TABLE OF CONTENTS

ABSTRACT	ix
ACKNOWLEDGMENT	xi
I. INTRODUCTION	1
II. REVIEW OF HZ-Her X-1 DATA	2
A. The 1.7-Day Binary Period	2
B. The 35-Day Phase	6
C. The Long Term Behavior	13
D. X Rays and Matter in the Binary System	15
E. Evolutionary Considerations	15
III. EXPERIMENTAL INSTRUMENTATION, PROCEDURE AND PRELIMINARY	
DATA REDUCTION	18
A. Instrumentation -- Synopsis	18
1. Instrumentation -- Detail	19
2. Instrumentation -- Calibration	25
B. Run Procedure	26
C. Data Analysis -- Preliminary	27
1. Blocking Data	27
2. Fourier Transforming	28
3. Converting Local Significance to a Signal Level . . .	30
4. Software Calibration	32
IV. THE DATA	33
A. Introduction	33
B. Summary	34
C. Simple Power Spectra	35
1. Her X-1 Line of Sight Results	35
2. Velocities of Features	37
3. Limb Features	38
4. Simultaneous and Anomalous Peaks	39

TABLE OF CONTENTS (continued)

D. Results from All Power Spectra	40
1. The Two Classes of Power Spectra.	40
2. The Sum of All the Power Spectra	42
3. Population Histograms	43
4. Spectra vs. 1.7-Day Orbital Phase	43
5. Mean Pulsed Fraction and Statistical Screens	44
E. Results From the Strong Detections	47
1. Precise Values of Frequency and its Time Derivative	47
2. The Long Term Stability of Features	49
3. Relating $\dot{P}$ to Positional Coordinates	51
F. The 35-Day Phase	53
1. Establishing the Epochs	53
2. Features I, II, and III in the 35-Day Phase	56
3. The 1.7-Day - 35-Day Template	58
4. The Spatial Origin of Features vs. 35-Day Phase	59
V. DISCUSSION OF THE OPTICAL PULSATION DATA	62
A. Model Calculations	62
1. The Detailed Calculations	62
2. Goals for the Calculations	68
3. Results of the Calculations	71
a. The two profiles	71
i) The amplitude profile	71
ii) The frequency profile	74
b. The effects of varying some parameters	75
i) Effects of mass ratio	75
ii) Prograde vs. retrograde pulsar rotation	76
iii) The effects of contour chosen	78
iv) The effects of the frequency multiple	80
v) Effects of orbital inclination	82

TABLE OF CONTENTS (continued)

c.	Some early conclusions from the models:	84
i)	The mass ratio vs. inclination from the velocities	84
ii)	The size of coherent areas	87
iii)	The "apparent" location of the pulsations	89
d.	Systematics of illumination affecting the determination of the mass ratio	91
i)	The equatorial shadow	91
ii)	The equatorial band	92
e.	The low and high energy x ray reprocessing limits and their effect on the mass ratio determination	93
i)	The physical reprocessing of x rays into optical light	93
ii)	Calculations of the reprocessing limb- darkening law	98
f.	Tidal lobes	105
4.	Quantitative Experimental Considerations and the Model Calculations	107
a.	Reflection heating and pulsed efficiency	107
b.	The efficiency limits of pulsed emission reprocessing	109
c.	The pulsed heating reprocessing	113
5.	Summary of Results of the Analysis of Features I and II	119
B.	Feature III	122
1.	The Stream-Disk Interpretation	122
a.	The geometrical shape of the cloud	125
b.	The width of the cloud	128
c.	The height of the cloud	130
d.	The reprocessing mechanisms and the physical conditions of the cloud	131
2.	Feature III and the 35-Day Phase.	145
C.	The Origin of the Anomalous Feature	145

TABLE OF CONTENTS (continued)

D. The 35-Day Phase and Remaining Mysterious	151
1. Backward-Marching Aspects	151
2. The Her X-1 Mystery and Beam Precession	154
E. Odds and Ends	162
F. Summary	163
VI. CONCLUSIONS	166
VII. APPENDIX	170
A. Fourier Transforms	170
1. Introduction	170
2. Noise	170
a. Parseval's theorem	174
b. Local power level	176
3. Fourier Transform Morphology	177
4. The Length of the Fourier Transforms	180
5. Signals	181
a. Delta function in frequency	181
b. Off-center delta function	182
c. Non-sinusoidal periodic signals	184
6. The Convolution Theorems	185
a. The transform of the convolution	185
b. The transform of the product	185
7. The Finite, Discrete Transform Process	186
8. Statistical Methods for Power Spectra	189
a. Single bins	189
b. Multiple bin sums	191
c. Fractional bin sums	192
9. Real Signals with Noise	193
a. Statistics	193
b. Deducing the signal strength	197
c. Systematics on the deduced amplitude	198

TABLE OF CONTENTS (continued)

B. Signal Folding	201
1. Relation to Fourier Transform	201
2. Measuring the Frequency	203
a. The A-B measure	203
b. The fluctuations in the f^0 measurement	205
c. The fluctuations of the cosine terms	208
d. The A+B check	210
e. The difference of complex phases	210
f. The pulse phase and its errors	211
g. Monte Carlo calculations	211
h. The measurement of frequency and the real data	212
3. Measuring $\dot{f}$	213
a. The A-B measure	213
b. Fluctuations of the cosine terms	215
c. The complex phases	216
d. Monte Carlo calculations	217
e. The measurement of $\dot{f}$ and the real data	217
f. Systematics on the $\dot{f}$ measurement	218
g. The $\dot{f}$ measurement in general	218
4. Summary of Measurement Procedures for f and $\dot{f}$	219
5. Odd Comments	221
C. Digital Photometric Methods	222
1. Advantages	222
2. Digitization Noise	222
3. The Poisson Minimum	225
4. Rate Limits	226
5. Afterpulsing	227
6. Afterbursting	228
REFERENCES	232
TABLES	239
FIGURE CAPTIONS	246

THE MEASUREMENT OF THE MASSES OF THE NEUTRON STAR, Her X-1,
AND ITS BINARY COMPANION, HZ Her, AS DERIVED FROM THE STUDY
OF 1.24-SECOND OPTICAL PULSATIONS FROM THE HZ Her - Her X-1
BINARY SYSTEM AND THE X RAY-TO-OPTICAL REPROCESSING REFLECTION
AND TRANSMISSION MECHANISMS

John Middleditch

Ph.D. Thesis

February 1975

ABSTRACT

Intermittent optical pulsations at the 0.1 - 0.3% level have been detected from this binary system in over 500 hours of optical observations. These pulsations are present only for well defined values of the 1.7-day (binary) and 35-day phases. Positions of the pulsation-emitting regions, projected onto the orbital plane, have been measured and three distinct regions have been resolved. A simple model is put forth which accounts for the observed binary behavior, which gives a direct determination of the mass ratio, $M_{\text{HZ Her}} / M_{\text{Her X-1}} = 1.69 \pm 0.05$, and which establishes that the spin of the pulsar is prograde. Additionally it is shown that HZ Her fills its critical Roche lobe. Using the above, the known x ray eclipse duration, and the mass function, the orbital inclination is calculated to be $i = 85^\circ \pm 5^\circ$ and the masses to be $M_{\text{Her X-1}} = 1.28 \pm 0.08 M_\odot$ and $M_{\text{HZ Her}} = 2.16 \pm 0.07 M_\odot$. Constraints on the physical parameters of the accretion stream and disk are derived from the data. The nature of the 35-day modulation of the data is discussed in relation to various models.

ACKNOWLEDGMENTS

This thesis is the result of years of development and support by the Lawrence Berkeley Laboratory in the field of astrophysics. I would like to thank Jerry Nelson for his support and guidance during my graduate years as well as Bob Kenney and Carl Helmholtz for their patience and support for this project. In addition, Terry Mast and later Gary Chanan were helpful with their discussions. I owe a great deal to the enthusiasm of John Saarloos who helped us design and then built the electronic components used in this experiment. Credit for the editing, typing and assembly of this thesis goes to the people of the Technical Information Department.

I would like to thank Chris Sunzeri for conscientiously performing the bulk of the observations on HZ Her. The people of the diner and the technical assistants at Mt. Hamilton also have my thanks for their help.

A large number of people at Berkeley, Santa Cruz and elsewhere deserve my thanks for their encouragement for this project. A few of these people are (alphabetically): Kinsey Anderson, Jon Arons, Ray Chiao, Frank Crawford, Dave Cudaback, Sandy Faber, Robert Kraft, Len Kuhi, Don Lamb, Fred Lamb, David Pines, Joe Silk, Hy Spinrad, Merle Walker, and Joe Wampler. Also special thanks go to Gene Commins as my thesis advisor.

This thesis is dedicated to Ellen Marie Lamb, my wife since this past November 8th.

ACKNOWLEDGMENTS

This thesis is the result of years of development and support by the Lawrence Berkeley Laboratory in the field of astrophysics. I would like to thank Jerry Nelson for his support and guidance during my graduate years as well as Bob Kenney and Carl Heilmholz for their patience and support for this project. In addition, Terry Mast and later Gary Chanan were helpful with their discussions. I owe a great deal to the enthusiasm of John Saarloos who helped us design and then built the electronic components used in this experiment. Credit for the editing, typing and assembly of this thesis goes to the people of the Technical Information Department.

I would like to thank Chris Sunzeri for conscientiously performing the bulk of the observations on HZ Her. The people of the diner and the technical assistants at Mt. Hamilton also have my thanks for their help.

A large number of people at Berkeley, Santa Cruz and elsewhere deserve my thanks for their encouragement for this project. A few of these people are (alphabetically): Kinsey Anderson, Jon Arons, Ray Chiao, Frank Crawford, Dave Cudaback, Sandy Faber, Robert Kraft, Len Kuhl, Don Lamb, Fred Lamb, David Pines, Joe Silk, Hy Spinrad, Merle Walker, and Joe Wampler. Also special thanks go to Gene Commins as my thesis advisor.

This thesis is dedicated to Ellen Marie Lamb, my wife since this past November 8th.

I. INTRODUCTION

The discovery of the two pulsing binary x ray sources, Centaurus X-3 and Hercules X-1 by the Uhuru satellite^{1,2} has provided astrophysicists with an unprecedented opportunity to study both the dynamics of close binary systems and the material inflow onto the degenerate members of these systems. The optical identification of the Her X-1 source by several groups³⁻⁵ with the variable star HZ Hercules⁶, which followed the x ray discovery by less than a year, has allowed a study of the binary system in the x ray and optical regions which has been so extensive that observational simultaneity between observers of the same or separate wavelengths has occurred frequently.

The intention of this thesis is to show that the study of one particular aspect of the HZ Her-Her X-1 system, the optical pulsations, has increased the understanding of the other data from this x ray-optical source and is essential to the interpretation of this phenomenon. This study has also provided the motivation to develop some new methods of data analysis. These methods may ultimately prove useful to the study of other discrete x ray sources which have become associated with white dwarfs, neutron stars and black holes. They may also have immediate applications to the periodicities associated with non-radial oscillations in white dwarfs seen by several astronomers.⁷

A review of the present understanding of the Her X-1 -- HZ Her system is presented first.

II. REVIEW OF HZ-HER X-1 DATA

A. The 1.7-Day Binary Period

The pulsed x ray flux from Her X-1 was observed by Tananbaum et al.² to have a frequency near 0.8078 Hz which was modulated by a small sinusoidal fraction. The period of this frequency modulation was found to be 1.70017 days. In addition, the entire x ray flux was observed to disappear for 0.24 ± 0.06 days at a time also with the same 1.7 day repetition interval. Finally, the negative going transit of the pulsation frequency across the average frequency was observed to lie midway between the eclipses of the x ray flux. The inescapable interpretation of these data was that the x ray source, Her X-1, was in a binary orbit with a non-degenerate companion star which occulted the x rays every 1.7 days when the line of sight to the earth passed through it.

From the doppler effect on the pulse frequency a maximum projected velocity, $v \sin(i)$, was measured to be 169.2 ± 0.4 km/sec. The inclination, i , of the orbital plane is defined as the angle between the normal to the plane and the line of sight to earth. Defining r_x to be the separation of the x ray source from the binary center of mass and P as the 1.7 day period (in seconds), the equation of circular motion gives $r_x \sin(i) \approx v \sin(i) \cdot P / 2\pi = 3.95 \pm 0.01 \cdot 10^{11}$ cm for the projected separation. By equating the centripetal accelerations of both components of the binary system to their mutual gravitational attraction one finds

$$4\pi^2 m_x r_x P^{-2} = G m_x m_c a^{-2} = 4\pi^2 m_c r_c P^{-2}$$

where m_x and m_c are the masses of the x ray source and companion, r_c is the separation of the companion from the common center of mass, $a = r_x + r_c$ and G is the universal gravitational constant.

From the above equations the relations for the mass ratio R are derived: $m_c/m_x = r_x/r_c = R$. Solving for the mass of the companion gives $m_c = 4\pi^2 a^3 r_x / P^2$. Using R and substituting from previous equations for r_x and a gives

$$m_c = [v \sin(i)]^3 \cdot P \sin^3 i \cdot (2\pi G)^{-1} (1 + 1/R)^2$$

The quantity $m_c \sin^3 i / (1 + 1/R)^2 = (v \sin(i))^3 P / 2\pi G$ is known as the mass function of a binary system and is the only constraint on the masses of the pair which is known from just the period and velocity information of one member. For the Her X-1 system, $m_c = 0.85 M_\odot (1 + 1/R)^2 / \sin^3 i$.

