cdot \nabla_{\alpha} \nabla_{\alpha} |v - v'| \cdot [\nabla_{\alpha} \chi' - \nabla_{\alpha} \chi] \right. \right. \\
 & \left. \left. - \frac{d^2}{d\beta^2} \int d^3 \chi \left\{ d^3 v' f_{\alpha}^{(0)} f_{\beta}^{(0')} \nabla_{\alpha} \chi \cdot \nabla_{\alpha} \nabla_{\alpha} |v - v'| \cdot \nabla_{\alpha} \chi \right\} \right] \quad (M-3)
 \end{aligned}$$

In this form it is easy to see that the inner product is unchanged by the interchange of χ and χ' , since $\nabla_{\alpha} \nabla_{\alpha} |v - v'|$ is symmetrical.

$$\nabla_{\alpha} \nabla_{\alpha} |v - v'| = \frac{\mathbf{I}(v - v') - (v - v')(v - v')}{|v - v'|^3} \quad (M-4)$$

Thus the self collision operator is symmetric.

$$(\chi, K_{\alpha\alpha} \chi) = (y, K_{\alpha\alpha} y) \quad (M-5)$$

How a similar relation will be given for the cross collision operator. The inner product is,

$$(\chi_\alpha, K_{\alpha\beta} \chi_\beta) = \frac{f_\alpha z_\alpha^2 e^2}{2\pi m_\beta} \int \frac{d^3 r}{|\mathbf{r}|^3} \left[\chi_\alpha \nabla_\alpha \cdot \frac{d^3 r}{f_\beta} \frac{f_\alpha^{(2)} f_\beta^{(2)'}}{f_\beta} \cdot \nabla_\beta \nabla_\beta |\mathbf{r} - \mathbf{r}'| \cdot \nabla_\alpha \chi_\beta' \right] \quad (M-6)$$

This equation is integrated by parts. The $\mathbf{v}$ and $\mathbf{v}'$ are interchanged, and the symmetric property of $\nabla_\alpha \nabla_\beta |\mathbf{r} - \mathbf{r}'|$ is used. Also as can be seen from equation (3-106) the coefficients can be written as follows.

$$\frac{f_\alpha}{m_\beta} z_\beta^2 e^2 = \frac{f_\beta}{m_\alpha} z_\alpha^2 e^2 \quad (M-7)$$

Thus equation (M-6) can be written as,

$$(\chi_\alpha, K_{\alpha\beta} \chi_\beta) = - \frac{f_\alpha z_\alpha^2 e^2}{2\pi m_\alpha} \int \frac{d^3 r}{|\mathbf{r}|^3} f_\beta f_\alpha^{(2)} \nabla_\beta \chi_\beta \cdot \nabla_\alpha \nabla_\alpha |\mathbf{r} - \mathbf{r}'| \cdot \nabla_\alpha \chi_\alpha' \quad (M-8)$$

In this form it is easy to see that this is the same results that would be obtained if the α and β were interchanged in equation (M-6) and integrated by parts. Thus the cross collision operator obeys the following relation.

$$(\chi_\alpha, K_{\alpha\beta} \chi_\beta) = (\chi_\beta, K_{\beta\alpha} \chi_\alpha) \quad (M-9)$$

Now the symmetry properties of the orbit operator will be given. The orbit operator as given by equation (3-99) can be written as,

$$H_a \chi = \int d^3x \left[\frac{1}{2} \left(\frac{\partial \chi}{\partial t} \right)^2 - \frac{1}{2} \left(\nabla \chi \right)^2 + \frac{1}{2} \left(\frac{\partial \chi}{\partial t} \right)^2 - \frac{1}{2} \left(\nabla \chi \right)^2 \right] \quad (M-10)$$

If $f_a^{(0)}$ is a Maxwellian, then the following relations are obtained.

$$\underline{b} \cdot \nabla f_a^{(0)} = 0 \quad (M-11)$$

$$u \frac{\partial f_a^{(0)}}{\partial u} - w \frac{\partial f_a^{(0)}}{\partial w} = 0 \quad (M-12)$$

