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Reduction of the Growth Rate of the $m = 1$ Resistive Magnetohydrodynamic Mode by Finite Gyroradius Effects

Bruce V. Waddell
Guy Laval
Marshall N. Rosenbluth

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REDUCTION OF THE GROWTH RATE OF THE
 $m = 1$ RESISTIVE MAGNETOHYDRODYNAMIC MODE
BY FINITE GYRORADIUS EFFECTS

Date Published: July 1977

Bruce V. Waddell*
Guy Laval
Marshall N. Rosenbluth

Institute for Advanced Study
Princeton, New Jersey

*Permanent Address: Oak Ridge National Laboratory
Oak Ridge, Tennessee 37830

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ABSTRACT

The reduction in the growth rate of the $m = 1$ resistive magneto-hydrodynamic (MHD) mode due to finite gyroradius effects is determined. Specifically, equilibrium diamagnetic flows, the Hall (or diamagnetic) term in Ohm's law, and ion viscosity are taken into account. The analysis is carried out in cylindrical geometry; toroidal corrections are included in only a very restricted sense. The calculation is valid in the limit where Braginskii's two-fluid equations are valid, i.e., where collision frequencies are large compared to the mode frequency and mean free paths are shorter than the relevant scale lengths. It is shown that the decrease in the mode growth rate due to ion viscosity is small for present-day tokamak parameters. On the other hand, diamagnetic drifts can significantly reduce the growth rate in both the ideal MHD and the resistive limits. In fact, in the ideal MHD limit, the mode can be completely stabilized if the pure MHD growth rate is less than half the ion diamagnetic drift frequency. Although the mode cannot be stabilized in the resistive limit, the growth rate is reduced by a factor of $\gamma_T^2 / |\omega_{*i} \omega_{*e}|$, where γ_T is the purely resistive (tearing) growth rate and ω_{*i} and ω_{*e} are respectively the ion and electron diamagnetic drift frequencies.

1. INTRODUCTION

As reported in Ref. 1, the soft x-ray signal from the plasma core of tokamaks exhibits a periodic or oscillatory time dependence; during a given cycle, the signal slowly rises for about a millisecond and then abruptly decays in less than a hundred microseconds. Because of the corresponding shape of the oscilloscope traces, signals of this type are referred to as sawtooth oscillations. The abrupt decay during a given cycle is called a minor or internal disruption. Superimposed upon the slow rise of a particular sawtooth and preceding the internal disruption, $m = 1$ oscillations with a growth rate of $1\text{-}5\text{ msec}^{-1}$ are observed.

The origin of this phenomenon is a matter of current theoretical interest. Because of the $m = 1$ precursor oscillations, an explanation most likely involves the $m = 1$ magnetohydrodynamic (MHD) mode which can occur in both toroidal and cylindrical geometry. In the regime where resistivity is unimportant and ideal MHD applies, this instability is referred to as the internal kink mode.² The same instability is called a tearing mode in the regime where resistivity dominates.^{3,4} We shall use this nomenclature often, but we emphasize that the two "modes" are really the same instability analyzed in two different limits of the resistivity parameter.

Recently it has been shown that during the nonlinear phase of the growth of the $m = 1$ tearing mode in cylindrical geometry, the toroidal current density and the safety factor profile flatten inside the singular surface where the safety factor equals unity.⁴⁻⁸ The time scale for this process is consistent with the time scale for the internal disruption. The growth rate of the tearing mode, however, is much larger than the

observed growth rate of the precursor oscillations.⁴ Furthermore, although the growth rate of the internal kink is smaller than the growth rate of the tearing mode, it is also larger than the observed value.¹

Consequently, we study the reduction in the growth rate of the $m = 1$ resistive MHD mode due to finite gyroradius. Specifically, in our analysis in cylindrical geometry, we include equilibrium diamagnetic flows, the Hall (or diamagnetic) term in Ohm's law, and ion viscosity. The analysis is valid in the limit where Braginskii's equations are valid, i.e., where collision frequencies are large compared to the mode frequency and mean free paths are shorter than the relevant scale lengths.⁹ Toroidal effects are included in only a very restricted sense. We find that the diamagnetic terms can substantially reduce the growth rate. Within the context of the present analysis, however, viscosity has little effect on the $m = 1$ resistive MHD mode in today's low β tokamaks.

In Sect. 2 a set of "generalized," helically symmetric, reduced MHD equations is derived, including ion viscosity and both the Hall and the resistive terms in Ohm's law. This set of nonlinear equations is obtained from the "two-fluid" equations in Ref. 9, using the techniques described in Refs. 4 and 10 to obtain the simpler set of reduced MHD equations. The equations are linearized in Sect. 3 and a set of two coupled partial differential equations is obtained. Then, in Sect. 4, the effect of the diamagnetic terms on the growth rate of the $m = 1$ resistive MHD mode is ascertained, ignoring ion collisional viscosity. The modification of the growth rate due to ion viscosity is analyzed in Sect. 5. Finally, in Sect. 6 a summary and the main conclusions are presented.

2. GENERALIZED, HELICALLY SYMMETRIC, REDUCED MHD EQUATIONS

In the limit where the electron and ion collision frequencies exceed the instability growth rate and frequency and where the electron and ion mean free paths are less than the relevant machine dimensions, the "two-fluid" equations derived in Ref. 9 by Braginskii can be employed. We define the convective derivative

$$\frac{d_j}{dt} = \frac{\partial}{\partial t} + \underline{V}_j \cdot \underline{\nabla},$$

where $\underline{V}_j$ is the fluid velocity of species j . The density and force balance equations are

$$\frac{dn_j}{dt} = 0$$

and

$$m_j n_j \frac{d_j \underline{V}_j}{dt} = -\underline{\nabla} P_j + Z_j e n_j (\underline{E}_j + \underline{V}_j \times \underline{B}) - \underline{\Pi}_j - \frac{Z_j}{|Z_j|} \underline{R}.$$

Here, n_j is the particle density, m_j is the mass, P_j is the pressure, Z_j is the charge number including the sign, e is the magnitude of the electron charge, $\underline{E}$ is the electric field, and $\underline{B}$ is the magnetic field. (The rationalized MKS system of units is employed.)