The absolute determination of the masses in the system is of theoretical interest because Her X-1 is probably a neutron star (due to the highly stable and relatively fast pulsation frequency) and because the masses of the system are important in the evolutionary history of such objects. The determination of both R and $\sin(i)$ necessitates the optical study of the system.

The optical identification of Her X-1 with HZ Her was facilitated by the revision of the Uhuru error box coordinates circa the end of May 1972* and by the extreme variation of the optical flux coming from HZ Hercules.†

*, †: See next page.

* The poorness of the AS & E positional resolution was hardly their fault. Rumors have it that NASA had arranged, over the great protest of AS & E, that Uhuru make a pass over the sun in order that some solar astrophysicist could briefly utilize the satellite. Consequently, as AS & E had predicted, the stellar aspect sensors were burned out and the only remaining method to determine positions was by combining the timing of the passes with the angular resolution of the detectors; $0.5^{\circ} \times 5^{\circ}$.

† Liller first suggested the candidate in IAU Circ. No. 2415 circa June 1972. Lamb and Sorvari⁸ announced HZ Her as a possible optical pulsar in IAU Circ. No. 2422. The strength of their reported pulsations (roughly 5-10% of the optical flux) has never been reproduced since then to within an order of magnitude. Without their report however, it is possible that Davidsen, Henry, Middleditch and Smith³ may never have pushed their resolution limits as far as they did. These pulsations were not confirmed by other observers until April of 1973 (see IAU Circ. No. 2523, Groth et al.).⁹

The Bahcalls noted the extreme optical variation of HZ Her in IAU Circ. No. 2427 of July 28, 1972 and Liller confirmed the Bahcalls' results in the same Circular.

Bopp, Grupsmitth and van den Bout were the first to report an optical spectrum of HZ Her in IAU Circ. No. 2424 of July 25, 1972. A feature in this spectrum had a radial velocity of 250 ± 40 km/sec, which they cited as corroboratory evidence to the report of Lamb and Sorvari that HZ Her was identified with Her X-1. This high radial velocity feature has also not, to the extent of my knowledge, been seen by other observers.

Many other reports on the features and activity of HZ Her followed in the IAU Circulars after the initial reports in Nos. 2422, 2424, and 2427.

The 1.7 day light curve of Davidsen et al.³ is shown in Figure 1 for the B band and U-B color of HZ Her with the behavior of the x ray flux shown below. Several other groups have published similar curves.^{4,5,10-12} The variation in B is about 1.8 magnitudes while the U-B varies by 1.0 magnitudes, with the highest B flux and the most ultraviolet colors coinciding with the orbital phase $0.5 = 180^\circ$ midway between the x ray eclipses when the x ray source is between its non-degenerate companion and the earth. This light curve is very unconventional by normal astronomical standards due to the absence of much of a secondary eclipse and due to the sharpness of the minimum.

A "reflection" effect of x rays heating the near face of the companion star is generally accepted as the primary cause of the unusual light curve.^{4,5} Wilson¹³ has pointed out that the phase law for the reflection effect does not agree with the observations and has suggested instead that a significant amount of the envelope of HZ Her was involved in the convective transfer of energy. However, Joss et al.¹⁴ and Pringle¹⁵ have invoked an extended upper atmosphere to explain the sharpness of the optical minimum. Others have maintained that it could be due to the gradual occultation of a bright disk surrounding Her X-1 (Crampton and Hutchings,¹⁶ Boynton et al.¹² and Strittmatter et al.¹⁷).

The detection of optical pulsations from HZ Her^{3,18,19} (the unmodified proper noun is used here to signify the non-degenerate companion to the x ray source, Her X-1) places limits on the optical thickness of any such extended atmosphere to the x rays. This is because the pulsations have been seen to originate closer to the center of mass of the companion than the inner Lagrangian point for any mass ratio $m_{\text{HZ}}/m_x \geq 1.3$.

Since all values for this ratio which have been derived from various data lie between 1.5 and 2.5, one must conclude that the pulsed x ray flux is penetrating to approximately where the photosphere would normally be located. Considerations of how often the pulsations are seen at various levels, which will be discussed later in the Review, will modify this objection to the upper atmospheres.

B. The 35-Day Phase

Tananbaum et al.² and Giacconi et al.²⁰ have shown that the x ray flux intensity also varies systematically over a 35 day interval. The x ray behavior is shown in Figure 2 taken from the paper of Giacconi et al.²⁰ This can be summarized by an 11 day active period where x ray flux is present (save for the periodic binary eclipses) followed by a 24 day interval where the x ray flux is characteristically undetected because of an intensity reduction by at least a factor of 20. Although Substantial hard x ray flux (4-12 kev) has been seen once in the middle of the 24 day "off" interval by the Copernicus satellite²¹, the lack of hard x rays is still the dominant behavior of this period.²⁰

The "turn-ons" which start the active cycle occur preferentially at the binary phases of 0.23 and 0.68 (where the phases 0.0 and 1.0 represent successive eclipses) are known to be abrupt and show significant absorption.²⁰ The x ray flux increases slightly for the first few binary cycles, rising to a smooth maximum in the envelope, and then begins to turn over, dropping down to zero usually by the end of the 7th binary cycle. There are also dips in the envelope other than the binary eclipses which do not drop all the way to zero flux but typically only to 0.5 to 0.1 of the

surrounding value. These dips first appear just prior to the regular eclipse and then appear progressively earlier in binary phase through the succeeding cycles until the end of the x ray active interval, at which time the backward march in phase has reached 0.65. An extra dip at phase 0.65 is seen just following the turn-ons which occur at the 0.23 binary phase.

The shape of the x ray flux envelope and the absorption at turn-on are both suggestive that activity is present for a few days before the turn-on^{*} and that the onset of x ray flux as seen from the direction to earth is caused by a "burn-through" effect. It will be seen later that optical pulsations have been systematically detected during this period preferentially at the binary phase interval of 0.17-0.30, although one observation of April 21, 1974, shows similar pulsation at the binary phase interval of 0.74-0.86.

The 1.7 day optical light curve has been shown by Kurochkin,¹¹ Chevalier et al.²³ and Boynton et al.¹² to vary systematically with the 35-day x ray behavior. The variations in the blue over parts of the 35-day phase can be seen in Figure 3 which has been reproduced from the paper of Boynton et al.¹² The Fig. 3 binary cycle number of the 35-day phase is indicated to the upper right of the figures; cycle number zero is fixed at the x ray turn-on. The results of Kurochkin¹¹ are similar but show that a sharp peak to the left of 0.5 in phase can occur at the location of the gap in

* Although the results of Sharma et al.²² from a balloon flight of UT 1973 for the energy band of 19-143 kev were negative, the observation was actually more than 7 days before the turn-on at Jan. 26.6 contrary to the authors' estimation of 5 days. Thus the question of hard x ray activity just before the turn-on remains as such.

Boynton's data for his cycle labelled '4' or just one cycle later. All three sets of published data^{11,12,23} have gaps for the phase that Boynton names as '7' around the binary phase 0.3-0.4 so it is not known how the sharp feature or flare appearing at approximately 5.3 in Kurochkin's data evolves until about 9, at which time the point appears at binary phase 0.55 in Kurochkin's own data. Boynton et al.¹² and others have shown that such sharp maxima are due to a very hot-colored component of the light.

It has been noted by almost everyone that the variation in the light curve is *more extreme* during the 24 day "quiescent" x ray interval. A closer examination of the light curves indicates that the variation begins to be quiescent at the time that the Uhuru 11-day envelope²⁰ would rise up from zero if it were extrapolated to a few cycles earlier than the turn-on. This is in good agreement with the "burn-through" interpretation of the x ray turn-ons so long as the actual onset of the hard x ray flux reaching the earth is constrained to have negligible effect on the optical light curve.

The average of the 35-day interval has been established by Giacconi et al.²⁰ at 34.88 ± 0.12 days although there seems to be a jitter approaching 1 binary cycle in the individual turn-on times. This jitter could be explained by uncertainties in the "burn-through" time, but the discrete binary phases of the turn-ons still require yet another mechanism for their explanation.

If the 35-day clocking mechanism is mechanical in nature such as free precession of the degenerate x ray source or precession of the non-degenerate companion star, then perhaps the 35-day and 1.7-day

periodicities are locked at exactly 20.5 binary cycles to a 34.88 day cycle. The revision of the period by 0.03 to 34.85 days is well within the errors. Such locking is not uncommon, for example, the planet Mercury has a rotation period which is exactly 2/3rds of its period of revolution about the sun.

None of the three optical light curve models mentioned previously -- the convective envelope, the extended upper atmosphere, or the bright disk models -- attempt to explain the 35-day clocking mechanism in much detail or to link that mechanism rigorously with the 35-day variations in the optical light and the features of the x ray behavior. A large number of mechanisms have been proposed to explain one or both of these sets of features; these are briefly mentioned below.

Pines et al.²⁴ have suggested a magnetic gate tied to the free precession of the neutron star (Her X-1). Arons and Bahcall²⁵ have referred to an ionization instability for the outer envelope which would almost periodically push matter through the inner Lagrangian point. Arons²⁶ has later described a self-excited flow mechanism which could be gated on and off if HZ Her had a magnetic field of about 200 gauss and was not quite corotating with the binary orbital motion; a steady soft x ray source is invoked from an accretion disk around Her X-1 to explain the persistence of the 1.7 day light curve. McClintok et al.²⁷ have used the energy requirement of the 1.7 day optical variation and knowledge of the hard x ray flux to set a minimum radius of any spherical soft x ray source at $5 \cdot 10^8$ cm with a $kT \leq 90$ eV. Pringle¹⁵ cites the results of Davidson and Ostriker,²⁸ Pringle and Rees²⁹ and Lamb et al.³⁰ which support the concept of material accumulation at an Alfvén surface where

the magnetic stress is equal to the material infall pressure. The Alfvén surface thus acts as a material reservoir, with accretion beginning when the weight of the reservoir overcomes the magnetic and centrifugal forces. He then mentions the ionization instability and the magnetic gating of a stellar wind as another possible explanation for the 35-day x ray behavior. Brecher³¹ has argued for a free precession of Her X-1 and gating the accretion with the direction of its magnetic axis relative to the plane of an accreting disk. And finally McCray³² has suggested that a phase-lock between the periastron of a 1% elliptic orbit and a timed delay of accretion from a more or less stable accretion disk of 24 days. The onset of accretion generates enough radiation pressure to terminate the spilling through the inner Lagrangian point; the exhaustion of the disk material after spilling turns the radiation pressure off and allows a big spill at the subsequent periastron. The clocking will be precise only as long as the delay of the onset of accretion from the disk remains within 0.85 days of 24.

The major failing of all the above models is the inability to explain why the variations in the optical binary light curve persist throughout the 24 day x ray off interval. Avni et al.³³ have suggested the existence of a steady soft x ray source as a possible explanation. Basko and Sunyaev³⁴ had earlier suggested that the hard x ray flux was always present over the entire 35-day interval, the variation being caused by either precession of beam direction or periodic narrowing of the beam by variations in an accretion shock front (due to variations in accretion rate) as cited from Komberg.³⁵ A more complex model of a precessing disk "lip" which periodically obscures the x rays from any nearly coplanar

direction has been advanced by Katz.³⁶ Unfortunately, it fails to explain the lack of absorption at the 35-day turn-off and it predicts a steady decrease of the 35-day period which has not been observed. Roberts³⁷ has advanced a model in which the precession of the non-degenerate companion star slaves the orientation of an accretion disk so that all directions lying close to the orbital plane are periodically obscured. This model suffers only from the latter objection to the previous model.

A satisfactory model has yet to be formulated to explain the characteristics of the 35-day x ray and optical variations. Observations of the soft x ray band during the 24 day low x ray state are essential to further understanding of this source,³⁸ although a 90 ev source as suggested by McClintok et al.²⁷ would never be seen directly due to interstellar absorption.

Other mechanisms worth mentioning devised to account for the persistence of the reflection effect on HZ Her have been advanced separate from speculation of the 35-day clock mechanism. Rosenbluth et al.³⁹ have cited sub surface fusion of accreted hydrogen as a steady source of low energy x rays from the neutron star. Unfortunately, the hard x ray flux produced from the 10% efficient accretion dictates that the flux produced from the 1% efficient nuclear burning is lower by a factor of ten. Nuclear burning also fails to explain the optical pulsations seen from HZ Her during the 24 day x ray off. Brecher and Morrison⁴⁰ have suggested that the heating of the surface is due to large fluxes of proton cosmic rays accelerated by Her X-1's magnetic field lines (corotating with the 1.24 second period). Then the heating can be transferred around the star by a moderate magnetic field whose strength at HZ Her's surface

need only be 10-100 gauss. A large gamma ray flux, steady over the 35-day period, is predicted from this model. The major objections to this model are the difficulties in efficiently accelerating large masses of protons (roughly 10^{17} grams/sec as derived from the x ray energy table with a 10 km radius neutron star) and in keeping the pulse period from increasing due to the braking torques of the accelerating mechanism. Earlier models of Brecher and Morrison⁴¹ and Lamb and Van Horn⁴² have argued that the Her X-1 source would be a degenerate white dwarf and therefore would be more likely to produce cosmic rays. As mentioned before, the observed stability of the x ray pulsation frequency argues against the white dwarf hypothesis.