The inner product of the orbit operator after integrating by parts with respect to space in the first term and with respect to velocity in the second term and using equations (M-11) and (M-12) becomes,

$$(y, H_a \chi) = \int d^3x \int d^3v \left[-u f_a^{(0)} \chi \underline{b} \cdot \nabla y + \frac{w u}{2} \chi f_a^{(0)} \underline{v} \cdot \underline{b} \frac{\partial y}{\partial w} - \frac{w^2}{2} \chi f_a^{(0)} \underline{v} \cdot \underline{b} \frac{\partial y}{\partial u} \right] \quad (M-13)$$

By comparing this with the equation (3-99) for the orbit operator is found to be antisymmetric that is,

$$(y, H_a \chi) = -(\chi, H_a y) \quad (M-14)$$

APPENDIX II

HOMOGENEOUS EQUATION

In this appendix a solution to the homogeneous equation will be given, and it will be shown that it does not make any contribution to the diffusion coefficients. The homogeneous equations are,

$$H_a z_{a0}^+ + K_{aa} z_{a0}^- + K_{ap} z_{p0}^- + C_a E_{a0} = 0 \quad (H-1)$$

$$H_a z_{a0}^- + K_{aa} z_{a0}^+ + K_{ap} z_{p0}^+ = 0 \quad (H-2)$$

$$H_p z_{p0}^+ + K_{pp} z_{p0}^- + K_{pa} z_{a0}^- + C_p E_{p0} = 0 \quad (H-3)$$

$$H_p z_{p0}^- + K_{pp} z_{p0}^+ + K_{pa} z_{a0}^+ = 0 \quad (H-4)$$

The sum of the two collision operators are,

$$\begin{aligned} K_{aa} z_{a0}^- + K_{ap} z_{p0}^- &= \Gamma_a \int d\phi \nabla_v \cdot \int d^3v f_a^{(a)} f_a^{(a')} \nabla_v \nabla_{v'} |v-v'| \cdot \left\{ \frac{\nabla_v z_{a0}'}{m_a} - \frac{\nabla_{v'} z_{a0}}{m_a} \right\} \\ &+ \Gamma_a \int d\phi \nabla_v \cdot \int d^3v f_a^{(a)} f_p^{(a')} \nabla_v \nabla_{v'} |v-v'| \cdot \left\{ \frac{\nabla_v z_{p0}'}{m_p} - \frac{\nabla_{v'} z_{p0}}{m_p} \right\} \end{aligned} \quad (H-5)$$

The following choice of z_{a0} and z_{p0} are solutions to the homogeneous equations.

$$z_{a0} = m_a (\delta + \gamma v^2) \quad (H-6)$$

$$z_{p0} = m_p (\delta + \gamma v^2) \quad (H-7)$$

One of the terms in the collision integral is,

$$\left\{ \frac{\nabla_v z_{p0}}{m_p} - \frac{\nabla_{v'} z_{p0}}{m_p} \right\} = 2\beta (v-v') \quad (H-8)$$

The collision integral is zero due to,

$$\nabla_v \nabla_{v'} |v-v'| \cdot (v-v') = 0 \quad (H-9)$$

this yields,

$$H_a z_{a0} = 0 \quad (H-10)$$

Therefore, δ and γ must satisfy,

$$\nabla \delta \cdot b = 0$$

(N-11)

$$\nabla \gamma \cdot b = 0$$

(N-12)

Thus, δ and γ are functions of ψ and t . The homogeneous solution for E_{bo} is zero.

The quantities $h_{\mu\alpha}$ and $h_{T\alpha}$ are odd in the space coordinate λ . The homogeneous solutions $j_{\alpha 0}$ are independent of λ thus even. Therefore the following inner products are zero.

$$(j_{\alpha 0}, h_{T\alpha}) = 0$$

(N-13)

$$(j_{\alpha 0}, h_{\mu\alpha}) = 0$$

(N-14)

Thus the homogeneous solution makes no contribution to the transport coefficients.

The quantities δ and γ can be chosen in such a way that the first order perturbation from the Maxwellian does not make any contribution to the total number of particles, or the total energy. In other words; the total particle density N and pressure P are determined to second order by the 0th order Maxwellian. In this work there is no need to calculate the quantities δ or γ .

APPENDIX 6

ONSAAGER RELATIONS

The purpose of this appendix is to derive the three Onsager relations. The derivation of one of these relations will be given, the others are obtained in a similar manner.