The α component of the viscous drag $\underline{\Pi}_j$ is given by

$$\Pi_{j\alpha} = \frac{\partial \pi_{j\alpha\beta}}{\partial x_\beta},$$

where, suppressing the j subscript, the viscosity tensor is given by

$$\pi_{\alpha\beta} = -\eta_0 W_{0\alpha\beta} - \eta_1 W_{1\alpha\beta} - \eta_2 W_{2\alpha\beta} + \eta_3 W_{3\alpha\beta} + \eta_4 W_{4\alpha\beta}.$$

Here x_α is a three-dimensional vector denoting the spatial coordinate, and implied summation notation is used. We define the unit vector as $\underline{b} \equiv \underline{B}/B$ and the tensor $W_{\alpha\beta}$ as

$$W_{\alpha\beta} = \frac{\partial V_\alpha}{\partial x_\beta} + \frac{\partial V_\beta}{\partial x_\alpha} - \frac{2}{3} \delta_{\alpha\beta} \underline{V} \cdot \underline{V},$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. Then the subtensors of the viscosity tensor are given by

$$W_{0\alpha\beta} = \frac{3}{2} (b_\alpha b_\beta - \frac{1}{3} \delta_{\alpha\beta}) (b_\mu b_\nu - \frac{1}{3} \delta_{\mu\nu}) W_{\mu\nu},$$

$$W_{1\alpha\beta} = (\delta_{\alpha\mu}^\perp \delta_{\beta\nu}^\perp + \frac{1}{2} \delta_{\alpha\beta}^\perp b_\mu b_\nu) W_{\mu\nu},$$

$$W_{2\alpha\beta} = (\delta_{\alpha\mu}^\perp b_\beta b_\nu + \delta_{\beta\nu}^\perp b_\alpha b_\mu) W_{\mu\nu},$$

$$W_{3\alpha\beta} = \frac{1}{2} (\delta_{\alpha\mu}^\perp \epsilon_{\beta\gamma\mu} + \delta_{\beta\nu}^\perp \epsilon_{\alpha\gamma\mu}) b_\gamma W_{\mu\nu},$$

$$W_{4\alpha\beta} = (b_\alpha b_\mu \epsilon_{\beta\gamma\nu} + b_\beta b_\nu \epsilon_{\alpha\gamma\mu}) b_\gamma W_{\mu\nu},$$

where $\delta_{\alpha\beta}^\perp = \delta_{\alpha\beta} - b_\alpha b_\beta$ and $\epsilon_{\alpha\beta\gamma}$ is the antisymmetric unit tensor. For electrons ($j = e$) and ions ($j = i$), the viscosity coefficients are given by

$$\eta_0^e = 0.73 n_e T_e \tau_e$$

$$\eta_0^i = 0.96 n_i T_i \tau_i$$

$$\eta_1^e = 0.51 \frac{n_e T_e}{\Omega_e^2 \tau_e}$$

$$\eta_1^i = \frac{3}{10} \frac{n_i T_i}{\Omega_i^2 \tau_i}$$

$$\eta_2^j = 4\eta_1^j$$

$$\eta_3^e = -\frac{1}{2} \frac{n_e T_e}{\Omega_e}$$

$$\eta_3^i = \frac{1}{2} \frac{n_i T_i}{\Omega_i}$$

$$\eta_4^j = 2\eta_3^j.$$

Here, T_j is the species temperature, Ω_j is the gyrofrequency, and τ_j is the collision time defined in Ref. 9. The electron coefficients are valid only for $Z_i = 1$. The viscosity arising from the η_0^j , η_1^j , and η_2^j terms is called collisional viscosity because it depends on the collision frequency. The viscosity arising from η_3^j and η_4^j is called magnetic viscosity because it involves the gyrofrequency but not the collision frequency.

The friction term $\underline{R}$ is given by

$$\underline{R} = \underline{R}_u + \underline{R}_T,$$

with

$$\underline{R}_u = en_e (\underline{J}_\parallel / \sigma_\parallel + \underline{J}_\perp / \sigma),$$

and

$$\underline{R}_T = -0.71 n_e \underline{\nabla}_\parallel T_e - \frac{3}{2} \frac{n_e}{\Omega_e \tau_e} (\underline{b} \times \underline{\nabla} T_e),$$

where $\underline{J}_\parallel$ ($\underline{J}_\perp$) is the current density parallel (perpendicular) to the magnetic field $\underline{B}$ and $\underline{\nabla}$ denotes the gradient parallel to $\underline{B}$. The perpendicular and parallel conductivities are given by

$$\sigma_\perp = \frac{e^2 n_e \tau_e}{m_e} = \eta_\perp^{-1}$$

and

$$\sigma_\parallel = 1.96 \sigma_\perp = \eta_\parallel^{-1}.$$

The electron and ion heat balance equations are discussed later.

First, it is necessary to obtain a generalized Ohm's law and a generalized fluid momentum balance equation. We consider only electrons and ions. Multiplying the momentum equations by $Z_j e/m_j$, adding, and neglecting terms of order m_e/m_i yields the generalized Ohm's law

$$\begin{aligned} \underline{E} = & -\underline{V} \times \underline{B} + \frac{1}{ne} \underline{J} \times \underline{B} + \frac{1}{ne} \underline{R} \\ & - \frac{1}{ne} \underline{\nabla} p_e - \frac{1}{ne} (\underline{\Pi}_e - \frac{m_e}{m_i} \underline{\Pi}_i) \\ & + \frac{m_e}{ne^2} \left[\frac{d\underline{J}}{dt} + \frac{1}{n} \underline{J} \cdot \underline{\nabla} n \underline{V} - \frac{1}{ne} \underline{J} \cdot \underline{\nabla} \underline{J} \right]. \end{aligned} \quad (1)$$

Here, $\underline{V} \equiv (m_e \underline{V}_e + m_i \underline{V}_i)/(m_i + m_e) \approx \underline{V}_i$ is the single-fluid velocity, $n \equiv n_e = Z_i n_i$, and $\underline{J}$ is the total current density. Adding the momentum balance equations gives

$$\rho \frac{d\underline{V}}{dt} = -\underline{\nabla} p - \underline{\Pi} + \underline{J} \times \underline{B} \quad (2)$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \underline{V} \cdot \underline{\nabla},$$

$\rho \equiv m_e n_e + m_i n_i \approx m_i n_i$ is the mass density, $P \equiv P_i + P_e$ is the total pressure, and $\underline{\Pi} \equiv \underline{\Pi}_e + \underline{\Pi}_i \approx \underline{\Pi}_i$ is the total viscous drag.

Following Refs. 4 and 10, we employ cylindrical geometry and consider initial perturbations which are functions only of r and $\tau = m\theta + kz$, where r is the radial coordinate, m the poloidal mode number, θ the poloidal coordinate, k the toroidal mode number, and z the toroidal coordinate. Subsequently, as the system evolves in time, all quantities will remain spatial functions only of r and τ , i.e., they will retain the original helical symmetry.