A spherically symmetric, optically thin accretion model of Buff and McCray⁴³ has shown that the "sonic" surface of the infalling plasma is critically dependent on the luminosity and spectrum of the x ray source. They allude to this as a possible switching mechanism which could account for the modulation of accretion. Furthermore, they find that an accretion disk shadow may be essential to such thin accretion; Alme (private communication) has also found that thin accretion is impossible for an unshielded x-ray source. Evidence for such a shadow from the high x ray cutoff energies of eclipsing binary systems has been previously given by Jones et al.⁴⁴ However, evidence of optical pulsations from the near limb of HZ Her described below sets a definite angular limit to the extent of this shadow in the direction normal to the orbital plane. Arguments will be given in the section of this thesis discussing the implications of the optical pulsations that will indicate such a shadow must often be missing.

C. The Long Term Behavior

The 1.7 day optical variation of HZ Herculis has been shown by Jones et al.⁴⁵ to spend periods of years at a reduced level characterized by a double-humped 1.7 day light curve with a double minimum at 15.05 ± 0.1 magnitudes. This is the same level as the deep single minimum of the "active" 1.7 day light curve. Figure 4 shows the data given in Jones et al.⁴⁵ The double maximum only rises to 14.75 ± 0.1 magnitudes, almost 1.4 magnitudes fainter than the active state maximum. Similar results have been reported by Wenzel and Hessner,⁴⁶ whose peak to trough amplitude of 0.3 magnitudes agrees well with the 0.28 ± 0.06 quoted in Jones et al.⁴⁵

Whelan⁴⁷ has fitted this light curve with a model which incorporates a gravity darkening law due to von Zeipel⁴⁸ and several assumptions. The major assumptions are that the Roche lobe is filled and that the eclipse times from the x ray data during the active years can be used to fit the data for the quiescent years. If the star were to stay in a quiescent state, it would be unlikely that the photosphere could be very near the Roche lobe if some of the previously quoted instabilities were in effect. Likewise the extrapolation of the x ray eclipse duration would be systematically high. On the other hand, the indication from Figure 4 that the eclipse near binary phase 0.5 is deeper than the eclipse of 0.0 supports the Roche interpretation of the quiescent light curve (the gravity at the inner Lagrangian point is zero). Also if the Roche lobe were significantly underfilled during the years of optical quiescence then the explanation of the magnitude variations would require an even more unlikely set of assumptions. Since Alme and Wilson⁵¹ have found that a star sitting behind the Roche lobe cannot possibly have a self-excited stellar wind,

then it is reasonable to allow the star to come close with no x ray excitation of its surface and require it not to accrete material onto the dormant x ray source.

The mass ratio that Whelan derives from these calculations is 1.8, with an orbital inclination of 85° . This is in good agreement with the spectroscopic results of Crampton and Hutchings¹⁵ and Crampton⁴⁹ who derive a value of 1.7 ± 0.3 and with the 1.69 ± 0.05 value from the results and calculations to be presented later in this Thesis.

The evolution of HZ Her has been suggested by Lawrence Anderson (private communication) as the possible explanation for the rekindling of the x ray behavior (along with the "active" 1.7 day light curve) after the extended optical quiescence. His calculations⁵⁰ show that HZ Her must be expanding at a rate which would otherwise double its radius in 10^3 years (were it not for the action of the potential well of Her X-1) in order to provide enough material for accretion. Thus no optical quiescent state can be expected to last longer than the time taken by HZ Her to expand to its Roche lobe. For a lobe which is only underfilled by roughly 1% this time is only 10 years.

The onsets of the quiescent states are more difficult to explain, but may involve the temporary exhaustion of the outer atmosphere.

It is possible that the photoelectric study of HZ Herculis during some future quiescent interval when the x ray heating is absent will provide a very accurate determination of the mass ratio and inclination of the orbit. X ray activity and optical pulsations are both presumably absent during the years of quiescence.

D. X rays and Matter in the Binary System

Basko and Sunyaev³⁴ and Arons²⁶ have discussed the effect of the x ray flux in the atmosphere of HZ Her and its ability to drive a stellar wind. Numerical calculations of Alme and Wilson⁵¹ indicate that a self-excited accretion powered mechanism would not be sustained by the induced mass loss rate unless the photosphere were close to the Roche limit. They point out that the flow at the inner Lagrangian point may well be shielded from the x rays by intervening matter.

The question of shielding is important because it affects the interpretation of the optical pulsation data and the 35-day periodicity. The model of McCray³² and those of other authors previously cited depend on radiation pressure to regulate the flow of material through the inner Lagrangian point. Significant intervening material could easily shield this position from the radiation pressure just as it could shield the photosphere of the star from the x ray pulsations. In this case the actual level of optical pulsations would be a good indicator of such material. This possibility is elaborated in Section V of this thesis.

E. Evolutionary Considerations

The possible evolution of the binary system with the Centaurus X-3 source as one member⁵² has been sketched by van den Heuvel and Heise.⁵³ The sequence of evolution depends critically on the amount of mass in the binary system. From the mass function quoted by Schreier et al.⁵² of $15M_{\odot}$, the Centaurus system is certainly many times more massive than the Hercules system. Therefore the conclusions about the Centaurus evolution do not apply to HZ Her.

One consideration which must be taken into account is that a white dwarf can be pushed beyond its Chandrasekhar limit by the accretion of material and thus form a neutron star without a supernova explosion.⁵⁴ The amount of time spent after the formation of a neutron star can be estimated from the secular speedup rate of Giacconi et al.²⁰ assuming that the speedup has persisted in a quasi regular fashion. This interval can be no more than 10^6 years. Thus it is likely that most of the evolution took place before the neutron star formation.

The results of Anderson⁵⁰ indicate that HZ Her must have already evolved through the main sequence so that it can now be in an expanding stage. Otherwise the 10 year quiescence would last forever. If we then assume that the mass of HZ Her is $2.0 M_{\odot}$ and that the evolution proceeds as $M^{-2.5}$, the estimated main sequence time for the star is given by

$$T_{ms} = 2^{-2.5} \cdot 10^{10} \approx 1.8 \cdot 10^9 \text{ yr}.$$

Since the accretion rate derived from the estimates of the x ray flux and distance is roughly $10^{-9} M_{\odot}/\text{yr}$, another 10^9 years would have to pass before the average white dwarf could be accreted up to $1.4 M_{\odot}$.

If the neutron star was formed by a supernova explosion, it is likely that the parent of the remnant was a more massive star which evolved more rapidly than its companion (HZ Her) and accreted material onto HZ Her prior to the explosion. In this case the evolutionary time quoted above would be uncertain first because the accretion of material onto HZ Her would affect its evolution and second because an undetermined amount of material would be blown off in the explosion.

The radial velocity of HZ Her was measured by Crampton⁴⁹ to be 60 km/sec toward the sun and 40 km/sec toward the local star group which is orbiting

about the center of the Galaxy. Luyten⁴⁹ has measured the proper motion of HZ Her to be 0.043 ± 0.02 arc seconds per year at 160° . At the distance of 3 kpc of Crampton⁴⁹ and Arons²⁶ this velocity is 600 ± 300 km/sec mostly tangent to the line of sight.

With this velocity, the binary system has gone not farther than 1 kpc in the 10^6 years or less since the formation of the neutron star. However, if the formation of the neutron star did not affect this velocity, then the HZ Her binary system would have come almost a megaparsec since its formation. Thus, if the proper motion measurements are not revised and if the neutron star formation did not appreciably disturb the motion, then HZ Her could become known as a visitor from outside our own galaxy.

III. EXPERIMENTAL INSTRUMENTATION, PROCEDURE AND PRELIMINARY DATA REDUCTION

This section will first discuss the instrumentation used for the HZ Hercules experiment. The on-line calibration of these electronics is discussed briefly. Procedures used in the data gathering and their rationale are outlined. The preliminary data analysis used to generate the power spectra of the data is discussed with the help of the results of the Appendices. Finally, the calibration of this analysis is discussed.

A. Instrumentation -- Synopsis

The instrumentation is shown in block diagram in Figure 5. In Section A the light is converted to photopulses by the photomultiplier. The pulses are amplified, selected and height standardized by the amplifier-discriminator. Finally, the pulses are converted to a +4 volt level by the "dead time generator".

Section B includes all of the scalers used in the monitoring of the pulse count rate at various locations of the electronics. This monitoring is accomplished with a General Radio 1120 frequency counter and two TSI 1501 dual scalers.

The pulse rate coming from the dead time generator is scaled by the digital divider of Section C. This divider emits one output pulse for every M input pulses, where M is adjustable. The output pulses enter a digital subtractor which counts a certain number, N (also adjustable), of input pulses before opening its output gate to succeeding input pulses. The counting of this subtractor is reset by the timing pulses from the

clock of Section F.

The output of the subtractor leads to the tape interface of Section D. This interface accumulates the pulses in one six-bit register for the duration of one clock pulse interval (time window) while another six-bit register with the previous time window's accumulants are written as a six-bit byte (seven counting parity) on magnetic tape by the incremental recorder of Section E.

The parameters M and N of the divider and subtractor of Section C are adjusted so that the pulse sums for the time window intervals are unambiguously represented in the 0-63 (six-bit) count range. The clock of Section F supplies all of the timing requirements of the TSI scalers of Section B, the subtractor of Section C, the tape interface of Section D and through this interface the requirement of the incremental recorder of Section F.

1. Instrumentation -- Detail

Light from the object is gathered by the 61 cm f/18 cassegrain reflector of Lick Observatory and passes through a small diaphragm in the focal plane, a fused silica Fabry lens and into the face of the phototube.

Photoelectrons from photons detected by the corrugated S20 $0.75" \times 0.75"$ photocathode are amplified in the 12 stages of the EMI 9658R photomultiplier tube in a transit time of 55ns with a typical pulse width of 10-15ns. Figure 6 shows a plot of the typical spectral response of similar phototubes; the insert shows that the extended red response of the corrugated S20 of the EMI 9658 tubes falls below the response of the extended S20 cathodes for wavelengths longer than 7250 Å, where the quantum efficiency of both is about 4%.

To reduce the dark current of the EMI tube (which is 30,000 to 100,000 counts s^{-1} at room temperature), the aluminum cold box in which the tube is mounted has a jacket which is filled with solid CO_2 . The operating temperature of the tube is thus reduced to below $-60^{\circ}C$ and its dark current is reduced to 10-25 counts s^{-1} .

The pulses from the phototube are led from the cold box to the Lick Observatory Electronics Lab. amplifier-discriminator via tubing cable. The photopulses are first amplified by a large factor and then selected according to voltage level by the discriminator. The output pulses from the discriminator have a negative 1.5 volt level and a width of 50ns.

The discriminator threshold was set by lowering the level until it experienced continuous oscillation (at roughly 10 MHz) and then raising the level back by some minimal safe margin. The helipot readings for the point of continuous oscillation and for the safe running level were 6.50 and 5.60 respectively.

The pulses are led from the LOEL dicr-amp, which rides piggyback on the EMI cold box, down to the lower level of the 61-cm dome via 50 ohm cable to another discriminator labeled "dead time generator".

The dead time after the pulses is adjusted so that no pulse is accepted within .01-.02 pulse intervals of the previously accepted pulse, e.g., 10 to 20 microseconds for a pulse rate of $1000s^{-1}$ (see appendix C on digital photometric methods). The output of the dead time box goes through 1000 ohms and about 50 feet of 50 ohm cable to a General Radio 1120 frequency counter located upstairs where the observer uses it to assist in centering, guiding and comparison star measurements. The GR counter was operated in the frequency mode with storage so that each previous second's count was displayed

during the current second. Since the drifting of the reactive input to the GR counter makes it extremely sensitive to cable ringing, the input signal was terminated with 125 ohms and attenuated X10 so that threshold drift due to temperature fluctuations or observer mishandling would either not change the count at all or cause it to drop suddenly to zero. To ensure the accuracy of the comparison star measurements, the internal clock of the GR counter was checked in 1972 against a Hewlett Packard Rubidium Vapor Frequency Standard and found to be within 2 parts in 10^6 of UTC.

The output signal from the dead time generator was also checked by one channel of a TSI 1501 dual scaler. The conversion of the +4 volt output signal to a negative level compatible with the input was accomplished with a 500 ohm series resistor and an inverter closest to the TSI input.

The TSI input threshold had to be reduced to 100mv so that the smaller inverted pulses could be detected. The scaler unit was gated for one second of the external quartz clock time while the pulses from the dead time generator were counted. A subsequent ungated interval of 0.8 quartz clock seconds enabled the experimenter to read the accumulated counts before it was reset by a pulse from a separate quartz clock spigot prior to the beginning of the next accumulated interval at 2.000... seconds. This scaler was used to check the raw count rate per second of the GR counter upstairs and for the convenience of the experimenter since any passing clouds or guiding lapses would show up immediately as a non-statistical change of this count rate.

The third branch of the dead time output signal went to the input of the digital divider box. This divider could be set to divide by powers of 2 from 1 to 4096 and powers of 3 from 1 to 3. This divider was set to the lowest number which allowed a $\pm 4-5\sigma$ fluctuations of the counts in the 40ms time windows. No reset was incorporated into this divider nor was any dependence upon the external quartz clock.

The pulse count was also divided by 250 (or 1000) and was displayed on another TSI 1501 dual scaler unit with the previously described level conversion, gate and reset. This enabled the observer to estimate raw counts per 4ms or 1ms time window for other more rapid runs and was helpful for the 40ms time window runs on HZ Her since the observer could mentally shift a decimal point.