Starting with the N set of equations, equation (5-41a) is multiplied by $\bar{z}_{a1}$, equation (5-41b) by $\bar{z}_{a1}^+$, equation (5-41c) by $\bar{z}_{\beta 1}$, equation (5-41d) by $\bar{z}_{\beta 1}^+$ and equation (5-41e) by E_{b1} , these equations are summed, the result is,

$$\begin{aligned} & (\bar{z}_{a2}, H \bar{z}_{a1}) + (\bar{z}_{a2}, K_{aa} \bar{z}_{a1}) + (\bar{z}_{a2}, \lambda_{a0} \bar{z}_{a1}) \\ & + (\bar{z}_{a2}, C_a E_{b1}) - (\bar{z}_{a2}, H_a \bar{z}_{a1}) - (\bar{z}_{a2}, \lambda_{aa} \bar{z}_{a1}) - (\bar{z}_{a2}, \lambda_{a0} \bar{z}_{a1}) \\ & + (\bar{z}_{\beta 2}, H_{\beta} \bar{z}_{\beta 1}) + (\bar{z}_{\beta 2}, h_{\beta\beta} \bar{z}_{\beta 1}) + (\bar{z}_{\beta 2}, K_{\beta a} \bar{z}_{a1}) \\ & + (\bar{z}_{\beta 2}, \lambda_{\beta 0} E_{b1}) - (\bar{z}_{\beta 2}, H_{\beta} \bar{z}_{\beta 1}) - (\bar{z}_{\beta 2}, \lambda_{\beta a} \bar{z}_{a1}) - (\bar{z}_{\beta 2}, \lambda_{\beta 0} \bar{z}_{\beta 1}) \\ & + (\bar{z}_{a2}, \lambda_{a0} E_{b2}) + (\bar{z}_{\beta 2}, \lambda_{\beta 0} E_{b2}) = \\ & (\bar{z}_{a2}, h_{aa}) + (\bar{z}_{a2}, h_{\beta a}) - \sqrt{E_{b1}} \bar{z}_{a1}^+ \end{aligned} \quad (5-42)$$

This is the same result that would be obtained if one started with the T set of equations, equations (5-41a) to (5-42e) and multiplied by $\bar{z}_{a2}$, $\bar{z}_{\beta 2}$ and E_{b2} and using the symmetry properties of the $\bar{z}$ and $\bar{z}^+$ operator given in Appendix M, thus,

APPENDIX I COLLISIONAL TRANSPORT COEFFICIENTS

In this appendix the equations given by Braginskii [28] are used to derive transport coefficients of the same form as those given in Chapter 5.

The conservations of particle, momentum and energy equation for electrons and ions as given by Braginskii are,

$$\frac{\partial N_e}{\partial t} + \nabla \cdot (N_e u_e) = 0 \quad (I-1)$$

$$\frac{\partial N_i}{\partial t} + \nabla \cdot (N_i u_i) = 0 \quad (I-2)$$

$$m_e N_e \frac{\partial u_e}{\partial t} + m_e N_e u_e \cdot \nabla u_e = -\nabla P_e - e N_e \left[\frac{u_e \times B}{c} \right] + R \quad (I-3)$$

$$m_i N_i \frac{\partial u_i}{\partial t} + m_i N_i u_i \cdot \nabla u_i = -\nabla P_i + \frac{e}{c} N_i \left[\frac{u_i \times B}{c} \right] - R \quad (I-4)$$

$$\frac{3}{2} N_e \left\{ \frac{\partial T_e}{\partial t} + u_e \cdot \nabla T_e \right\} + P_e \nabla \cdot u_e = -\nabla \cdot q_e + Q_e \quad (I-5)$$

$$\frac{3}{2} N_i \left\{ \frac{\partial T_i}{\partial t} + u_i \cdot \nabla T_i \right\} + P_i \nabla \cdot u_i = -\nabla \cdot q_i + Q_i \quad (I-6)$$

Assuming equal ion and electron temperature, only one of the last two equations are needed. The vector R is given by,

$$R = e N_e \left\{ \frac{E}{c} + \frac{u \times B}{c} \right\} + \frac{3}{2} \frac{N_e}{\Omega_e \tau_e} \frac{u \times B}{c} \quad (I-7)$$