Faraday's law may be written as

$$\frac{\partial \underline{A}}{\partial t} = - \underline{E} + \underline{\nabla} \phi ,$$

where $\underline{B} = \underline{\nabla} \times \underline{A}$ and ϕ is a scalar potential. Because the unit vector $\hat{h}$ in the direction of the helix defined by τ is given by $\hat{h} = \hat{z} - \delta \hat{\theta}$, with $\delta \equiv kr/m \ll 1$, $\hat{h} \cdot \underline{\nabla} \phi = 0$ and

$$\frac{\partial \psi}{\partial t} = -\hat{h} \cdot \underline{E} ,$$

where $\psi \equiv \hat{h} \cdot \underline{A}$ is the net flux through the helical ribbon defined by the axis of the cylinder and the helix intersecting the point r, θ, z .

Substituting for $\underline{E}$ from Eq. 1, noting that $\hat{h} \cdot (\underline{D} \times \underline{B}) = -\underline{D} \cdot \underline{\nabla} \psi$ (where $\underline{D}$ is an arbitrary vector), and neglecting terms of order δ^3 and $\delta\beta$ [where $\beta \equiv 2\mu_0(n_i T_i + n_e T_e)/B_z^2$], we obtain

$$\frac{d\psi}{dt} = -\eta(J_z + 2\tau_e \frac{dJ_z}{dt}) + \frac{1}{ne} \underline{J} \cdot \underline{\nabla} \psi + 0.71 \frac{1}{e} \nabla_{\parallel} T_e + \Pi_h ,$$

where $\Pi_h = \hat{h} \cdot (\underline{\Pi}_e - m_e \underline{\Pi}_i / m_i) / (ne)$ and $\nabla_{\parallel} = \hat{b} \cdot \underline{\nabla}$. From Ampere's law, the toroidal current density J_z is given by

$$J_z = -(\nabla^2 \psi + \frac{2kB_z}{m}) / \mu_0 . \quad (3)$$

Applying the operator $\hat{z} \times$ to Eq. 2 yields

$$\underline{J} \cdot \underline{\nabla} \psi = \frac{1}{B_z} (\hat{z} \times \underline{\nabla} P + \underline{\Pi}) \cdot \underline{\nabla} \psi ,$$

provided terms of order γ/Ω_i are neglected, where γ is the growth rate of the instabilities of interest. Consequently, the flux equation can be written as

$$\frac{d\psi}{dt} = -\underline{V}^* \cdot \underline{\nabla}\psi + \frac{1}{neB_z} \hat{\Pi} \cdot \underline{\nabla}\psi + \Pi_h - \eta(J_z + 2\tau_e \frac{dJ_z}{dt}) , \quad (4)$$

with

$$\underline{V}^* \equiv \frac{1}{neB_z} \underline{\nabla}P \times \hat{z} + 0.71 \frac{I}{eB_z} \underline{\nabla}T_e \times \hat{z},$$

and

$$\hat{\Pi} = \hat{z} \times \underline{\Pi} .$$

If we assume that B_z is constant in space and time, then magneto-acoustic waves will be automatically eliminated from consideration. Furthermore, the toroidal component of Faraday's law combined with Ohm's law (Eq. 1) implies that

$$\underline{\nabla} \cdot \underline{V} = \frac{1}{eB_z} \hat{z} \cdot [\underline{\nabla}\psi \times \underline{\nabla}(J_z/n)] - \frac{1}{en} \underline{\nabla}n \cdot \underline{J} + \frac{0.71}{eB_z} \hat{z} \cdot (\underline{\nabla} \times \underline{\nabla}_{\parallel} T_e) . \quad (5)$$

However, evaluation of this expression shows that $\underline{\nabla} \cdot \underline{V}_{\perp} \sim \delta^2 |\underline{V}_{\perp}|/r_w$, where $\perp$ means perpendicular to z and r_w is the radius of the cylinder. Thus, in reducing the momentum balance equation, we take $\underline{\nabla} \cdot \underline{V} = 0$. Later, in the electron heat balance equation, $\underline{\nabla} \cdot \underline{V}$ can be included as a correction term; in fact, in obtaining Eq. 5 we have neglected terms which are small far from the tearing layer where the $\underline{\nabla} \cdot \underline{V}$ correction is important for $m > 1$ modes.

In addition to the fact that $\underline{\nabla} \cdot \underline{V} = 0$ to order δ , an analysis of the momentum balance equation shows that V_z may be neglected because $V_z \sim \delta |V_{\perp}|$. Consequently, we introduce the stream function A for the velocity,

$$\underline{V} = \underline{\nabla}A \times \hat{z}.$$

Then, taking the toroidal component of the curl of Eq. 2 gives

$$\rho \frac{d}{dt} \nabla^2 A + \underline{\nabla} \rho \cdot \frac{d}{dt} \underline{\nabla} A = \hat{z} \cdot \left[\underline{\nabla} \psi \times \underline{\nabla} J_z \right] + \Pi^*, \quad (6)$$

where $\Pi^* \equiv \hat{z} \cdot (\underline{\nabla} \times \underline{\Pi})$ and all the spatial derivatives are perpendicular to z .

3. LINEARIZED EQUATIONS

Equilibrium quantities vary only in the radial direction. The equilibrium velocity is in the $\hat{z}$ direction and is given by $V_{z0} = -P'_{i0}/(enB_{\theta 0}) + E_{r0}/B_{\theta 0}$, where the prime denotes the radial derivative, the "o" subscripts denote equilibrium (unperturbed) quantities, and E_{r0} is the equilibrium (or ambipolar) electric field. We consider perturbations $\tilde{f}$ of the form

$$\tilde{f} = f_1(r)e^{-i\bar{\omega}t + im\theta + ik_z z} = f_1(r)e^{-i\bar{\omega}t + i\omega t},$$

where ω is the frequency. The electric field piece of V_{z0} simply introduces a Doppler shift to $\bar{\omega}$. Thus, we may omit $E_{r0}/B_{\theta 0}$ in V_{z0} provided we replace $\bar{\omega}$ by ω , where $\omega \equiv \bar{\omega} - \omega_D$, $\omega_D \equiv kE_{r0}/B_{\theta 0}|_{r_s}$, and r_s is the radius of the singular surface. We must remember, however, that the frequencies found in the following sections must be incremented by an amount ω_D when referring to the laboratory coordinate system.

In obtaining the linearized momentum balance equation both the magnetic and collisional parts of the viscosity term must be retained. The magnetic part cancels with the part of the inertial term arising from the equilibrium velocity, and inside the tearing layer Eq. 6 linearly reduces to

$$\rho_0(\omega - i\mu \frac{\partial^2}{\partial x^2}) A_1' = i \frac{m\psi_0'}{\mu_0 r_s} \psi_1', \quad (7)$$

where we have made the standard assumption that $mf_1/r \ll \partial f_1/\partial x$.