The output of the divider went to the input of a digital subtractor. The subtractor was reset by the 25Hz timing pulses from the quartz clock after which a set number of input pulses would have to be received before

a gate opened the output to following input pulses. The subtractor was set with a vertical column of "baseball bat" toggle switches whose representative values ascended in powers of 2 from 1 until 1048576 at the bottom end of the column. The unit always subtracted the individual sums of the toggle switches, plus one. A subtraction constant of 0 could be achieved by teeing through the input with the output cable.

The pulses from the output of the subtractor are routed into the tape interface and to another peripheral divider unit. This divider can be set to any number between 1 and 99; it is nominally set to 25 when the 40ms time windows are used. The output of this divider is routed into a third unit of a TSI 1501 scaler which is reset and gated exactly as the previous two described. The scaler thus reads the counts per 40ms time window that are passed into the tape interface. When the divider and subtractor have been set correctly, this scaler should be indicating 31 or 32-- the center of the 0-63 range of the six bit registers of the tape interface-- or less for the count rates which are not high enough to necessitate division or subtraction.

At any instant of time, one of the two six bit registers of the tape interface is accumulating while the other is being used to write a seven-track bit on magnetic tape by a Kennedy 1400 or 1600 incremental recorder, the seventh track being parity. The registers interchange roles synchronously with the 25Hz clock signal from the quartz oscillator. A START-STOP switch which initiates or terminates the writing of data is thrown by the experimenter at the appropriate times. A flip-flop circuit is located behind this mechanical switch to eliminate the possibility of "bounce". The starting and stopping of the runs can also be controlled by an external

+4 volt gate signal.

At the start of the data writing, a gate signal from the tape interface unit allows the fourth TSI 1501 unit to count quartz clock pulses which have been driven from within the tape interface box. This scaler functions as a word counter for the run and is manually reset by the experimenter before the start of the next run.

The tape interface box also contains a test routine and an interval divider which can be 2, 4 or 8 when used in the test mode (1, 2 or 4 otherwise). This routine writes groups of 2, 4 or 8 numbers sequentially from 0-63 so that an easily recognized pattern can be seen on the magnetic tape when it is viewed with a tape viewer. This is done to ensure that the recorder is writing on all channels.

Seven lights located on the interface box indicate the activity of the six numerical bits and the parity bit. Each light flashes for the duration of the time window when a bit is written on the corresponding track. A distribution which is well centered between 31 and 32 will tend to light all the lights uniformly in the time average. A hopelessly wide distribution which wraps around the 0-63 range several times will also light the bulbs uniformly. Off-center but non-overlapped distributions are indicated by the excessive brightness or dimness of the 32-bit light for the above-center or below-center cases respectively. These lights are also the only check on the internal divider of the tape interface since the third TSI scaler unit only looks at the subtractor-output to interface-input part of the circuit.

The quartz clock (Teledyne) had a short term stability of $1 \cdot 10^{-8}$. This clock was set on its date of installation circa 1971 at $1 \cdot 10^{-6}$ parts

different from the UTC rate so that the aging process would correct this offset. The clock was calibrated in 1972 to be within 1 part in 10^6 of UTC and again in 1974 when it was found to be within 2 parts in 10^7 of UTC. For the start of the HZ Herculis program in the Summer of 1972, the quartz clock was slaved to a Hewlett Packard Rubidium Vapor Frequency Standard. Since the RVFS drifted less than 50 microseconds in several months, it is therefore unlikely that the quartz clock was off by more than $1 \cdot 10^{-6}$ from the UTC rate at any time during the HZ Her program from 1972 to 1974.

The clock unit also provided the gate and reset levels to 3 of the 4 units of the two TSI 1501 dual scalers, although the reset pulses had to be boosted up to higher positive voltage by the logic level converter box in order that the units accept them.

During the 1972 observing season, the above described electronics were used in parallel with the electronics described in Nelson et al.⁵⁵ which were used to fold the pulse signal synchronously with the apparent frequency of the Her X-1 source. After September of 1972, the signal averaging electronics were returned to the Lawrence Berkeley Laboratory since they were non-essential to the operation of the experiment.

2. Instrumentation -- Calibration

The above system was calibrated on-line using the Crab Pulsar in 1971 (April 22 and October 13) and in 1972 (February 28 and August 11) with run times of $2^{18} \cdot 1$ ms, $2^{20} \cdot 4$ ms, $2^{21} \cdot 1$ ms, and $2^{16} \cdot 10$ ms respectively. Simultaneously, the photopulses were synchronously signal averaged in a 1024 channel multiscaler with channel advances every 25 microseconds (and start sweeps every 33.1 ... ms). The light curves obtained

from folding the continuously recorded data at the apparent frequency of the Crab Pulsar were consistent with those obtained on the multiscaler in both height, shape, and phase.

B. Run Procedure

A comparison star (indicated in Figure 8) was measured using the GR1120 frequency counter with sky subtracted. Since the runs which lasted from 3 to 6 hours consisted almost entirely of one stretch continuously recorded data for each night, the comparison star was measured both before and after each run. To compensate for the non-linearity of the cosecant of the zenith angle of HZ Her as a function of time, a third point was estimated to be used in a quadratic interpolation.

During the run, any slow drift in the count rate which could threaten the centering of the data in the six bits was removed by the experimenter adjusting the digital subtractor at the maximum rate of one unit of change every several minutes. In this way no significant spurious power was introduced to the frequencies higher than about 0.1 Hz.

The guiding was on an offset guide star several arc minutes to the southwest of HZ Her (Figure 8). With the use of the GR 1120 counter the observer adjusted the verniers for the location of the eyepiece reticle until the center of the diaphragm used corresponded to the known position of the guide star with respect to the 19.5 arc second diameter circular eyepiece reticle. The centering of HZ Her in the 11 or 17 arc second diaphragm could then be checked by viewing the guide star at any time during the night. The eyepiece was covered when the guide star was not being checked to ensure that light from the dome did not enter the

optical system.

Lapses in the guiding where the edge of the diaphragm begins to cut into the image of HZ Her are easily detected by a change in the count rate upstairs on the GK counter or the TSI 1501 scaler. In the data analysis these lapses appear as a drop in the instantaneous count rate, and more significantly as an increase in the instantaneous variance at the boundaries of the guiding lapse. Factors of 3-10 are typical for this increase. The cause of the increase is partly the chopping of the star by the diaphragm edge and partly the variance trying to readjust to a new average count level.

Fortunately, the count rate of HZ Her in comparison to the sky background is high enough so that any such guiding lapse usually drops the divided and subtracted counts per time window down to zero. Such lapses can then be considered harmless to the sensitivity of the run by a simple fortran statement which restores the "dropped out" 6-bit words to the average count rate. Appendix section A4 shows why this procedure is superior to the insertion of Monte-Carlo noise.

C. Data Analysis -- Preliminary

1. Blocking Data

The tapes from the Kennedy recorder were blocked from the continuous format into records of 4096, 60-bit words, which contained 40960 of the time windows of data. The blocking program was used on the CDC 6600 and 6400 and functioned in the sequence of READ, REWIND-SKIP-READ, REWIND-SKIP-READ,... until the record was exhausted. The rewind was necessary to reset the File Environment Table (FET) and so that the tape drive knew

where it was. Each successive skip jumped past the previously read data before reading in such a way that the first and last two 60-bit words of successive records were overlapped. This was done so that one or more time windows could not be accidentally lost.

If such dropouts did exist and were left uncorrected then the coherence of any pulsations in the data would be damaged, with the higher frequencies suffering more. The CDC tape drive units were reproducible enough so that when a silver load point was located into the physical record the unblocking would only jitter by one or two bytes (=time windows). This meant that data could be salvaged from tapes which were damaged or which contained unreadable spots.

2. Fourier Transforming

The blocked data are then analyzed with a modified version of the Cooley-Tukey fast Fourier transform algorithm. The typical time taken for a transform of 2^{20} or 10^6 data points (all real numbers) for Fortran coding was 200 and 700 seconds for the CDC 6600 and 6400 respectively. The code was recently rewritten by the author in assembly language, making optimum use of the multiplexing inherent to the 6600, so that the time taken per 10^6 points is now 50 seconds, half of which is used in the 3500 random accesses to the disk.

The power spectrum is obtained by summing the squares of the amplitudes of the complex Fourier coefficients. A running local average was used to determine the *significance* of individual bins. The exponential significance is equal to the power of the bin in question divided by the average of the power in a small frequency interval centered about the

bin (not including the bin itself and a few of its nearest neighbors). For example, if this ratio is 10 then the probability of a given bin accidentally exceeding 10 is $e^{-10} = 4.54 \cdot 10^{-5}$. As long as the behavior of the power spectrum in such small regions is reasonably smooth, then this local definition of significance can be expected to hold.

Figure 7 shows the features of a typical power spectrum. The horizontal axis shows the frequency in Hertz with the section from 1 to 12 Hz cut out. The vertical axis shows the power on a log scale. Ordinarily, width of a power spectrum bin is $1/T$ Hz where T is the run duration in seconds. However, each bin in the histogram shown is the sum of 128 of these adjacent, resolution-limited bins. This summing process is used to decrease the relative fluctuations of the resultant bins and to emphasize only the coarser features. Since the duration of the run was roughly 10^4 seconds, the frequency resolution obtained from this summing process is therefore $128 \cdot 10^{-4} = 10^{-2}$ Hz. The arrow points to the location of a signal detected from HZ Her whose significance of 23 average bin levels is swamped by the coarser averaging.

The power spectrum shown, like the rest of the spectra obtained, contains no unusual gross features above 0.2 Hz. The peak which increases toward 0.0 Hz is caused by the longer term variations in the count rate. These variations can be introduced by slow changes in the light level from the star or background, in the transparency of the sky, in the direction to HZ Her, and in the scintillation of light from HZ Her and out of the diaphragm used.

In addition to checking the gross shapes of the spectra obtained, a histogram was constructed from the derived statistical significances

for all of the resolution-limited bins for each power spectrum which would be flat only for a perfectly exponential distribution with an average of unity. These histograms were invariably flat not only for the HZ Her results but were also flat even for the extremely sloped spectra obtained from runs on bright stars.

3. Converting Local Significance to a Signal Level

The amplitude, a , in counts s^{-1} of a sinusoidal signal present in the data which is necessary to produce a locally-defined significance of P_0 average power spectrum levels (as described in the preliminary data analysis section) is given by the formula.

$$a = \sqrt{4 \cdot MSD \cdot LL(f) \cdot (P_0(f)-1) \cdot T^{-1} e^{-2}} \quad \text{counts } s^{-1}$$

From Appendix Section A9b, MSD is defined as the variance (mean squared deviation from the average) of the raw data in counts²s⁻¹. $LL(f)$ is the ratio of the average power level near the frequency f to the average power level over the entire AC range, $1/T$ to $f_{\text{samp}}/2$ (Hz). The parameter, f_{samp} , is the sampling frequency and is given by the reciprocal of the time window widths ($1/t$ Hz). T is the run duration in seconds.

The quantity, $P_0(f)-1$, is referred to as the "excess power" at the frequency f . The product $MSD \cdot LL(f)$ is by definition the variance of the raw data at the frequency f (as measured from the power spectrum analysis).

The efficiency, e , is used to eliminate the effects of the variable transmission of the earth's atmosphere, telescope mirror reflectivity, and photomultiplier quantum efficiency.

In the HZ Her experiment, e is defined by observations on a star of similar magnitude (12.7) and blue color. Figure 8 shows this comparison star in a field of 100 square arc minutes of sky near HZ Her. This print was taken from a 10x magnification of the appropriate Palomar Sky Survey print.

The operating definition of e is

$e \equiv$ current comparison star rate/fiducial comparison star rate. The fiducial comparison star rate was measured to be 5000 counts s^{-1} on an excellent night in July of 1972 with the star directly overhead.

Once the amplitude (a) for a particular result was calculated, the pulsed fraction was arbitrarily defined as

$$pf = a/4000.$$

The reference rate of 4000 counts s^{-1} was chosen because it approximates both the 35-day phase average of the 1.7 day maximum count rate from HZ Her on the 61 cm telescope and the 1.7 day phase average of the total count rate of the star and sky in the 11 and 17 arc second diaphragms used.

An algorithm which converts excess power into a magnitude for any telescope and any detector with a similar wavelength response is given by

$$m = 16.0 - 1.25 \log_{10} \{ 4 \cdot MSD \cdot LL(f) \cdot [P_0(f) - 1] \cdot T^{-1} (225e)^{-2} \}.$$

The parameter e has been grouped with the nominal count rate for 16.0 magnitudes to indicate that the product of the two can be replaced with one comparison rate sample from a star of known magnitude. The rate obtained for the comparison is then extrapolated to $m = 16.0$ and used directly in the formula above.

A typical run on HZ Her with $MSD = 4000 \text{ counts}^2 s^{-1}$ lasting

$2^{18} \cdot 40 \text{ ms} = 10,485.76 \text{ s}$ with $e = 1$ would yield $P_0 = 14$ for $m = 20.3$, and typical upper limits obtained for a similar run with $P_0 \leq 2.0$ would be $m = 21.7$.