All terms with gradients parallel to the magnetic field have been dropped. Also $Z_i = 1$ is assumed in this appendix. The q_i are given by,

$$\underline{j}_e = \frac{e N_e}{4\pi} \left(\underline{v}_e + \frac{3}{2} \frac{N_e T_e}{e B} \nabla \times \underline{v} \right) - \frac{3}{2} \frac{e N_e T_e}{e B} \nabla \times \underline{v} \quad (\text{P-8})$$

$$\underline{j}_i = -\frac{N_i}{4\pi} \nabla T + \frac{3}{2} \frac{N_i T}{e B} \nabla \times \underline{v} \quad (\text{P-9})$$

τ_e is the electron-ion collision frequency,

$$\tau_e = \frac{3}{4\pi n_e} \frac{\sqrt{m_e}}{\ln \Lambda} \frac{1}{e^2 N} \quad (\text{P-10})$$

The conductivities are given by,

$$\sigma_{\perp} = \frac{e^2 N_e \tau_e}{m_e} \quad (\text{P-11})$$

$$\sigma_{\parallel} = \frac{1}{4\pi} \sigma_{\perp} \quad (\text{P-12})$$

$\underline{v}$ is the relative velocity of the two fluids.

$$\underline{v} = \underline{u}_e - \underline{u}_i \quad (\text{P-13})$$

The current is given by,

$$\underline{j} = -e N_e \underline{v} \quad (\text{P-14})$$

In the equal temperature case the Q_i are:

$$Q_e = \frac{j_{||}^2}{\epsilon_{||}} + \frac{j_{\perp}^2}{\epsilon_{\perp}} + \frac{1}{N_e e} j_{\perp} \cdot \left\{ \frac{3}{2} \frac{N_e}{N_e T_e} b \times \nabla \right\} \quad (P-15)$$

$$Q_i = 0 \quad (P-16)$$

The thermal conductivities can be written in terms of the electrical conductivities.

$$\mathcal{K}_1^e = 4.66 \frac{N_e^2 T_e c^2}{B^2 \sigma_{\perp}} \quad (P-17)$$

$$\mathcal{K}_1^i = \sqrt{\lambda} \frac{N_e^2 T_e c}{B^2 \sigma_{\perp}} \sqrt{\frac{m_i}{m_e}} \quad (P-18)$$

The conservation of particles equation (P-1) is integrated over a flux surface.

$$\int d\mathbf{r}^3 \frac{dN_e}{dt} = \int d\mathbf{r}^3 \nabla \cdot \nabla N_e + \int \frac{d\mathbf{r}^3}{|\nabla\psi|} \nabla\psi \cdot u_e N_e \quad (P-19)$$

The ordering described in Chapter 3 is used on the momentum equation (N-3). The second order part is,

$$c N_e \left\{ \underline{E} + \frac{u_e \times B}{c} + \underline{K} \right\} = 0 \quad (P-20)$$

An equation for $\nabla \psi \cdot \underline{u}_e$ can be obtained from equation (P-20). First a cross product of equation (P-20) with $\underline{B}$ is taken, then a dot product with $\nabla \psi$ is taken. Equation (P-7) is used for $\underline{R}$.

$$N_e \underline{u}_e \cdot \nabla \psi = \frac{c N_e}{B^2} [\underline{E} \times \underline{B}] \cdot \nabla \psi - \frac{c^2 |\nabla \psi|^2}{B^2} \frac{N_e P}{\sigma_{\perp}} \left\{ \frac{d \ln N}{d \psi} + \frac{1}{4} \frac{d \ln T}{d \psi} \right\} \quad (\text{P-21})$$

This is substituted into equation (P-19). The $\underline{V} \cdot \nabla \psi$ term is eliminated from the equation in the same way as used in Section IV. The result is,

$$\frac{d}{dt} \left\{ \int d^3 N_e \right\} = -N_e c D \left\langle \frac{\underline{E} \cdot \underline{B}}{B^2} \right\rangle - \frac{N_e P c^2}{\sigma_{\perp}} \left\langle \frac{|\nabla \psi|^2}{B^2} \right\rangle \left\{ G_1 + \frac{7}{4} G_2 \right\} \quad (\text{P-22})$$