The viscosity parameter μ is given by $\mu = 3a_i^2 v_i/5$, where $v_i = \tau_i^{-1}$ and $a_i^2 = T_i/(2m_i \Omega_i^2)$. The quantity $x \equiv r - r_s$, with r_s the radius of the singular surface specified by $q(r_s) = m/n$ (where q is the safety factor).

One can show that the viscosity and perturbations in the resistivity are negligible in the linearized helical flux equation. Consequently,

inside the tearing layer the linearized version of Eq. 4 becomes

$$i(\omega - \omega_{*e}^p - 0.71\omega_*^T) \psi_1 - \psi_0' V_{r1} = i \frac{m}{eB_z r_s} (P_1/n + 0.71T_{e1}) \psi_0' - \frac{\eta_0}{\mu_0} [1 - 2i\tau_e(\omega - \omega_{*i}^p)] \psi_1' , \quad (8)$$

where $\omega_{*i}^p \equiv mP_{oi}'/(enr_s B_z)$, $\omega_{*e}^p \equiv -mP_{oe}'/(enr_s B_z)$, $\omega_*^T \equiv -mT_{eo}'/(er_s B_z)$, and $V_{r1} \equiv mA_1/r$. In order to obtain expressions for P_1 and T_{e1} , we consider the heat balance equations, ignoring the transport terms which are important but make the analysis intractable. The $\nabla \cdot \underline{V}$ and viscosity terms may be ignored because they are important only in obtaining a solution for $m > 1$ modes far from the tearing layer.¹¹ Thus, we have $dP_i/dt = 0$, and

$$\frac{d_e P_e}{dt} = \frac{dP_e}{dt} - \frac{1}{ne} \underline{J} \cdot \nabla P_e .$$

The current density $\underline{J}_z$ is given by Eq. 3. Consequently, upon linearization we obtain $P_1 = -iV_{r1}P_o'/\hat{\omega}_{pi}$ and $T_1 = -iV_{r1}T_{eo}'/\hat{\omega}_{pi}$, where $\hat{\omega}_{pi} = \omega - \omega_{*i}^p$. Upon substituting for P_1 and T_{e1} in Eq. 8 and defining $\omega_{*i} \equiv \omega_{*i}^p$, $\omega_{*e} \equiv -(\omega_{*e}^p + 0.71\omega_*^T)$, and

$$\eta^* \equiv [1 - 2i\tau_e(\omega - \omega_{*i})] \frac{\omega - \omega_{*i}}{\omega + \omega_{*e}} ,$$

we obtain

$$i(\omega - \omega_{*i})\psi_1 + V_{r1}|\psi_0'|x = -\eta_0\eta^*\psi_1'/\mu_0 , \quad (9)$$

where we have observed that the expansion in x of ψ_0 gives $\psi_0' = -|\psi_0'|x$.

We notice that Eqs. 7 and 9 reduce to those obtained in Ref. 12, provided $\bar{\mu}^2 = \tau_e \omega = \tau_e \omega_{*i} = 0$. This is what one would expect, because in Ref. 12 collisionless effects are treated carefully but viscosity is neglected, while in this article collisionless effects are not treated rigorously but viscosity is retained.

Outside the tearing layer, the inertial and viscosity terms may be neglected in the momentum balance equation. Therefore, Eq. 6 implies that J_z is a function only of ψ , and expansion of Eq. 3 yields

$$\psi_1'' + \psi_1'/r - \frac{m^2}{r^2} \psi_1 = - \frac{J_{z0}'}{\psi_0'} \psi_1.$$

For $m = 1$, the solution of this equation is simply

$$\psi_1 = \begin{cases} C\psi_0'; & r < r_s \\ 0; & r \geq r_s \end{cases},$$

where C is a normalizing constant.

It is convenient to write Eqs. 7 and 9 in dimensionless form. The two relevant time scales are the poloidal MHD time $\tau_{Hp} \equiv m(\mu_0 \rho_0)^{1/2} / k B_z$ and the skin time $\tau_R \equiv r_w^2 / \eta_0$. We normalize the time t to τ_{Hp} , x and r to r_w , and ψ to $k B_z r_w^2 / m$. Defining the quantity $S \equiv \tau_R / \tau_{Hp}$, the viscosity parameter $\bar{\mu}^2 \equiv \mu \tau_R / r_w^2$, and the shear parameter $\alpha \equiv m |\psi_0''| / r_s = \left. \frac{dq_0}{dr} \right|_{r=r_s}$, where $|\psi_0''|$ and r_s are in dimensionless form and q_0 is the safety factor, we obtain

$$(\gamma + i\omega_{*i})\psi_1 - \alpha x A_1 = \eta^* \psi_1' / S \quad (10)$$

and

$$\gamma A_1' - \bar{\mu}^2 A_1^{IV} / S = -\alpha x \psi_1' \quad (11)$$

Here $\gamma \equiv -i_\omega$ and the fourth derivative with respect to x is denoted by the superscript IV.

4. EFFECT OF DIAMAGNETIC DRIFTS

In the limit that the viscosity parameter $\bar{\mu}^2 = 0$, we want to determine the effect of finite gyroradius on the $m = 1$ resistive MHD mode. We define $X \equiv x/\epsilon$, where ϵ is the characteristic singular layer width which must be determined from the solution of the equations. We also define $\lambda \equiv S\gamma_i \epsilon^2/\eta^*$, where $\gamma_i \equiv \gamma + i\omega_{*i}$, and then implicitly determine ϵ by requiring $\lambda^2 = \gamma\gamma_i/(\alpha\epsilon)^2$. Then, defining $\gamma_e \equiv \gamma - i\omega_{*e}$, we have

$$\lambda^3 \gamma_T^3 = \frac{\gamma\gamma_i\gamma_e}{1 + 2\tau_e\gamma_i},$$

$$\epsilon = \left[\frac{\gamma\gamma_i(1 + 2\tau_e\gamma_i)}{\alpha^2 S^2 \gamma_e^2} \right]^{1/6},$$

where $\gamma_T = \alpha^{2/3}/S^{1/3}$ is the $m = 1$ tearing mode growth rate in the pure MHD limit, i.e., the growth rate ignoring diamagnetic drifts and viscosity. For the modes of interest, the quantity $\tau_e |\gamma + i\omega_{*i}| \ll 1$ and may be neglected. Consequently, in terms of the quantity $\Psi \equiv -\psi/(\epsilon|\psi'_0|)$ and the fluid displacement $\xi = mA_1/(r_s\gamma_i)$, Eqs. 10 and 11 become Eqs. A1 and A2 in the Appendix. As described in Ref. 13 and derived in detail in the Appendix, the general dispersion relation or eigenvalue equation is

$$\Gamma\left(\frac{\lambda^{3/2} + 5}{4}\right) / \Gamma\left(\frac{\lambda^{3/2} - 1}{4}\right) = \frac{\lambda^{5/4}\gamma_K}{8\alpha\epsilon}. \quad (12)$$

where Γ denotes the gamma function. The quantity γ_K is the growth rate of the $m = 1$ ideal internal kink mode in the pure MHD limit² and in general can include the effects of noncircular poloidal flux surfaces

and finite inverse aspect ratio. In Refs. 14 and 15, attempts have been made to calculate γ_K in toroidal geometry but, unfortunately, the results do not agree.