4. Software Calibration

A Monte-Carlo program provided a direct calibration of the FFT analysis. The waveform used was a 30 Hz sine wave with an amplitude of 50 counts s^{-1} . This waveform was Monte-Carloed into 2^{18} , 4ms time windows with a Poissonian background of 8000 counts s^{-1} . A peak of 11-12 σ significance was produced in the transform (this is roughly 65-82 excess average power spectrum levels; the lower results occur when the frequency splits between two spectral bins). These results confirm the previous analysis of the sensitivity of the Fourier Transform.

IV. THE DATA

A. Introduction

Before describing the many facets of the optical pulsation data for the HZ Her system, it is appropriate to summarize the classes of information which have been extracted from the data and the possible correlations for which the information can be tested. The patterns which emerge from this study will be useful in choosing among possible theories and perhaps in formulating new theories.

The information which could be extracted from the rapid photometry on any object includes amplitudes, phase, frequency, harmonic content and, as is shown in the Appendix, the time derivative of the frequency and some indication of the time profile of the pulse amplitude.

In practice, it is almost impossible to connect the phases between the observations because of the weak and intermittent nature of the features to be discussed. Moreover, the large extent of the possible region of emission of optical pulsations (the component separation and size of the non-degenerate companion are both about a dozen light seconds) may make any attempts at such phasing physically impossible because of intrinsic scatter in the location of emission.

Thus the information gathered and considered for this study will be restricted to amplitude, frequency and its first derivative. However, the harmonic content of the data will not be discussed here because the presence of harmonics and subharmonics in the data is a relatively rare and weak occurrence. [Earlier reports of Davidsen et al.³, Middleditch and Nelson¹⁸ and Groth¹⁹ have established the existence of harmonics

and subharmonics.]

The HZ Her system has been shown to vary on four timescales: 1.24 seconds, 1.7 days, 35 days, and roughly ten years. The optical data of the first timescale can be correlated to the other three timescales. Unfortunately, since HZ Her has as yet (Sept. 1974) to change its behavior according to the last and longest timescale, only correlations relating the 1.24 second, the 1.7 day and 35 day behaviors may be made. This still leaves six possible relations, all of which will be discussed with the help of the next several dozen figures and a few tables. Some of the patterns have only recently become evident (after two years of observations) since the gathering of information for the 35 day phase is a Herculean task.

B. Summary

Correlations of amplitude and frequency of the 1.24s optical pulsations with the 1.7 day orbital phase are discovered and investigated with the help of power spectra from the Fourier transforms displayed in Figures 9-30, 35-46 and, in less explicit fashion, Figures 47-51. Three strong features and one weaker feature are discovered in the orbital phase-frequency plane. The possible change of the velocity of the three strong features is investigated in Figure 52. No change is obvious -- a good limit is placed on the frequency rate of change of the Her X-1 pulsar. The behavior of these features over the 35 day phase (the 35 day phases are calculated with the Uhuru ephemeris with the aid of the optical pulsation data) is investigated with the help of Figures 54-62; a definite 17½ day (half of the 35 day period) behavior is seen in the three strong features.

The 35 day phase itself is seen to be consistent with a 34.85 ± 0.05 day pattern over 697 days.

The correlations of frequency to its time derivative and of both to the 1.7 day orbital phase are investigated with the help of Figure 53. The origin of the optical pulsations is seen to be from either near Her X-1 or from discrete positions on the emerging or receding limbs of HZ Her. A further investigation through Figures 63-73 extends the tests for correlations of the above variables to include the 35 day phase. The possibilities here become so numerous that more data will be needed before many of the possible correlations can be resolved.

A subject cross reference is given for all figures and tables at the end of this *Data* section.

C. Power Spectra

The continuous strings of data taken on HZ Her are first blocked and the Fourier analyzed as described in Sections C1 and C2 of Section III. After the raw power spectrum is formed by summing the squares of the complex coefficients, a scan for significant peaks is initiated which uses a running local average to determine the significance of each bin. During this scan, a small region of frequency centered around the Her X-1 average is saved for later analysis.

1. Her X-1 Line of Sight Results

Figure 9 shows such a sample section of Run 10. Power is plotted vertically, while the upper horizontal scale shows the measured frequency corrected to the solar system barycenter for the direction to HZ Her.

The lower horizontal scale is derived from the upper by a doppler argument, $v = (f - f_0)c/f_0$, and so represents the velocity relative to the center of mass of the binary system projected along the line of sight to earth. Note that an object moving toward the earth is defined as positive.*

The horizontal dotted line near the bottom scale represents the average power spectrum for two much larger regions of the same power spectrum which begin just beyond the sides of this frame; this defines unity for the vertical scale. The three dashed vertical lines mark the limits and center of the doppler motion of Her X-1 at ± 169.2 and 0.0 km s^{-1} .

The circle in the upper center represents both the tracking of the Her X-1 pulsar frequency and the Her X-1 orbital position as a function of orbital phase. Terrestrial observers look at the orbit circle from the bottom with x ray eclipse occurring when Her X-1 is on the far side of the circle at 0° . Thus the orbital progression shown is for the left-handed sense of rotation. *The actual track of Her X-1 over the duration of this run is shown by the darkened part of the circle near 90° .*

The companion star to Her X-1 may occupy a roughly tear-shaped region with the point of the tear (the inner Lagrangian point) near the center and the body extending away from Her X-1 toward the opposite side of the circle and perhaps farther.

The right hand vertical scale gives the $\log_{10}$ of the probability of any given bin accidentally exceeding a vertical height. Due to the exponential nature of the distribution bin heights in a power spectrum, this scale is derived from the left hand power scale with a fixed ratio.

*This is contrary to Hubble, but Hubble was a measurer of wavelengths while the current research deals with frequency.

The title paragraph at the upper left gives the power excess over the expected power for the number of bins, N , in the power spectrum shown. For a spectrum which has been derived from data with no signals present, the excess power is normally distributed about zero with a sigma of $\sqrt{N}$. The sum of the variances of all of the bins is also given as the "chi-square". In the absence of signals this chi-square is normally distributed about N with a sigma of $\sqrt{8N}$. The time limits given for the run are in UTC.

Figure 9 shows an event (the large spike in the spectrum) with a relative velocity near 175 km s^{-1} and with enough power to give a probability of less than 10^{-6} of being an accident. It may be interpreted, to first order, as originating somewhere along the line of sight to Her X-1 by x ray excitation of gas in the binary system or from Her X-1 directly. In similar plots Figures 10 and 11 show two more such examples at binary phases near 290° and 320° .

2. Velocities of Features

The rationale behind the above interpretation is that the feature (not necessarily always the same gas atoms) which is producing the pulsations is assumed to be corotating with the same angular frequency of the binary system. Letting ϕ represent the binary phase of the feature (which may be a different phase from that of Her X-1) at the time of the observation, the projected velocity in the direction to Earth is given by $v = w r \sin \phi$ where w is the orbital rotation frequency, $2\pi P^{-1}$, and r is the distance of the feature from the center of mass. Since the velocity is measured from the doppler effect explained above and w is known, then we have $r \sin \phi = v/w = \text{Constant}$. This is also the equation of a vertical line

through the orbital circle of Figures 9-11, namely, $x = \text{constant}$. When the errors on the measured velocity are included, the possible region of emission for the feature becomes a vertical band located directly above the spike in the power spectrum. Therefore any result in the power spectrum can be mapped onto a region of the orbital circle using the assumption of corotation.

3. Limb Features

Figure 12 shows the detailed spectrum of another feature which cannot be coming from the Her X-1 source because of the wide disparity in velocities. It could, however, originate from the limb of the companion star, HZ Her, or from gas anywhere else along a vertical band through the center of the orbit circle. The radius vector from the center of mass to the coorbiting feature cannot be too large, however, since the run lasted for some 25.7° or orbital motion (2.91 hours). During this time, any such feature with a large radius would change enough in frequency to smear the power across more than one of the bins of the power spectrum. Thus narrow features, like the one appearing in Figure 12, are more likely to originate nearer to the center of the orbit circle than from the circle itself or outside of the circle.

Figures 13-18 show similar results obtained near the same binary phase, $\phi = 0.20 - 0.25$. Figures 19-22 show again more results which are probably coming from the limb which is seen when the x ray source is receding at binary phase $\phi = 0.80$.

4. Simultaneous and Anomalous Peaks

Occasionally the two types of results, limb and x ray source, are observed simultaneously as in Figures 23-25. At other times, an anomalous spike is observed to appear simultaneously with a regular feature. Figure 26 shows two peaks, one which might be classified with the usual features coming from the limb of the companion to the x ray source. The other peak is seen to have a higher velocity than the x ray source. However, the power level of this feature is low enough so that approximately one could be expected as an accident for the region bounded by ± 2 Her X-1 doppler limits.

Possible explanations for the appearance of the anomalous peaks include the temporary presence of relatively dense gas in the binary system which is excited by the x ray pulsar or the breakdown of the usual assumption of corotation for one of the regular features. Fortunately, the anomalous peaks are numerically few in comparison to the two categories of features discussed above.

Figure 27 shows the strongest anomalous peak ever detected, with a velocity just lower than the feature seen to be from the emerging limb of HZ Her. Since the separation of the two features is well-resolved, it is beyond question that the smaller feature could be a sideband of the larger feature. However, in the analysis associated with Figures 51-74, which will be described below, the sidebands of the larger feature are sufficiently large to contaminate the measurement of the smaller. The smaller result will therefore be excluded from consideration for these analyses.

There are other Lagrangian points ($L_2 - L_5$) which exist for the binary

system. While none of these points is stable*, it is quite possible for amounts of gas to persist for long intervals because of the absence of deviating gradients. Such a gas accumulation at or near the forward libration point may be responsible for the appearance of the anomalous feature shown in Figure 27.

D. Results from All Power Spectra

1. The Two Classes of Power Spectra

In choosing the length of data for Fourier analysis, the experimenter has to make a painful choice. On the one hand, the spectra should be made of data strings as long as possible in order to give the best possible frequency resolution. On the other hand, transforms of shorter runs are better for indicating trends in the pulsational amplitude of any features.

For this analysis, a benevolent compromise has been reached where either good frequency-resolution spectra or reasonable time-resolution spectra are used depending upon the nature of the information desired. Therefore two ensembles of power spectra exist: one of data broken into short runs to determine pulsed-fraction, and one of longer runs for the display of the features on the frequency scale. The general rule for the first group is that none may be longer than a 2.91 hour run; the data on any given night is usually represented by one or more of these spectra. In the cases where a time overlap is required for two or more such spectra to completely cover the night's data, the overlap is usually no more than necessary (less than one hour). A few exceptions exist to the total

* The libration or Trojan points, L_4 and L_5 , each of which makes an equilateral triangle with the two masses of the system, do in fact become stable for mass ratios greater than roughly 26 (or less than $1/26$) due to the action of the Coriolis force.⁵⁶

length rule in the cases where there is little or no evidence for pulsation anywhere near the Her X-1 frequency limits.

In the second group, the transforms are made of data strings with the maximum length possible and thus frequently cover the whole night's run. Exceptions to this rule occur where features are strong in one part of the night and weaker in another part, *and* the frequency location of the feature is such that the longer transform does not show an improved frequency resolution over the two shorter transforms. In this case the two shorter transforms would be used for both ensembles of data.

Run 71 is used as an example of the selection process for the two ensembles. Figure 28 shows a detail spectrum of the first 2.91 hours of the night's data. A feature from the limb of HZ Her is suspected but is neither significant nor well-resolved because the power may be splitting between bins. Figure 29 shows a detail spectrum of the last 2.87 hours of the 4.69 hours total. Perhaps a little power yet remains in the location of the feature.

Figure 30 shows the detail spectrum which resulted from the addition of 1.13 hours of constant pulse count data to the end of the original 4.69 hour data string and then Fourier transforming the whole 5.82 hours. This has resulted in greatly improved resolution and significance in spite of the fact that the feature still occupies two bins (these bins are only half as wide as those previously shown). If the two bins of the feature of Figure 28 were really not associated with the same feature, then the result would most probably have been a dilution of the significance of the separate features because of the increased resolution--perhaps with a cancellation effect toward the midpoint of the two.

The spectra of Figures 28 and 29 were consequently used only to determine pulsed fraction and that of Figure 30 was used to represent the feature in frequency space.

2. The Sum of All the Spectra

In order to test the total effect of the results observed, the frequency-resolved power spectra have been summed together and displayed in Figure 31. Each spectrum in the sum was sliced so that the pieces would fall into the highly resolved bins of the frequency template of the sum spectrum (with the jitter of ± 1 bin due to the earth's motion about the sun removed from each contributor). For example, a 90-minute spectrum would contribute, on the average, $\frac{1}{4}$ of the value of each of its own bins into four of the "six-hour" bins (the exact time length is actually 20971.52 seconds = 5.82 hours).

The left vertical scale tells exactly how many such "six-hour" run equivalents have been added into the spectrum. The horizontal dotted line has been drawn across the spectrum at this number. Error bars are plotted on the bins with the lowest and highest frequencies of the spectrum. These errors have been adjusted so that the expectation value for the chi-square mentioned in the title line at the upper left is equal to the number of bins in the spectrum, N , in the absence of any signals. The distribution of this number should be a normal one, with a sigma of $\sqrt{2N}$.

The relative proportions of the features seen below the Her X-1 orbit circle would be quantitative only if the data coverage over binary phase were uniform. However, nonuniformities in the data coverage will later be seen to exist up to a factor of two.