The $\underline{E} \cdot \underline{B}$ term can be evaluated since the current parallel to the magnetic field is known, equation (2-16), and the parallel conductivity is known. Equation (P-22) becomes,

$$\frac{d}{dt} \left\{ \int d^3 N_e \right\} = -D_1^N G_1 - D_2^N G_2 - D_3^N G_3 \quad (\text{P-23})$$

Where the transport coefficients for $Z=1$ are,

$$D_1^N = c^2 N_e P \left\{ \frac{D^2}{\sigma_{\parallel}} \left\langle \frac{1}{B^2} \right\rangle + \frac{1}{\sigma_{\perp}} \left\langle \frac{|\nabla \psi|^2}{B^2} \right\rangle \right\} \quad (\text{P-24})$$

$$D_2^N = c^2 N_e P \left\{ \frac{5}{2} \frac{D^2}{\sigma_{\parallel}} \left\langle \frac{1}{B^2} \right\rangle + \frac{7}{4} \frac{1}{\sigma_{\perp}} \left\langle \frac{|\nabla \psi|^2}{B^2} \right\rangle \right\} \quad (\text{P-25})$$

$$D_3^N = \frac{c^2 N_e D^2}{\pi \sigma_{\parallel}} \quad (\text{P-26})$$

The transport equation for the pressure is obtained in a similar manner, it is,

$$\frac{d}{dt} \left\{ r^3 \frac{3}{2} p \right\} = -D_1^E G_1 - D_2^E G_2 - D_3^E G_3 + S_e \quad (E-27)$$

Where the diffusion coefficients and source term are,

$$D_1^E = 5T D_1^N - \frac{3}{2} \frac{c^2 \rho^2}{\sigma_1} \left\langle \frac{10 \Psi^2}{B^2} \right\rangle \quad (E-28)$$

$$D_2^E = 5T D_2^N + \frac{c^2 \rho^2}{\sigma_2} \left\langle \frac{10 \Psi^2}{B^2} \right\rangle \left\{ \frac{\sqrt{\frac{m_e}{m_i}}}{2\sqrt{2}} - 0.71 \right\} \quad (E-29)$$

$$D_3^E = 5T D_3^N \quad (E-30)$$

$$S_e = \frac{1}{2} \left\langle dr \left\{ u_e \cdot \nabla \Psi \frac{d\rho_e}{d\Psi} + u_i \cdot \nabla \Psi \frac{d\rho_i}{d\Psi} \right\} \right\rangle + \rho \langle V \cdot \nabla \Psi \rangle \quad (E-31)$$

In order to derive the transport equation for the toroidal magnetic flux one needs only to know the parallel current and the parallel conductivity. The transport equation and diffusion coefficients are,

$$\frac{d}{dt} \left\{ r^3 \frac{1}{r^2} \right\} = -D_1^D G_1 - D_2^D G_2 - D_3^D G_3 \quad (E-32)$$

$$D_1^D = \frac{c^2 D P \langle i \rangle}{\sigma_{||}} \quad (E-33)$$

$$D_2^D = \frac{5}{2} \frac{c^2 D P \langle 1 \rangle}{q_v}$$

(P-34)

$$D_3^D = \frac{c^2 D}{4\pi q_v} \langle B^2 \rangle$$

(P-35)

This completes the set of transport equations for the collisional or Pfirsich Schluter regime.

APPENDIX Q

SUMMARY OF EQUATIONS FOR A-TWO-SPECIES PLASMA

The case that is considered here is one in which the ratio of the particle density of species α to the particle density of species β is independent of ψ . In this case the fluxes can be written in terms of the gradient of $PN^{5/3}$, temperature, T , and the toroidal flux, Φ , as in the two species case. This is a simplifying assumption which allows the transport coefficients to be computed with only minor modification of the present computer program. The results are to be taken to indicate the initial diffusion coefficients of such a plasma. In an actual time dependent problem, one must know the fluxes of both ion species, since there is no reason to believe that the fluxes of both ion species will diffuse in a manner to maintain a constant density ratio. In the more complicated case where this assumption is not made an additional gradient must be included in the calculation.