First, we determine the effect of diamagnetic drifts on the $m = 1$ MHD mode in the ideal MHD (internal kink) limit. We consider the limit of Eq. 12 as $S \rightarrow \infty$, i.e., as $\lambda \rightarrow \infty$. Employing the formula

$$\lim_{Z \rightarrow \infty} \frac{\Gamma(Z + a)}{\Gamma(Z + b)} = Z^{a-b} \left[1 + \frac{(a-b)(a+b-1)}{2Z} + \dots \right],$$

we obtain

$$\gamma(\gamma + i\omega_{*i}) = \gamma_K^2 \left[1 - \left(\frac{\gamma_T}{\gamma_K} \right) \left(\frac{\gamma_T}{\gamma_e} \right)^{1/2} \right]^2.$$

Then, if $\gamma_T/\gamma_K \sim \gamma_T/|\omega_{*i}| \sim \gamma_T/|\omega_{*e}| \ll 1$,

$$\text{Re}\gamma \approx \left\{ \gamma_K^2 \left[1 + 2 \left(\frac{\gamma_T}{\gamma_K} \right) \left(\frac{\gamma_T}{\omega_{*i} + 2\omega_{*e}} \right)^{1/2} \right] - \frac{\omega_{*i}^2}{4} \right\}^{1/2}$$

and

$$\text{Im}\gamma \approx -\frac{\omega_{*i}}{2} + \frac{\gamma_K \gamma_T}{\text{Re}\gamma} \left(\frac{\gamma_T}{\omega_{*i} + 2\omega_{*e}} \right)^{1/2}.$$

Therefore, we conclude that diamagnetic drifts reduce the growth rate of the internal kink mode. In fact, the mode can be completely stabilized; the stability criterion is $\gamma_K < |\omega_{*i}|/2$ when the resistive term is negligible. We also observe that resistivity increases the internal kink growth rate, as one would expect.

Now, we consider the effect of diamagnetic drifts on the $m = 1$ tearing mode by analyzing the dispersion relation in the limit $|\gamma_K/(8\alpha\epsilon)| \ll 1$. Then λ is nearly unity and Eq. 12 becomes

$$\gamma(\gamma + i\omega_{*i})(\gamma - i\omega_{*e}) = \gamma_T^3 [1 + \gamma_K/(\pi^{1/2}\alpha\epsilon)]^2. \quad (13)$$

Of course, for the case $\gamma \gg \omega_{*i}, \omega_{*e}$,

$$\gamma \approx \gamma_T [1 + \gamma_K/(\pi^{1/2}\alpha\epsilon)]^{2/3} \approx \gamma_T.$$

For the cases $\gamma \approx -i\omega_{*i}$ and $\gamma \approx i\omega_{*e}$, there are no unstable roots. The most interesting case is $\gamma \ll \omega_{*i}, \omega_{*e}$ for which we obtain

$$\gamma = \frac{\gamma_T^3}{\omega_{*i}\omega_{*e}} F^2 \left[1 - i \frac{\gamma_T^3 (\omega_{*i} - \omega_{*e})}{(\omega_{*i}\omega_{*e})^2} F^2 \right], \quad (14)$$

where

$$F \equiv 1 + \frac{\gamma_K}{\pi^{1/2}\alpha\epsilon_0}$$

and

$$\epsilon_0^{-1} = (1 - i)(S|\omega_{*e}|/2)^{1/2}.$$

Similar results have been obtained by Bussac et al.¹⁶ and Basu et al.¹⁷

If $\gamma_K = 0$ and $\omega_{*i} = \omega_{*e} \equiv \omega_*$, the unstable root of Eq. 13 is purely real and in general is given by

$$\gamma = \gamma_T(z_2 + z_2^*), \quad (15)$$

where $z_1 = (\bar{z} + 1/2)^{1/3}$, $z_2 = -(\bar{z} - 1/2)^{1/3}$, and $\bar{z} = [(\omega_*/\gamma_T)^6/27 + 1/4]^{1/2}$.

In Fig. 1, γ/γ_T as given by Eq. 15 is plotted as a function of ω_*/γ_T .

The dotted line denotes the approximate result given by Eq. 14. We conclude that the inclusion of diamagnetic or finite gyroradius effects in the analysis of the $m = 1$ tearing mode reduces its growth rate a significant amount for $\gamma_T < \omega_{*i}$, ω_{*e} . However, the instability remains essentially purely growing, except for the Doppler shift discussed in Sect. 3.

Now we consider the limit where $|\gamma_K/8\alpha\epsilon| \gg 1$ but $\gamma_K < 0$. A negative γ_K could arise from toroidal¹⁴⁻¹⁵ and/or noncircular cross section effects. Then the appropriate limit of the dispersion relation is $\lambda \ll 1$, yielding

$$\gamma(\gamma + i\omega_{*i})(\gamma - i\omega_{*e})^3 = \frac{2^4 \gamma_T^9}{\gamma_K^4}.$$

The most interesting unstable root is such that $\gamma = i\omega_{*e} + \delta\gamma$, where $|\delta\gamma| \ll |\omega_{*e}|$ and

$$\text{Re}\delta\gamma = 2^{1/3} \frac{\gamma_T^3}{[\omega_{*e}(\omega_{*e} + \omega_{*i})]^{1/3} \gamma_K^{4/3}}.$$

This case is analyzed in more detail in Ref. 17. There is another unstable root with frequency $-\omega_{*i}$ but the growth rate is smaller. These solutions are valid provided $|\delta\gamma/\omega_{*e}| \ll 1$ and $\lambda \ll 1$; i.e., we must have

$$|\gamma_K| > |\omega_{*i,e}| > \left(\frac{\gamma_T}{|\gamma_K|}\right)^{4/5} \gamma_T.$$

If the preceding inequalities are not satisfied, we find no unstable

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Fig. 1. Growth rate of the $m = 1$ MHD mode in the resistive or tearing limit, plotted as a function of ω_*/γ_T .

solutions. These results indicate that the effects of toroidicity and noncircular cross section on γ_K warrant further consideration.