3. Population Histograms

A population histogram for some of the bins used from the individual spectra to form the sum spectrum is shown in Figure 32. Only those bins with a frequency lower than $f_0 - 1.2\Delta f$, where $\Delta f = f_0 \cdot 169.2 \text{ km s}^{-1} \cdot \text{c}^{-1}$ is the unit Her X-1 doppler amplitude, have been used for this histogram. The vertical population scale is logarithmic and the horizontal power scale is linear, so that the exponential distribution expected for white noise should fall along a straight line of fixed slope as drawn. The only free parameter, the y-intercept, is fixed by the total number of bins in the population. The graph shows that this lower cut in frequency appears to be without significant signals.

Figure 33 plots the population-power histogram for the central frequency region for the bins of the contributing spectra, from $f_0 - 1.2\Delta f$ to $f_0 + 1.2\Delta f$. Excess population is apparent for power above 5.0 levels and may persist at lower levels but cannot be distinguished from the normal because of the nature of this plot. Figure 34 gives the population-power histogram of the upper frequency section, from $f + 1.2\Delta f$ upward (to roughly $f_0 + 1.2\Delta f$). This histogram is consistent with being normal. It can therefore be concluded that almost all of the behavior seen from the HZ Her system falls within modest velocity limits ($\pm 200 \text{ km s}^{-1}$).

4. Spectra Versus 1.7 Day Orbital Phase

Figures 35-46 show the sum of spectra for each 30° section of binary phase over the whole 360° of binary phase (0° = eclipse). Each section is identified by the darkened track on the Her X-1 orbit circle. Since the frequency-resolved ensemble of spectra was used to form the sums, some of

the stronger features appearing in one frame may "spill over" into the adjacent frames because of the length of the runs.

Near-limb features occur at binary phases $60-120^\circ$ and $270-330^\circ$ and a line-of-sight feature occurs at $270-330^\circ$ (and a weak one at $60-120^\circ$). A marginal result appears at $30-60^\circ$ in Figure 36 where one feature actually has a higher velocity than the x ray source (this is in part due to Run 5 which was discussed previously in Figure 26). Another feature in the near limb position for binary phase $120-150^\circ$ shown in Figure 39 could be due in part to the spillover from the $90-120^\circ$ runs.

Another weak feature is seen between the positive near-limb position and the position Her X-1 doppler limit in Figures 41-43 from $180-270^\circ$ orbital phase. This cannot be caused by the spillover of any other feature and seems to persist at least as long as the three stronger features.

The weak feature appearing just lower in velocity than the Her X-1 track from $330-360^\circ$ is, in Figure 46, on the other hand, likely to be due to the spillover effect. This hypothesis is substantiated by its similarity in shape and location to the Her X-1 feature of the immediately preceding frame.

5. Mean Pulsed Fraction and Statistical Screens

It is of some interest to ask whether the anomalous features seen in the preceding figures are due to the accumulation of many small results, or just a few large results. It is also of interest to inquire about the behavior of the mean pulsed fraction in the 1.2-day phase (with the systematics of uneven data coverage divided out).

Figure 47 has been constructed to help answer the above questions.

This figure shows all of the data folded over modulo the binary cycle. The horizontal axis scale gives the binary phase (nominally labelled 10.00 to 11.00 because this is the midpoint of the 20.5 cycles of the long 35-day phase). Eclipse is at 0.00 and 1.00. The top frame shows the data coverage; the histogram contents indicate the number of times the particular 15° segment of binary phase has been covered by the data. The lowermost frame shows the power spectrum bins which pop through a statistical screen over all of the spectra in the region between the two horizontal dashed lines (representing twice the extent of Her X-1's doppler amplitude in relative velocity or frequency). The power threshold used in the screen is 10.0; the expectation value for the number of accidental boxes to appear through the screen is roughly 0.16 for the current data ensemble. There are, in fact, some 33 boxes shown. The frequency-resolved ensemble of spectra was used to construct both the data coverage histogram and the lowermost frame.

The track of Her X-1's pulsation frequency in binary phase is shown by the sine wave. The dotted lines about this track indicate the region which was used to derive the pulsed fraction results which are plotted in the middle frame. The vertical scale has units of thousandths of the mean broadband optical brightness of HZ Her for binary phase near 0.3 and 0.7 as extrapolated using a nearby comparison star of similar color and magnitude. The phase-resolved ensemble of data was used to compute the mean pulsed fraction with the unevenness of the data coverage divided out.

The results of the lower frame indicate three regions in binary phase-frequency space where significant excess power is repeatedly observed. Two regions correspond to the near-limb features seen near binary phases

0.2 and 0.8 and the third corresponds to the Her X-1 line-of-sight pulsations seen near phase 0.85. The main feature displayed in the middle frame is the Her X-1 like pulsation occurring from phase 0.79 to 0.92 in uniform intensity.

Figure 48 is exactly like Figure 47 except that two changes have been made. The first is that the statistical screen of the lower frame has been lowered to 8.0 levels. The expected number of accidental boxes is now 1.35, with 40 boxes actually shown. The second change is that the dotted lines indicating the location of the pulsed fraction experiment has been changed from tracking the Her X-1 source to covering the near-limb features discussed above.

The results of the middle frame indicate the behavior of the strength of the near-limb features in binary phase. The marginal feature which pops up between 0.50 and 0.58 is probably the same anomalous feature which was discussed previously for the sums of spectra; here the tracking of the dotted lines lose the feature by 0.58 binary phase. Since the dotted lines of Figure 47 dip below the zero velocity line more steeply than those of Figure 48, the effect was less pronounced in Figure 47.

Figure 49 is identical to Figure 47 except that the statistical screen in the lower frame has dropped to 7.0 levels. Some 3.7 accidental boxes are expected from noise; 50 boxes are shown. Figure 50 is identical to 48 except that the screen is now down to 6.0 levels with some 10 accidental boxes expected and 66 actually appearing.

E. Results from the Strong Detections

1. Precise Values of Frequency and Its Time Derivative

It is shown in the Appendix that the frequency can be measured to an accuracy of

$$df = 3\pi^{-1} (6P_0)^{-1/2} T^{-1} \text{ (Hz) } ,$$

where P_0 is the power associated with the feature and T is the duration of the feature in seconds (the analysis also checks this T and it is changed if the analysis so indicates). Thus for a 5 σ result ($\Delta 1:1.7 \cdot 10^6$ accidental likelihood, or $P_0 = 14.37$) $df = 0.11 T^{-1}$. This is almost a factor of five improvement over the half-bin errors associated with displays of the power spectrum.

Likewise, the Appendix also shows how f' can be measured to a precision of

$$df' = 9.6 (90P_0)^{-1/2} T^{-2} \text{ (Hz s}^{-1}\text{)} .$$

Thus with $P_0 = 14.37$, the precision which can be attained in f' is 80% of T^{-2}

Accordingly, the 29 strongest features detected have been measured in f and f' and plotted in Figure 51 in a similar fashion to the lower frames of Figures 47-50. The left vertical scale gives the relative velocity while the right vertical scale shows its frequency counterpart. The lower horizontal axis plots binary phase in degrees while the upper axis shows decimal cycles. The boxes plotted in the frame have been given the slope according to their measured frequency derivative. The vertical

errors are ± 1 sigma and the horizontal extent of the boxes is due merely to the duration of the feature.

Actually, the frequency errors associated with the extrapolation of the errors in f' from the centers of the boxes to the ends (a change of $T/2$) will make the net error at these points about 4 times the error at the center. It would be more accurate then to plot the boxes more in the shape of a double horn (or one-half Maltese cross). This procedure has been omitted for the sake of simplicity.

The features seen in Figure 51 are the three which have been discussed above: near-limb at 0.2 (I) and 0.8 (II) and Her X-1 at 0.85 (III). Only two boxes appear separate from these features: one is almost 40 km s^{-1} above the Her X-1 track at 0.2 and the other is near $+60 \text{ km s}^{-1}$ at binary phase 0.13. These two boxes will not be considered as belonging to features I, II and III in the rest of this text.

It is interesting to note the good agreement between the centers of Feature III and the doppler track of Her X-1. The three boxes with earlier binary phase do not match the tracking of frequency as well as the three later boxes.

For Features I and II, it is interesting to ask if the frequency derivatives are consistent with some average value. For Feature I near the 0.2 phase the average $f' = -3.8 \pm 7.8 \cdot 10^{-10} \text{ Hz s}^{-1}$ and has a chi-square of 19.2 for 11 degrees of freedom (or a 7% confidence). For the feature near 0.8 phase, the average $f' = -2.1 \pm 1.5 \cdot 10^{-9} \text{ Hz s}$ with a chi-square of 7.7 for 8 degrees of freedom. Thus these features seem to have reasonable internal consistency in f' .

The velocities of features I, II and III were also checked for consistency with their averages. The only consistency found was for Feature III

(the velocities for this feature only were computed as residuals from the Her X-1 track). The average velocity residual for the six points of Feature III is $-2.1 \pm 1.2 \text{ km s}^{-1}$ with excellent confidence. The two near-limb features, however, do not hold together quite as well; Feature I averages $16.9 \pm 0.6 \text{ km s}^{-1}$ with a chi-square of 32.8 for 11 degrees of freedom and the later phase limb feature averages $-24.2 \pm 1.0 \text{ km s}^{-1}$ with a chi-square of 53.5 for 8 degrees of freedom.

Table I lists the parameters of the strong results which belong to Features I, II and III. Column 1 gives the run number, column 2 gives the Universal calendar date. Column 3 gives the date of the center of the run feature in JD - 2,440,000.5. Column 4 gives an integer cycle associated with the 35-day phase (x ray "on" at 6.68 or 7.23 and "off" by 14.0); the 1.7 day phase is given after the decimal point. Columns 5 and 6 give the full feature duration in days and 1.7 day cycles respectively. Columns 7 and 8 give the frequency and its error in Hz. Columns 9 and 10 give the f' and its error in Hz s^{-1} . Column 11 gives the power associated with the feature after optimization for f and f' . The minimum power shown is 11.6 levels and the maximum power is 38.7 levels.

2. The Long Term Stability of Features

The poor internal consistencies in velocity for Features I and II could be caused by a systematic change in the Her X-1 pulse frequency. The possibility of such a change in frequency over the relatively short binary phase intervals covered by the Features is ruled out because of the general lack of strong frequency derivatives for the features.

To test for a long term systematic change, Figure 52 has been

constructed to display the velocities of the points associated with the three Features as a function of time. The horizontal axis gives the date in JD - 2,440,000.5. The vertical axis gives the velocity relative to the center of mass of the system except for the six points seen near the line of sight to Her X-1. These have had the Her X-1 tracking subtracted from their velocities and thus represent residuals. These six points are also the six which fall nearest to the zero-velocity dotted line.

The 12 points associated with Feature I fall above the zero-velocity line between +10 and +25 km s⁻¹. The 9 points associated with Feature II are the nine points in Figure 52 with the most negative velocities. The one point which appears just below another point which covers the zero-velocity line near T = 1800 belongs to the nine of Feature II and is not associated with any of the six of Feature III.

The 12 points of Feature I (near-limb at phase 0.2) do not seem to show any systematic trend in time. The average velocity of the two points before T = 1600 is 15.7 ± 2.5 km s⁻¹ while the average of the five points after T = 2200 is 15.0 ± 1.1 km s⁻¹. On this basis, it is possible to rule against a systematic increase of frequency as reported by Giacconi et al.²⁰ extrapolated to a two year period. The increase allowed by these points of -0.7 ± 2.8 km s⁻¹ is less than $+4.9$ km s⁻¹ per two years or less than $8 \cdot 10^{-6}$ parts per year at the 95% confidence level. This is consistent with the revised mean limit of $5 \cdot 10^{-6}$ parts in the period from February 1972 to March 1973 from Uhuru⁵⁷ (Schreier, Tananbaum, private communication, 1974).

3. Relating $\dot{f}$ to Positional Coordinates

In a manner similar to that described previously for f , $\dot{f}$ may be related to a positional coordinate under the assumption of corotation of the source. Starting with the doppler relationship $v = c (f - f_0) f_0^{-1}$, the time derivative is taken to get $a = c \dot{f} f_0^{-1}$, where a is the acceleration along the line of sight to earth.

Now letting ϕ represent the binary phase of the corotating feature, and r represent the radius of its orbit, then $a = w^2 r \cos \phi$ and

$$r \cos \phi = a w^{-2} = \text{constant},$$

where w is the orbital frequency in radians s^{-1} . The above equation is that of a horizontal line with respect to the Her X-1 orbital circle (i.e., parallel to the line joining the phases 0.25 and 0.75). The two equations by which the position of the feature can be determined are then:

$$r \sin \phi = c (f f_0^{-1} - 1) w^{-1}$$

$$r \cos \phi = c \dot{f} f_0^{-1} w^{-2}.$$

The first determines the position along a line parallel to the line joining 0.25 and 0.75 in phase, and the second determines the position along a line parallel to the line joining 0.0 and 0.50 phase.

The assumption of *circular* motion of the feature follows because an upper limit of 0.1 has been established for the eccentricity of the Her X-1 orbit.² The only other assumption which is made is that the measured values of f and $\dot{f}$ for a finite interval of phase, $d\phi$, are good values for the median phases of the runs. This assumption is probably good because none of the measured results cover more than 50° in binary phase. On this scale, the chord is a good approximation to the arc segment.

In this way, the parameters measured for the stronger features have been plotted in Figure 53. The left vertical and lower horizontal scales give the Y and X positional coordinates in millions of km. The upper and right hand scales give the conversion of position to frequency and velocity respectively -- serving partially to indicate the scaling involved.