The linearized kinetic equation for the α species is,

$$H_{\alpha}^2 + K_{\alpha\alpha} \chi_{\alpha} + K_{\alpha\beta} \chi_{\beta} + K_{\alpha T} \chi_T - h_{\alpha} \left\{ \left(\frac{d\chi}{d\psi} \right) + \left(\frac{d\chi}{d\Phi} \right) \right\} + h_{\alpha} \left(\frac{d\chi}{dT} \right) = 0 \quad (Q-1)$$

The coupled set of kinetic equations are,

$$H_\alpha \dot{z}_\alpha + K_{\alpha\alpha} \dot{z}_\alpha + K_{\alpha\beta} \dot{z}_\beta + K_{\alpha\gamma} \dot{z}_\gamma + C_\alpha E_b = h_{\alpha\alpha} G_1 + h_{\alpha\gamma} G_2 \quad (1-7)$$

$$H_\beta \dot{z}_\beta + K_{\beta\beta} \dot{z}_\beta + K_{\beta\alpha} \dot{z}_\alpha + K_{\beta\gamma} \dot{z}_\gamma + C_\beta E_b = h_{\beta\alpha} G_1 + h_{\beta\gamma} G_2 \quad (1-8)$$

$$H_\gamma \dot{z}_\gamma + K_{\gamma\gamma} \dot{z}_\gamma + K_{\gamma\alpha} \dot{z}_\alpha + K_{\gamma\beta} \dot{z}_\beta + C_\gamma E_b = h_{\gamma\alpha} G_1 + h_{\gamma\beta} G_2 \quad (1-9)$$

$$Z_\alpha e \int d^3u f_\alpha^{(0)} \dot{z}_\alpha + Z_\beta e \int d^3u f_\beta^{(0)} \dot{z}_\beta + Z_\gamma e \int d^3u f_\gamma^{(0)} \dot{z}_\gamma = h_{\alpha 0} G_3 + h_{\gamma 0} G_4 \quad (1-10)$$

The new functional used in the variational principle is,

$$\begin{aligned} \mathcal{L}\{\xi_\alpha, \xi_\beta, \xi_\gamma, E, h\} = & (\xi_\alpha^+, K_{\alpha\alpha} \xi_\alpha^+) - (\xi_\alpha^-, K_{\alpha\alpha} \xi_\alpha^-) + (\xi_\beta^+, K_{\beta\beta} \xi_\beta^+) \\ & - (\xi_\beta^-, K_{\beta\beta} \xi_\beta^-) + (\xi_\gamma^+, K_{\gamma\gamma} \xi_\gamma^+) - (\xi_\gamma^-, K_{\gamma\gamma} \xi_\gamma^-) + 2(\xi_\alpha^+, K_{\alpha\beta} \xi_\beta^+) \\ & - 2(\xi_\alpha^-, K_{\alpha\beta} \xi_\beta^-) + 2(\xi_\beta^+, K_{\beta\alpha} \xi_\alpha^+) - 2(\xi_\beta^-, K_{\beta\alpha} \xi_\alpha^-) + 2(\xi_\alpha^+, K_{\alpha\gamma} \xi_\gamma^+) \\ & - 2(\xi_\alpha^-, K_{\alpha\gamma} \xi_\gamma^-) + 2(\xi_\beta^+, K_{\beta\gamma} \xi_\gamma^+) - 2(\xi_\beta^-, K_{\beta\gamma} \xi_\gamma^-) + 2(\xi_\gamma^+, K_{\gamma\alpha} \xi_\alpha^+) \\ & - 2(\xi_\gamma^-, K_{\gamma\alpha} \xi_\alpha^-) + 2(\xi_\gamma^+, K_{\gamma\beta} \xi_\beta^+) - 2(\xi_\gamma^-, K_{\gamma\beta} \xi_\beta^-) \\ & - 2(C_\alpha \xi_\alpha^-, E) - 2(C_\beta \xi_\beta^-, E) - 2(C_\gamma \xi_\gamma^-, E) \\ & - 2(\xi_\alpha^+, h_\alpha) - 2(\xi_\beta^+, h_\beta) - 2(\xi_\gamma^+, h_\gamma) - (E, \psi_{\text{ext}}) \end{aligned} \quad (1-11)$$