5. EFFECT OF VISCOSITY

In this section, we determine the damping effect of viscous drag on the $m = 1$ resistive MHD mode. We first consider the quantity

$$\bar{\mu}^2 = \frac{\mu \tau_R}{r_w^2} = \frac{1}{10} \left(\frac{T_e}{T_i} \frac{m_i}{m_e} \right)^{1/2} \beta_e Z_i^2,$$

where $\beta_e \equiv 2\mu_0 n T_e / B_z^2$ and Z_i is the ion charge. For typical tokamak parameters, $T_e/T_i = 2$, $Z_i = 1$, and $\beta_e = 5 \times 10^{-3}$; hence, $\bar{\mu}^2 \approx 1/30$. Thus, in the following analysis, we shall exploit the fact that $\bar{\mu}^2$ is small.

In terms of ξ and ψ , Eqs. 10 and 11 become

$$\psi + X\xi = \frac{1}{\lambda} \frac{d^2 \psi}{dX^2} \quad (16)$$

and

$$\left(1 - \frac{\mu^*}{\lambda} \frac{d^2}{dX^2} \right) \frac{d^2 \xi}{dX^2} = \frac{X}{\lambda^2} \frac{d^2 \psi}{dX^2}, \quad (17)$$

where $\mu^* \equiv \gamma_e \bar{\mu}^2 / \gamma$ and $\gamma_e \equiv \gamma - i\omega_{*e}$. We introduce the Fourier transform f_p of an arbitrary function $f(X)$

$$f_p = \frac{1}{2\pi} \int_{-\infty}^{\infty} dX e^{-ipX} f(X).$$

Then, the Fourier transforms of Eqs. 16 and 17 are

$$\psi_p + i \frac{d}{dp} \xi_p = - \frac{p^2}{\lambda} \psi_p$$

and

$$p^2(1 + \mu^* p^2/\lambda) \xi_p = i \frac{1}{\lambda^2} \frac{d}{dp} p^2 \psi_p .$$

Eliminating ψ_p and introducing the function ϕ defined by

$$\phi \equiv p \xi_p / \sqrt{p^2 + \lambda^2} ,$$

we obtain

$$\frac{d^2 \phi}{dp^2} = Q(p) \phi ,$$

where

$$Q(p) = \frac{(\lambda + \mu^* p^2)(p^2 + \lambda)^3 - 3\lambda}{(p^2 + \lambda)^2} .$$

First, we consider the somewhat unrealistic limit where $\gamma_K = \omega_{*i} = \omega_{*e} = 0$ and $\bar{\mu}^2 \gg 1$, which corresponds to $0 < \lambda \ll 1$. In this limit a WKB solution can be constructed. The function Q has two real turning points given by $p = \pm p_0$, where $p_0 = (3\lambda/\bar{\mu}^2)^{1/8}$. The eigenvalue condition is

$$\int_{-p_0}^{p_0} dp [-Q(p)]^{1/2} = \pi(n + 1/2) .$$

In the limit $\bar{\mu}^2 \gg 1$, this condition reduces to

$$\int_{-y_0}^{y_0} dy \frac{[3 - N^2 y^2(1 + y^2)]^{1/2}}{(1 + y^2)} = \pi(n + 1/2) ,$$

where $y^2 \equiv p^2/\lambda$, $y_0 = p_0/\lambda^{1/2}$, and $N^2 \equiv \bar{\mu}^2 \lambda^3$. Thus, N must be a quantity of order unity and

$$\gamma = \left(\frac{N}{\bar{\mu}}\right)^{2/3} \gamma_T .$$

This expression shows that in the tearing limit, an unrealistically large viscosity can significantly reduce the growth rate of the tearing mode, although it cannot stabilize it completely. However, the preceding WKB analysis is not sufficient to insure the existence of the mode; the proof of existence is developed in the following paragraphs.

In order to include the diamagnetic drifts and analyze the realistic case where $\bar{\mu}^2 \ll 1$, we must consider a quadratic form. In terms of y ,

$$\frac{d^2 \phi}{dy^2} = \left[\frac{\lambda^3 (1 + \mu^* y^2) (1 + y^2)^3 - 3}{(1 + y^2)^2} \right] \phi , \quad (18)$$

and the quadratic form is

$$\lambda^3 = \frac{\int dy \left[\frac{3\phi^2}{(1 + y^2)^2} - \left(\frac{d\phi}{dy} \right)^2 \right]}{\int dy \phi^2 [(1 + y^2)(1 + \mu^* y^2)]} . \quad (19)$$

Notice that λ^3 is variational with respect to ϕ . Thus, for $|\mu^*| < 1$, the eigenvalue λ may be obtained by employing ϕ_0 as the trial function for ϕ , where ϕ_0 is the eigenfunction for $\mu^* = 0$. We obtain

$$\lambda^3 = \frac{I_1/I_2}{1 + \mu^* I_3/I_2} ,$$

where

$$I_1 = \int dy \left[\frac{3\phi_0^2}{(1 + y^2)^2} - \left(\frac{d\phi_0}{dy} \right)^2 \right] ,$$

$$I_2 = \int dy \phi_0^2 (1 + y^2) ,$$

$$I_3 = \int dy \phi_0^2 y^2 (1 + y^2) .$$

Obviously, for $\mu^* = 0$, $I_1/I_2 = \lambda_0^3$, where λ_0 is the solution of Eq. 12, and hence

$$\lambda^3 = \frac{\lambda_0^3}{1 + \mu^* I} ,$$

where $I \equiv I_3/I_2$ is of order unity.

First, we observe that if $\gamma_K = \omega_{*i} = \omega_{*e} = 0$, then $\mu^* = \bar{\mu}^2$, $\lambda_0 = 1$, and the integrals can be taken along the real axis so that

$$\lambda = \frac{1}{1 + \bar{\mu}^2 I}$$

with $I > 0$. Thus, we observe that $0 < \lambda < 1$ or $0 < \gamma < \gamma_T$, which proves the existence of the mode in the limit analyzed in the preceding paragraphs, provided $\bar{\mu}^2$ is small enough for λ to be obtained by substituting ϕ_0 for ϕ in Eq. 19. Next, we consider the more general case where the diamagnetic frequencies are not zero. Because $\bar{\mu}^2 \ll 1$, viscosity can be important only if $|\gamma_e I / \gamma| \gg 1$. Thus, from the results of Sect. 4, viscosity has an important effect only if $|\gamma_K / (\alpha \epsilon)| \ll 1$, i.e., when $\lambda_0 \approx 1$, because in the other limits $|\gamma_e / \gamma| \ll 1$. It can be trivially verified that

$$\phi_0 = (y^2 + 1)^{-1/2} e^{-y^2/2}$$

is the solution of Eq. 18 for $\mu^* = 0$ and $\lambda = 1$. Since $\lambda = 1$, y and ϕ_0 are real and the integrations may be taken along the real axis.