The system center of mass and that of Her X-1 are fixed at (0.,0.) and (3.95,0.) -- in this sense the picture projects away the difference in orbital inclination, i , from 90° . The location of the HZ Her c.m. and the Roche lobes drawn about HZ Her and Her X-1 are dependent on an assumed value for the mass ratio of 1.70 ($r = q^{-1} = m_{\text{HZ}}/m_{\text{X}}$).

Since the frame of reference is taken to be rotating left-handedly with the binary system, the earth and the rest of the universe are seen from this frame to be rotating about the system every 1.7 days in a counter clockwise fashion. The direction to earth at the midpoint of each observation is indicated by the pointers at the ends of the boxes shown. The binary phase of the midpoint of each observation may thus be computed with the use of the directional pointers at the upper right.

Each observation is indicated by a box with a pointer. The narrow dimension of the boxes represents $\pm 2 \sigma$ of frequency about the center. The long dimension of the boxes represents $\pm 1 \sigma$ in f' about the central value. The majority of the boxes are seen to originate either from near the Her X-1 source or from the limb of HZ Her. The absence of boxes near the inner Lagrangian point anywhere in between this point and the x ray source is notable. Since the boxes indicate the observation of optical *pulsations*, the appearance of a box indicates that the path from Her X-1 to the box is due to the exciting radiation and that the path from the

box to the earth is thin in the visible band.

The results which appear to be seen through the HZ Her Roche lobe could indicate also matter above or below the orbital plane, or a feature which was not quite corotating with the orbital motion, but had some net acceleration with respect to this motion.

In general, Figure 53 does confirm the supposition of optical pulsations originating either near the x ray source or from the limb of the companion. The lack of results from binary phase $0.5 = 180^\circ$ is also notable, as it has been in the previous Figures (47-51). The lack of results at binary phase 0. is, of course, as expected for both the limb and x ray features.

F. The 35-Day Phase

1. Establishing the Epochs

The 35 day phases of the data have been computed following the Uhuru ephemeris²⁰ and one observed fact of the optical data. This is that the feature observed from the limb at binary phase 0.2 is always observed before the x ray "turn on". Once the x ray "on" has commenced, then optical pulsation features from Her X-1 are seen at binary phase near 0.85. After this, strong limb features at phase 0.2 are not detected. Since an observation one night of 0.2 in phase can be conveniently followed by one at 0.8 the next night, this hypothesis was tested on several occasions. A strong limb event was never observed just after strong Her X-1 events

(that is, for the times when the phase of 0.2 was covered right after the scheduled "turn-on"). The few times during the observing program when Uhuru knew of a "turn-on", Her X-1 events were seen during the "on" and never before, and limb events at 0.2 were seen before, but never during.

The ephemeris used to determine the 35-day phase was an extrapolation forward from $T = 1501.18$ ($T = \text{JD} - 2,440,000$) with a period of 34.88 days. The 1.7 day phase was computed for each extrapolated epoch and compared to the favored phases of 0.23 and 0.68 (according to Giacconi et al.²⁰). The actual "turn-on" orbital phase was decided by whichever of the two was closest to the extrapolation. The origin of the 35-day interval was then set to be either 6.68 or 7.23 binary cycles before this "turn-on" when the orbital phase was at 0.0 (and thus is the "eclipse" nearest the middle of the x ray "off"). The 35-day phases were then computed in cycles from the origin for all data gathered between this origin and the succeeding origin. This interval would thus always be an integral number of binary cycles.

When a "turn-on" was seen in the Uhuru x ray data, a new fiducial epoch was set accordingly and all succeeding epochs were calculated accordingly. Occasionally the optical pulsation results from the limb near 0.2 in phase (Feature I) and from Her X-1 near 0.85 (Feature III) dictated similar minor revisions in the "turn-on" cycle.

Table II gives the results of this complicated process for the 25 such intervals encompassing the two year span of the optical observations. Column 1 numbers the intervals from 1 to 25. Column 2 gives the times of the interval starts ($T = \text{JD} - 2,440,000.5$) which fall at 0.0 binary phase near the middle of the x ray "off" interval. Column 3 gives the time of

the "turn-ons" which occur roughly 1/3 the way through the 35-day interval. Column 4 lists the phase in binary cycles of the "turn-on" established for the interval. Columns 5 and 6 give the separation of the beginning of the interval from the beginnings of the immediately succeeding and preceding intervals respectively.

The x subscripts in column 3 mark the established x ray "turn-ons" from Tananbaum (private communications, 1972-1974). The asterisk superscripts in column 3 mark the points where the "turn-ons" were established from the optical pulsation data according to the method described above.

In two cases, at $T = 1778.31$ and $T = 1814.01$, the "turn-ons" were set at 7.23 because observations of optical pulsations near 6.85 were weaker than expected (the usual strength expected is near $1.6 \cdot 10^{-3}$). The simultaneous data cited in Groth et al.⁶⁹ were used to establish the turn-on at 1778.31.

The turn-ons at 1606.59 and 1674.60 have been interpolated evenly between the x ray observed ones at 1571.65 and 1708.60 (and 1639.68 from McClintok et al.¹⁸) for the sake of continuity alone. Since no optical observations were taken during these intervals, the exact "turn-on" dates do not matter.

The two turn-ons observed from the optical pulsations at 1535.95 and 2233.02, which are the most widely spaced of the observations, are exactly 20 ± 20.5 binary cycles apart. Thus, although the 35-day rate has been observed to vary considerably for shorter periods, an average over some 20 such intervals spanning 697 days is consistent with a 34.85 day interval. If a one day jitter is allowed for such a long-period average, then the value is 34.85 ± 0.05 days. Unfortunately, various intervals such as

the period from $T = 1600$ to $T = 1700$ contain a large accumulated jitter from this average.

It is worth noting that the recently computed turn-ons at $T = 2267.95$ and $T = 2302.72$ are consistent with the observations of Her X-1 from Copernicus,⁵⁸ although the interpretation given here does not associate these epochs with any recent unusual jittering behavior -- contrary to the interpretations given for the Copernicus data. It is also interesting that the dates listed in Table II for the "turn-on" cycles are compatible with a 20 or 21 binary cycle interval and never longer or shorter.

2. Features I, II, and III in the 35-Day Phase

Once the 34.85 day phases have been determined, it is possible to test the stronger features which appear on the binary phase-frequency plane of Figures 47-51 for variability over this timescale. Figure 54 displays the results of such a test on the pulsed fraction for Feature I between phases 0.167 and 0.29 ($60-105^\circ$). Pulsed fraction for the results as previously defined is plotted vertically in thousandths while the binary cycle number through the 21 cycles of the 35 day period is plotted on the horizontal. The legend in the upper right gives the binary phase-frequency region used to select the spectra and regions within the spectra respectively from which the pulsed fraction points were derived. The horizontal scale is discontinuous above each tick mark because the small region of binary phase selected (45°) has been mapped linearly onto the whole region between each pair of consecutive integer cycles. This has been done so that the data do not overlap each other too closely and so that any short term variability within each of the 45° binary phase

regions would be more easily discerned.

The x ray "on" starts at 7 (6.68 or 7.23) and lasts roughly until 14. The results for this region indicates two 35-day phases where the activity seems to be strongest, one near 5.2 and 6.2 and the other at 16.2. The two regions are separated by exactly half of the 35-day cycle. Both of these regions of maximum pulsed fraction are outside the active x ray region.

Figure 55 shows a similar plot for Feature II from the limb near binary phase 0.8. Again two regions which are separated by roughly 10 cycles appear to be prominent. Unlike those of Figure 54, however, the activity seems to persist over three or more binary cycles and at several points (Cycles 9-11 and 2) seems to be erratic in nature. Also, unlike the feature at 0.2, the 0.8 feature seems to precede the 0.2 feature by 4 or 5 cycles (or to lag it by 6 cycles). At the exact places where Feature I becomes prominent, Feature II has nearly dropped to zero. (Thus the phases where the *lack* of pulsing is seen unambiguously is again spaced ten cycles apart.) Figure 56 shows the 35-day characteristics of the Her X-1 type pulsations at binary phase 0.167 to 0.25 ($60-90^\circ$). Only one point out of 30 is not consistent with zero, and this is the singular result of Run 10.

On the other hand, Figure 57 displays a Feature III which seems to be reproducible over the 1.7 day phase 35-day phase matrix. It covers phases 0.75-0.92 along the Her X-1 track (binary phases--and each cycle of the 35-day phase of course). The results seem to occur with a ten cycle repetition, since optical pulsation was detected at cycle 20.68, and indicate that for one region of 35-day phase such pulsation only occurs during the first half of the x ray "on" period.

Figure 58 shows the data which covers the region of the anomalous feature discussed previously in the section dealing with the sum spectra over the 1.7 day phase. Out of the 31 points plotted, all but six had a positive level of pulsation associated with an excess of power. The probability of such a distribution occurring accidentally is about 10^{-3} . The four highest points are located between cycles 11 and 17. The data seems to be sporadic in nature, since other points in this region were not as high.

Figures 59-61 show the 35 day characteristic for random phase-frequency regions which are more typical of the majority of those which show no unusual activity.

3. The 1.7-Day, 35-Day Template

In order to assess any weak patterns in the data which may have been missed by the preceding displays, Figure 62 has been constructed by using the ensemble of data used for pulsed fraction. The 21 cycles of the 35-day phase have been taken three at a time and the data at each binary phase section for the three cycles has been treated as occurring at the same time. The horizontal scale is thus labelled for the cycles used to form each of the seven frames which are stacked end-to-end. The x ray "on" period of cycles 7-14 is covered by the third, fourth, and fifth frames (6-15). The vertical scale gives the binary phase; the results have been segmented into 12 sections over one binary cycle. The horizontal extent of each frame is used to display the frequency or relative velocity scale from -2 Her X-1 doppler shifts to +2 Her X-1 doppler shifts. This velocity range has been segmented into 21 velocity regions, each of which

is 0.2 Her X-1 doppler shifts wide. In each of these innumerable rectangular regions, lines have been drawn so that the number drawn is roughly proportional to the mean pulsed fraction in the region. The exact legend is:

- 0 lines : insufficient phase coverage
- 1 line : less than 0.0 thousandths mean pulsed fraction
- 2 lines : 0.0 - 0.2 thousandths mean pulsed fraction
- 3 lines : 0.2 - 0.4 thousandths mean pulsed fraction
- etc.

Thus the darker the sections appear, the higher the mean pulsed fraction. Also, since the lines are drawn to partition the imaginary rectangles equally but do not reach the boundaries of the rectangles, the boxes with higher mean pulsed fraction tend to appear more closely packed. The track of Her X-1 has been drawn up each of the seven frames for reference.

The usual three phase-frequency features are seen in the frames, with their 17 day behavior evident. The contributions to the anomalous feature seem to be distributed over several boxes -- as is consistent with Figure 59. The only other concentration which could be considered unusual is the tendency to get pulsation at high negative velocities at binary phase 0.42-0.50 and cycles 18-21. Unfortunately, since the results of only one run contributed to the region, the level of pulsation is more uncertain than the average for this figure.

4. The Spatial Origin of Features Versus 35-Day Phase

In order to display any change in the location of the pulsational features in the binary system over the 35-day phase, Figures 63-73 have been constructed. They are the same as Figure 53 in every way except that

the data which appears on each has been restricted to a two-cycle phase interval starting with 0-2 at Figure 63. In this way, some of the boxes which could not be seen in Figure 53 can be more easily picked out. The dates for the boxes shown may be found in Table I, although the positional coordinates derived are not listed in the table.

Since the presence or absence of results in Figures 63-73 depends upon the data coverage, Figure 74 has been constructed to make this coverage over the 35-day interval easy to see. The coverage histograms shown are similar to those constructed in Figures 47-50. They differ only because these 11 histograms, each of which covers two binary cycles (except the last), have been stacked on top of each other for compactness. The horizontal axis gives the binary phase in cycles from 0.00 to 1.00 (both ends are at x ray eclipse). The vertical scale of each histogram has been set to a maximum of 3; the horizontal dotted lines are drawn across the histograms at unity. The numerical labels near the vertical axis give the cycle range of each histogram in the 35-day phase.

Only one feature is seen in Figure 63 at binary phase near 0.8. Figure 64 is blank because no significant features were seen in the interval 2.0 - 4.0. The data coverage in this interval is sparse, but is sufficient to cover any possible features near phase 0.2. The phase near 0.8 is not well covered for this region of 35-day phase.

Figure 65 shows seven boxes in the 4.0 to 6.0 region: two were taken near phase 4.2, two are from phase 4.8 and the remaining three at 5.2

Figure 66 which covers 6.0 to 8.0 (and the x ray "turn-on" near 7.0) has four boxes from the limb region at 6.2, and one from the Her X-1 region at 6.85. Figures 57 and 58 have indicated that more results have actually

occurred near 6.8 and 7.8 with similarly high pulsation, but none others were significant enough to analyze due to less optimum observing conditions.

Figure 67 covers 8.0 to 10.0 and has three boxes near Her X-1, two of which were gathered near 9.85 with a third gathered near 8.85. The fourth box which appears to be seen through the non-degenerate star was seen simultaneously with the third box from Her X-1 at phase near 8.85. These two boxes are parallel as a consequence of the equality of their phases.