The transport equation for the total number of particles becomes,

$$\frac{d}{dt} \left\{ \int d^3r N \right\} = - (h_{\alpha N}, \chi_\alpha) - (h_{\beta N}, \chi_\beta) - (h_{\gamma N}, \chi_\gamma) \\ - e D N \left\langle \frac{\mathbf{E} \cdot \mathbf{B}}{B^2} \right\rangle - \left\langle \nabla \psi \cdot \mathbf{b} \times \left\{ d^3r \left(\frac{m_\alpha}{\mathcal{L}_\alpha} \frac{\partial f_\alpha}{\partial t_c} + \frac{m_\beta}{\mathcal{L}_\beta} \frac{\partial f_\beta}{\partial t_c} + \frac{m_\gamma}{\mathcal{L}_\gamma} \frac{\partial f_\gamma}{\partial t_c} \right) \right\} \right\rangle \quad (Q-12)$$

The pressure transport equation becomes,

$$\frac{d}{dt} \left\{ \int d^3r \frac{3}{2} P \right\} = - T \left\{ (h_{\alpha T}, \chi_\alpha) + (h_{\beta T}, \chi_\beta) + (h_{\gamma T}, \chi_\gamma) \right\} + \frac{5}{2} P e D \left\langle \frac{\mathbf{E} \cdot \mathbf{B}}{B^2} \right\rangle \\ P \left\langle \nabla \psi \cdot \mathbf{V} \right\rangle + \int d^3r \frac{dP}{d\psi} \mathbf{V} \cdot \nabla \psi + \int d^3r \frac{e}{4\pi} \frac{dD}{d\psi} \mathbf{E} \cdot \mathbf{B} \\ - \left\langle \int d^3r \nabla \psi \cdot \mathbf{b} \times \mathbf{V} \left\{ \frac{m_\alpha}{\mathcal{L}_\alpha} \frac{\partial f_\alpha}{\partial t_c} + \frac{m_\beta}{\mathcal{L}_\beta} \frac{\partial f_\beta}{\partial t_c} + \frac{m_\gamma}{\mathcal{L}_\gamma} \frac{\partial f_\gamma}{\partial t_c} \right\} \right\rangle \quad (Q-13)$$

The transport equation for the toroidal magnetic flux is unchanged. The variational principle computes the first four quantities on the right hand side of equation (Q-12) and (Q-13) in the same manner as in the two species case.

Adapting the results of Appendix I, the last term in equation (Q-12) can be written as,

$$\left\langle \nabla \Psi \cdot \underline{p} \times \underline{v} \right\rangle = 4\sqrt{\pi} c^2 \left\langle \frac{|\nabla \Psi|^2}{B^2} \right\rangle \sum_i \ln \Lambda N_i z_i e N_i$$

$$\sum_j \frac{z_j e^{1/2} \sqrt{\frac{m_j}{2\pi}}}{1 + m_j/m_i} \left\{ \frac{2}{3} \left[\frac{1}{z_i e} - \frac{1}{z_j e} \right] G_1 + \left[\frac{2}{3} \left\{ \frac{1}{z_i e} - \frac{1}{z_j e} \right\} + \frac{\left\{ \frac{1}{z_i e} - \frac{m_j/m_i}{z_i e} \right\}}{\left\{ 1 + m_j/m_i \right\}} \right] G_2 \right\}$$

(C-14)

The sums are over the subscript α, β and γ .

The last term in equation (C-13) can be written as,

$$\left\langle \nabla \Psi \cdot \underline{p} \times \underline{v} \right\rangle = d^3 \sum_i \frac{m_i}{N_i} \left\langle \frac{\partial f_i}{\partial t} \right\rangle \frac{v^2}{2}$$

$$\frac{c^2}{\pi^2} \left\langle \frac{|\nabla \Psi|^2}{B^2} \right\rangle \sum_i \ln \Lambda N_i z_i e \sqrt{\frac{2\pi}{m_i}} m_i \sum_j \left[V'_{2\alpha\beta} G_1 + V'_{1\alpha\beta} G_2 \right] \quad (C-15)$$

Some values of the quantities $V'_{2\alpha\beta}$ and $V'_{1\alpha\beta}$ are given in the table in Chapter 8.