Therefore, $I = 2$ and the dispersion relation becomes

$$[\gamma + 2(\gamma - i\omega_{*e})\bar{\mu}^2](\gamma + i\omega_{*i})(\gamma - i\omega_{*e}) = \gamma_T^3. \quad (20)$$

In order to solve Eq. 20 for γ , we set $\gamma = \gamma_0 + \delta\gamma$, where

$$|\gamma_0/\omega_{*i}| \sim |\gamma_0/\omega_{*e}| \sim |\delta\gamma/\gamma_0|. \quad \text{To lowest order,}$$

$$\text{Re}\gamma_0 = \frac{\gamma_T^3}{(1 + 2\bar{\mu}^2)[\omega_{*i}\omega_{*e} + b\omega_{*e}(\omega_{*i} - \omega_{*e})]}$$

and

$$\text{Im}\gamma_0 = \frac{b\omega_{*i}(\omega_{*e})^2}{\omega_{*i}\omega_{*e} + b\omega_{*e}(\omega_{*i} - \omega_{*e})},$$

where $b \equiv 2\bar{\mu}^2/(1 + 2\bar{\mu}^2)$. To next order, we find

$$\text{Re}\gamma = (\text{Re}\gamma_0) \left\{ 1 - \frac{2(\text{Im}\gamma_0)[(1 + b)\omega_{*e} - \omega_{*i}]}{\omega_{*i}\omega_{*e} + b\omega_{*e}(\omega_{*i} - \omega_{*e})} \right\} \quad (21)$$

and

$$\text{Im}\gamma = \text{Im}\gamma_0 + \frac{[(\text{Im}\gamma_0)^2 - (\text{Re}\gamma_0)^2][(1 + b)\omega_{*e} - \omega_{*i}]}{\omega_{*i}\omega_{*e} + b\omega_{*e}(\omega_{*i} - \omega_{*e})}.$$

For present devices, typically $\bar{\mu}^2 < \gamma_T^3/[2\omega_{*i}(\omega_{*e})^2]$ so that $\text{Re}\gamma \gg \text{Im}\gamma$ and the requirement that $|\mu^*I| < 1$ for Eq. 20 to be valid is satisfied. Consequently, we conclude that in present devices where $\bar{\mu}^2 \ll 1$ viscous stabilization does not occur. According to Eq. 21, however, viscous stabilization can occur for roughly $b \gtrsim 1/2$ or $\bar{\mu}^2 \gtrsim 1/2$. For a hydrogen plasma with $T_e/T_i = 2$, this corresponds to $\beta_e \gtrsim 8\%$. For such a large value of β_e , however, our assumption of $|\gamma_K/(\alpha\epsilon)| < 1$ is not valid. Furthermore, for $\bar{\mu}^2 \gtrsim 1/2$, $\text{Re}\gamma_0 \ll \text{Im}\gamma_0$ and the requirement of $|\mu^*I| < 1$ cannot be satisfied.

6. CONCLUSION

In general, we conclude that the growth rate of the $m = 1$ MHD mode in cylindrical geometry is reduced if diamagnetic drifts and collisional viscosity are included in the analysis. The decrease due to viscosity, however, is small for present-day tokamak parameters. On the other hand, diamagnetic drifts can significantly reduce the growth rate in both the ideal MHD and resistive limits. In fact, in the ideal MHD limit, the mode can be completely stabilized if the pure MHD growth rate is less than half the ion diamagnetic drift frequency. Although the mode cannot be completely stabilized in the resistive limit, the growth rate is reduced by a factor of $\gamma_T^2 / (\omega_{*i} \omega_{*e})$.

Finally, we make some observations concerning minor disruptions or sawtooth oscillations in tokamaks. In view of the results of Refs. 4-8, the nonlinear growth of the mode in the tearing limit provides a reasonable explanation of this phenomenon. Of the various growth rates obtained in this report, the one which most likely applies to the $m = 1$ precursor oscillations is given by Eq. 14, because generally we expect values of $|\gamma_K / \alpha \epsilon| < 1$ and $\gamma_T < |\omega_{*i,e}|$. The frequency of the precursor oscillations might be explained by the Doppler shift arising from the ambipolar electric field. We observe that the diamagnetic drifts reduce the growth rate to a value that is comparable with the experimentally observed growth rate. Consequently, the growth rate given by Eq. 14 is compared in detail with experimental data in a parallel publication¹⁸ with favorable results.

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APPENDIX

Consider the pair of equations,

$$\xi'' = \frac{X}{\lambda^2} \Psi'' \quad (A1)$$

and

$$\Psi = -\xi X + \frac{1}{\lambda} \Psi' , \quad (A2)$$

which are valid inside the tearing layer. The primes denote derivatives with respect to X . We must find a solution to Eqs. A1 and A2 which matches the solution obtained outside the tearing layer. As reported in Refs. 1 and 2, the general requirements are

$$\lim_{X \rightarrow \infty} \xi = 0 , \quad (A3)$$

$$\lim_{X \rightarrow \infty} \xi = \xi_0 , \quad (A4)$$

and

$$\lim_{X \rightarrow \infty} \frac{\xi'}{\xi} = \frac{D}{X^2} , \quad (A5)$$

where ξ_0 is a constant and

$$D = - \frac{\gamma_K}{\pi \alpha \epsilon} .$$

In the derivation of Eq. A5 in Ref. 1, it was shown that D arises from terms which are of higher order in δ than the terms retained in Sect. 2 of this article. One of these higher order terms is proportional to the pressure gradient which provides a driving force for the internal kink.

In general, γ_K can also include toroidal effects. The case $\gamma_K = 0$ corresponds to the pure tearing mode. We also observe that far from the singular surface, resistivity may be ignored and Eq. A2 implies that

$$\lim_{|X| \rightarrow \infty} \psi = -\xi X.$$

Finally, inspection of Eqs. A1 and A2 and the boundary condition of Eqs. A3-A5 reveals that ξ' must be even in X .

Introducing the variable

$$\chi = X\psi' - \psi$$

and integrating Eq. A1 yields

$$\xi' = (\chi - \chi_\infty)/\lambda^2, \quad (A6)$$

where $\chi_\infty = \lim_{|X| \rightarrow \infty} \chi$. Because $\chi' = X\psi''$,

$$\psi = -\chi + X \int_{\infty}^X dX^* \frac{\chi'}{X^*}. \quad (A7)$$

Thus,

$$\lim_{X \rightarrow -\infty} \xi = -\lim_{X \rightarrow -\infty} \frac{\psi}{X} = 2 \int_0^{\infty} dX \frac{\chi'}{X}$$

and

$$\lim_{X \rightarrow -\infty} \xi' = -\lim_{X \rightarrow -\infty} \frac{d}{dX} \frac{\psi}{X} = -\frac{\chi_\infty}{X^2},$$

so that Eq. A5 becomes

$$\chi_\infty = -2D \int_0^{\infty} dX \frac{\chi'}{X}. \quad (A8)$$

Now, we introduce $Z = X^2/\sqrt{\lambda}$, solve Eq. A2 for ξ , differentiate, and use Eqs. A6 and A7 to find the equation for χ ,

$$\frac{d^2\chi}{dZ^2} - \frac{1}{2Z} \frac{d\chi}{dZ} - \frac{\lambda^{3/2}\chi}{4Z} - \frac{\chi}{4} = -\frac{\chi_\infty}{4} \quad (A9)$$

Substituting

$$\chi = \sum_{n=0}^{\infty} e^{-Z/2} L_n^{-3/2}(Z) ,$$

where L_n^α are the generalized Laguerre polynomials satisfying

$$Z \frac{d^2 L_n^\alpha}{dZ^2} + (\alpha + 1 - Z) \frac{dL_n^\alpha}{dZ} + n L_n^\alpha = 0 ,$$

yields

$$\sum_{n=0}^{\infty} a_n Z^{-1} e^{-Z/2} L_n^{-3/2} [n + (\lambda^{3/2} - 1)/4] = \chi_\infty/4 .$$

Next, we multiply by $e^{-Z/2} Z^{-1/2} L_n^{-3/2}$ and use the orthogonality relation

$$\int_0^\infty e^{-Z} Z^\alpha L_m^\alpha L_n^\alpha dZ = \delta_{mn} \frac{(n + \alpha)!}{n!}$$

to find

$$a_n \frac{(n - 3/2)!}{n!} (\lambda^{3/2} - 1 + 4n) = \chi_\infty \int_0^\infty Z^{-1/2} e^{-Z/2} L_n^{-3/2} dZ. \quad (A10)$$

Employing $L_n^\alpha = L_n^{\alpha+1} - L_{n-1}^{\alpha+1}$ and the relation

$$\int_0^\infty e^{-Yt} t^\alpha L_n^\alpha dt = \frac{(n + \alpha)!}{n!} (Y - 1)^n \underline{Y}^{-(n + \alpha + 1)}$$

to evaluate the integral in Eq. A10 gives

$$\chi = \chi_{\infty} 2^{-1/2} e^{-Z/2} \sum_{n=0}^{\infty} (-1)^n (4n - 1) L_n^{-3/2}(Z) / (\lambda^{3/2} - 1 + 4n) .$$

Next, writing

$$(\lambda^{3/2} - 1 + 4n)^{-1} = \int_0^{\infty} d\beta e^{-\beta(\lambda^{3/2} - 1 + 4n)} ,$$

using the relation

$$\sum_{n=0}^{\infty} L_n^{\alpha}(Z) x^n = (1 - x)^{-\alpha-1} \exp \left(\frac{xZ}{x-1} \right) .$$

and introducing $y \equiv e^{-4\beta}$ results in an integral representation for χ ,

$$\chi = \chi_{\infty} - \chi_{\infty} 2^{-5/2} \lambda^{3/2} \int_0^1 dy y^{(\lambda^{3/2} - 5)/4} (1 + y)^{1/2} \exp \left[-\frac{Z}{2} \left(\frac{1 - y}{1 + y} \right) \right] . \quad (A11)$$

Equation A11 may be verified by direct substitution in Eq. A9. Finally, substituting Eq. A11 into Eq. A8 yields the eigenvalue equation Eq. 12.

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81. Dr. George Vahala, Physics Dept., College of William & Mary, Williamsburg, VA 23185
82. Dr. Robert E. Price, Division of Magnetic Fusion Energy, G-234, Energy Research and Development Administration, Washington, D.C. 20545

83. Dr. R. C. Davidson, Division of Magnetic Fusion Energy, G-234, Energy Research and Development Administration, Washington, D.C. 20545
84. Dr. Oscar P. Manley, Division of Magnetic Fusion Energy, G-234, Energy Research and Development Administration, Washington, D.C. 20545
85. Mr. E. E. Kintner, Division of Magnetic Fusion Energy, G-234, Energy Research and Development Administration, Washington, D.C. 20545
86. Dr. L. D. Pearlstein, L-388, Lawrence Livermore Laboratory, P.O. Box 808, Livermore, CA 94550
87. Dr. J. P. Friedberg, Los Alamos Scientific Laboratory, Los Alamos, NM 87544
88. Dr. David J. Rose, Dept. of Nuclear Engineering, MIT, Cambridge, MA 02139
89. Dr. Gareth E. Guest, General Atomic Co., P.O. Box 81608, San Diego, CA 92138
90. Dr. Claude Mercier, Service du Theorie des Plasmas, Centre d'Etudes Nucléaires, Fontenay-aux-Roses (Seine) France
91. Dr. J. B. Taylor, Culham Laboratory, UKAEA, Abingdon, Oxon, OX14 3DB, United Kingdom
92. Dr. D. Pfirsch, Institute for Plasma Physics, 8046 Garching bei München, Federal Republic of Germany
93. Dr. V. D. Shafranov, I. V. Kurchatov Inst. of Atomic Energy, 46 Ulitsa Kurchatova, P.O. Box 3402, Moscow, U.S.S.R.
94. Dr. A. Rogister, Institute for Plasma Physics, KFA, Postfach 1913, D-5170, Jülich 1, Federal Republic of Germany
95. Dr. J. G. Cordey, Culham Laboratory, UKAEA, Abingdon, Oxon, OX14 3DB, United Kingdom
96. Dr. David Baldwin, L-388, Lawrence Livermore Lab., P.O. Box 808, Livermore, CA 94550
97. CTR Reading Room, c/o Prof. Bruno Coppi, Dept. of Physics, MIT, Cambridge, MA 02138
98. Dr. Harold P. Furth, Princeton Plasma Physics Lab., Princeton Univ., P.O. Box 451, Princeton, NJ 08540
99. Dr. Paul H. Rutherford, Princeton Plasma Physics Lab., Princeton Univ., P.O. Box 451, Princeton, NJ 08540
100. Research & Technical Support Div., Oak Ridge Operations, Energy Research and Development Administration, P. O. Box E, Oak Ridge, TN 37830
- 101-127. Tech. Information Center, P. O. Box 62, Oak Ridge, TN 37830