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REFERENCES

1. L.A. Artamonov, Nuclear Fusion, 12, 313, (1972).
2. H.R. Furth, Nuclear Fusion, 15, 87, (1975).
3. L.A. Artamonov, et al, Nuclear Fusion Conference 1971. (4th Int Conf. Madison), IAEA-CN-23/1-3, 41. (1971).
4. C. Breton, et al, 6th European Conference on Controlled Fusion and Plasma Physics, Moscow, 1, 2, (1973).
5. C.F. Barnett, et al, in Plasma Physics and Controlled Nuclear Fusion Research, (Proc. 4th Int. Conf. Tokyo) 1, 347, (1971).
6. K. Bol, et al, 6th European Conference on Controlled Fusion and Plasma Physics, Moscow, 1, 13, (1973).
7. T. Ohkawa, et al, 6th European Conference on Controlled Fusion and Plasma Physics, Moscow, 1, 131, (1973).
8. D.D. Haddock, et al, Nuclear Fusion, 13, 271, (1973).
9. U. Appolloni-Fantoli, et al, in Plasma Physics and Controlled Nuclear Fusion Research, (Proc. 5th Int. Conf. Tokyo), 1, 191, (1974).
10. H. Fujisawa, et al, in Controlled Fusion and Plasma Physics, (Proc. 5th Int. Conf. Tokyo), 1, 2, (1974).
11. L. McDonalds, et al, in Plasma Physics and Controlled Nuclear Fusion, (Proc. 5th Int. Conf. Tokyo), 1, 1, (1974).
12. J. Kluber, et al, in Plasma Physics and Controlled Nuclear Fusion, (Proc. 5th Int. Conf. Tokyo), 1, 173, (1974).

28. B.I. Braginskii, *Reviews of Plasma Physics*, 1, 205, (1965).
29. S. Pfirsch and A. Schluter, *Max-Planck Institute Report*, MPY/PA/7/62, (1962).
30. S.D. Mozhdanov, *Plasma Physics*, 8, 314, (1966).
31. H. Maschke, *Plasma Physics*, 13, 905, (1972).
32. F.W. Rotherford, *Physics of Fluids*, 13, 482, (1970).
33. M. A. Litine and F.L. Hinton, *Physics of Fluids*, 16, 183, (1973).
34. S. Galeev and R.Z. Sagdeev, *Soviet Physics, JETP*, 26, 211, (1968).
35. Kovrizhnik, *Soviet Physics JETP*, 29, 475, (1969).
36. G. Friedman, *Physics of Fluids*, 13, 490, (1970).
37. Hinton and M.N. Rosenbluth, *Physics of Fluids*, 16, 935, (1973).
38. G. Friedman, F.L. Hinton, and M.N. Rosenbluth, *Physics of Fluids*, 16, 1645, (1974).
39. G. Friedman and W.B. Thompson, *Physics of Fluids*, 16, 1645, (1974).
40. G. Friedman, *Comments on Plasma Physics*, 2, 75, (1976).
41. H. Maschke, *Plasma Physics*, 14, 141, (1972).
42. G. Friedman, M.N. Byrne, and H.H. Klein, *Nuclear Fusion*, 15, 109, (1975).
43. G. Friedman, *Physical Review Letter*, 29, 462 (1972).
44. G. Friedman and I.B. Bernstein, *Physics of Fluids*, 18, 187, (1975).
45. G. Friedman, and I.B. Bernstein, *Annals of Physics*, 110, (1962).
46. G. Friedman, *Physics of Fluids*, 17, 547, (1974).

47. H.K. Meier, W.I. van Rij, C.O. Beasley, Jr., and J.E. McCune, Oak Ridge National Laboratory, ORNL/TM-5314.
48. C.O. Beasley, Jr., H.K. Meier, W.I. van Rij, and J.E. McCune, Oak Ridge National Laboratory, ORNL/TM-5317.
49. P.H. Rutherford, Physics of Fluids, 17, 1782, (1974).
50. A. Samain, Association Euratom-C.E.A., EUR-CEA-FC-744, (1974).
51. J.W. Connor, Plasma Physics, 15, 765, (1973).
52. F.L. Hinton and T.B. Moore, Nuclear Fusion, 14, 639, (1974).
53. J.W. Bond, Jr., K.M. Watson, and J.A. Welch, Jr., Atomic Theory of Gas Dynamics, 11, (1965).
54. T.J.M. Boyd and J.J. Sanderson, Plasma Dynamics, 283, (1969).
55. V.D. Shafranov, Reviews of Plasma Physics, 2, 103, (1966).
56. M.D. Kruskal and R.M. Kulsrud, Physics of Fluids, 1, 205, (1958).