Figure 68 has one box from Her X-1 seen near 10.8 and another from the limb of HZ Her near 11.8. Figure 69 has three boxes from the limb: one seen at 12.75 inside the limb and two more seen at 13.85 on the limb. Figure 70 covers cycles 14-16 and is the first frame since Figure 65 to cover an "inactive" x ray period. One result was seen from the emerging limb at 15.2 and the next result came on the following night from the limb at 15.8.

Figure 71 has two results which were both near cycle 16.25 in the 16-18 region. Figure 72 is blank; the coverage near phase 18.2 and 18.8 was adequate, thus the absence of boxes is significant.

Figure 73 covers the last cycle, 20-21, but the coverage in this cycle near 0.2 is not good. However, a box near Her X-1 was seen at 20.8.

V. DISCUSSION OF THE OPTICAL PULSATION DATA

Examination of the data in the previous section indicates that about 90% of the significant events observed from the HZ Her system can be divided into three distinct groups according to their frequencies and orbital phases. Prior to any discussion of optical pulsation data in relation to the properties of the HZ Her - Her X-1 binary system, the significance of the three features must be well understood.

To reach this understanding it is necessary to know: 1. Why no pulsations are seen from the limb of HZ Her during the phase interval 0.33-0.67 and 2. Why no pulsations are seen along the line of sight to Her X-1 outside of the phase region 0.79-0.92.

Detailed calculations are introduced to help answer question number 1. A recent observation of pulsed emission lines by Davidsen, Margon and Middleditch⁵⁹ -- henceforth DM² -- is used to help refine the understanding of the atmospheric processes which bear on the hypotheses under which the calculations are performed. With the insight gained from the calculations, question number 2 is discussed in its relation to stream-disk models through the medium of some less-detailed calculations.

Finally, with the mechanics of the optical pulsations well understood, a few simple models for the 35-day optical and x ray behavior are briefly investigated.

A. Model Calculations

1. The Detailed Calculations

The calculations performed utilize the solutions to the equations

for the equipotential surfaces of the corotating system as the trial shapes of the non-degenerate star, HZ Her. These equations take the form

$$\phi = \phi(x, y, z, R) M_*^{-1} G^{-1} = -\frac{1}{2} r^2 (1+1/R)^{-1} a_2^{-2} - r_*^{-1} - (R r_x)^{-1} \quad (1)$$

where the origin of the coordinate system is taken at the center of mass, the masses lie along the x axis with the x ray source on the positive side, and the z axis is perpendicular to the orbital plane. The variable, $R = M_*/M_x$, is the mass ratio, $r = (x^2 + y^2)^{1/2}$ is the distance from the center of mass projected onto the orbital plane (used above in the repulsive centrifugal term), $r_* = [(x+a_2 R^{-1})^2 + y^2 + z^2]^{1/2}$ is the distance of a field point from M_* and $r_x = [(x-a_2)^2 + y^2 + z^2]^{1/2}$ is the corresponding distance from M_x . The constant, $a_2 = 3.95 \cdot 10^{11} / \sin(i)$ cm, is the measured separation of Her X-1 from the system center of mass.²

The largest possible equipotential surface which closes only around HZ Her is the critical Roche equipotential, and it crosses the mass axis at the inner Lagrangian point, L1 (this is the saddle point which lies between the two masses). The position of L1 on the x axis, x_{L1} , may be found from the solution to the equation

$$R = \left[\frac{(1-b) - (1-b)^{-1}}{(b-b^{-2})} \right] \quad (2a)$$

$$\text{where } x_{L1} = b(R) \cdot a_2 \cdot (1 + 1/R) - a_2/R \quad (2b)$$

A few calculations were performed using the Tidal lobe model. The equations for the tidal equipotentials are

$$\phi = \phi(x, y, z, R) M_*^{-1} G^{-1} = x R^{-1} (1+1/R)^{-2} a_2^{-2} - r_*^{-1} - (R r_x)^{-1} \quad (3)$$

This model describes the behavior of a star which is stationary in space and not corotating with the binary motion. Unlike the Roche model,

this potential is only approximate insofar as the Coriolis force is assumed to be acting on circular velocity cross sections and the viscosity is assumed to be zero*.

The equation which gives the location where the critical Tidal lobe crosses the x axis is:

$$R = b^3 (2-b) (1-b)^{-1} \quad (4)$$

Note that for $b = 1/2$, $R = 3/4$, i.e. the analogue of $L1$ for this potential would be under the gravitational influence of the larger mass (x ray source) were it not for the effects of the Coriolis force. Again, $X_{L1} = b(R) \cdot a_2 \cdot (1+1/R) - a_2/R$ in the c.m. system of reference.

This model is used below primarily for the purpose of gauging how a slight degree of non-corotation may affect the results obtained for the Roche lobe model. It is unlikely that the gaseous outer atmosphere of a $4 \cdot 10^{11}$ cm diameter star only $4 \cdot 10^{11}$ cm from a $1 M_{\odot}$ neutron star could remain in a non-corotating state for more than a few months.

To describe the geometry of the calculations, Figure 75 has been constructed. The parameter $d\vec{a}$ represents the local area vector, $\vec{\Omega}$ represents the rotation vector as seen from the earth (parallel to the z axis) and $\hat{e}(t)$ represents the unit vector in the direction to earth at the orbital phase $\theta(t)$. Thus the earth and the rest of the universe proceed to rotate in a clockwise (left-handed) sense about the non-inertial

* Both the Roche and Tidal models assume the masses act like points; insofar as the outer atmospheres of stars in the late (expanding) stage of evolution are not dense, this assumption is a good one. On the other hand, the Coriolis force acting on the "sloshing" motion due to a star's deformity in the Tidal model is not likely to be a small effect. The Roche lobe model is not subject to this objection, since all the matter is in corotation.

frame of corotation of the system. The vectors $\vec{r}_*$, $\vec{r}$ and $\vec{r}_x$ are the vector analogues of the scalar quantities in Equation (1) above. The variable i shown in the figure is the inclination of the plane of the orbit to the line of sight.

The x ray power seen by the area element $d\vec{a}$ is given by

$$dE_x = L_x \cdot \text{Max}(d\vec{a} \cdot \vec{r}_x / r_x^3, 0) / 4\pi \quad (5)$$

where $d\vec{a}$ points outward from the closed surface and L_x is the x ray luminosity assuming an isotropic flux scaled up from the flux observed from the earth. The reradiated power seen from the earth at the orbital phase θ (assuming 100% instantaneous conversion of the x rays into optical energy) is given by

$$dE_{or} = dE_x \cdot \text{Max}(\hat{e} \cdot d\vec{a} / |d\vec{a}|, 0) \quad (6)$$

This energy has the form of pulsation

$$dE_{or} = dE_{or}(d\vec{a}, \theta, t) = dE_o \cdot \cos[2\pi f(d\vec{a}, \theta)t + \phi(d\vec{a}, \theta)] \quad (7)$$

where $f(d\vec{a}, \theta)$ and $\phi(d\vec{a}, \theta)$ are the frequency and phase of the pulsations of the particular area element $d\vec{a}$ at orbital phase $\theta(t)$. Orbital motion and the doppler effect make these variables functions of θ and t (ϕ also depends on the particular model used for the direction of the pulsar sweep). The variable, t , is of course the time in seconds.

What is desired as the answer is the Fourier transform of the integral over a certain binary phase interval $\theta_1 - \theta_2$ of the integral over the visible illuminated area of the surface of the star and normalized to the x ray

flux. This is given by

$$G(f) = [(\theta_2 - \theta_1)/L_x]^{-1} \int_{-\infty}^{+\infty} e^{-2\pi i f t} dt \int_{\theta_1}^{\theta_2} d\theta \int E_{or}(\vec{d}\vec{a}, \theta, t) d\vec{a} \quad (8)$$

The theoretical amplitude of optical pulsation is computed by converting the summed power, $\int |G|^2$, near the frequency of interest, $w_o/2\pi$, in a fashion similar to the conversion operation used on the power spectra of the real data. In effect, this is just taking the square root of the summed power times the appropriate conversion constant. The average frequency and width of the feature are obtained by taking the first and second moments of $|G(f)|^2$ near $w_o/2\pi$.

The calculation of $G(f)$ can be more explicitly demonstrated as

$$G(f) = (t_2 - t_1)^{-1} \int_{t_1}^{t_2} e^{-2\pi i f t} dt \int_A V(\cos\theta, \cos\psi) (4\pi r_x^2)^{-1} \\ \left\{ 1 + \cos \left[w_o t - w_o/c (r_x - z \cos i - r \sin i \cos \{ 2\pi (t - t_1) P^{-1} + \phi_o \}) \right] \right. \\ \left. \pm \left(\tan^{-1} [y(a_2 - x)^{-1}] - 2\pi (t - t_1) P^{-1} \right) \right\} \cdot da \quad (8a)$$

where t_1 and t_2 are the times of the beginning and end of run, V is the "visibility" function which is zero if the area element cannot be seen by either the earth or Her X-1 and is the product of $\cos\theta$ and $\cos\psi$ otherwise, θ is now the angle between the earth and the normal to the area element, ψ is the similar angle for Her X-1, P is the orbital period, w_o is the Her X-1 pulsar frequency, c is the velocity of light, x, y, z , and r are

defined with respect to the c.m. coordinates of the area element da , i is the angle of inclination of the observer to the orbital plane, r_x is the distance from da to Her X-1 as defined in Figure 75 and ϕ_0 is the initial orbital phase at $t = t_1$.

In Figure 75 the earth is understood to be rotating about the coordinate system fixed to the Roche lobe in a left-handed (clockwise) fashion.

The first term in the cosinusoidal pulsation for each area element is the usual pulsar frequency times t . The next term consists of what may be called "geometry"; the first two parts are constant delays and the third is the variable delay due to the circular motion of the area element. The last term is the result of having a pulsar beam which is either prograde (+), retrograde (-) or perpendicular (0) with respect to the orbital motion. The first part initializes the spin phase of each area element and the second term represents the loss of the spin phase due to orbital motion.

The $|G(f)|^2$ spectrum is then summed near $\omega_0/2\pi$ to get the equivalent average "instantaneous" pulsation amplitude as described above. Likewise for the average frequency and the width. The d.c. offset term which adds to the time varying cosine term is unimportant to the results of $G(f)$ near $\omega_0/2\pi$.

In practice the portion of the equipotential surface which is exposed to the x ray source, Her X-1, was divided into 5000-7000 area elements so that no area element significantly exceeded the size of 0.05 square light seconds ($0.45 \cdot 10^{20} \text{ cm}^2$) so that the linear dimensions of each area element were not much greater than $7 \cdot 10^9 \text{ cm}$.

The increments in orbital phase were taken to be the amount of phase traversed in 81.92 seconds so that a typical 2.91 hour run used 128 such

intervals in the integration.

2. Goals for the Calculations

Before proceeding further, it is advantageous to consider what one might hope to accomplish by comparing the results of the calculations with the data. Once the list of plausible questions has been made the list of parameters which will have to be used in the attempt to answer the questions will become more obvious.

For the experimental observations the first goal has already been stated — to explain the anisotropy of the results in orbital phase.

Other experimental questions which might be answered are:

2. The difference of duration of Features I and II over the 35-day phase (e.g. Figures 54 and 55)
3. The small asymmetry of velocities and accelerations of Features I and II with respect to the system center of mass (e.g. Figures 51 and 52).
4. The lack of higher harmonics in the data.

Some theoretical questions which might be answered by these calculations are:

1. What is the mass ratio of the system?
2. Which, if any, equipotential surface does HZ Her occupy?
3. What is the orbital inclination of the orbital system?
4. What is the efficiency of the process which converts x ray pulses into optical pulses:
5. What is the amount of absorption present between Her X-1 and HZ Her and how does it change with the 35-day phase?

The typical parameters which are varied for such integrations are listed at the head of the columns of Table III.

Column 1 gives the mass ratios used; these varied from 1.0 to 2.5. Column 2 gives the ratio of the distance between the point of intersection of the chosen equipotential contour with the mass axis and the HZ Her center of mass to the distance between the HZ Her center of mass and the inner Lagrangian point. Most calculations used the Roche lobe and thus most of the numbers of column 2 are 1.00; values of 0.95 and 0.85 were used occasionally.

Column 3 lists the profile of the beam strength of the pulsar in the direction normal to the orbital plane. Most calculations used a $\sin^2\theta$ modulation, where θ is the polar angle. Since the Roche lobes subtend angles which are almost always within $\pm 30^\circ$ of the orbital plane except in cases of extreme mass ratios, this modulation has little effect on the results. Moreover, the normalization to the total x ray luminosity, L_x , was not changed since the x ray flux measured from the earth almost surely represents a sample taken from close to the orbital plane because of the high orbital inclination of the HZ Her - Her X-1 system.

Column 4 gives the sense of the pulsar beam rotation: prograde, retrograde or perpendicular to the orbital plane. In the plane defined by the equator of the pulsar beam, the profile of the x ray intensity with the azimuthal angle, ψ , is given by $1 + \cos \psi$, with $\psi = 2\pi ft + \psi_0$. For the perpendicular beam sweep, the time-independent $\sin^2\theta$ modulation has been deleted.

Column 5 gives the duration of the run in units of 10485.76 seconds (or 0.0714 cycles = 25.7° orbital phase). Column 6 gives the model used

in the excitation of the surface. Most calculations use the normal instantaneous 100% efficient reprocessing (denoted ----). Some include the effects of shadows and other biases with high angles of emergence or incidence to the surface. The "low" designation refers to the deletion of the cosine factor in the direction to the observer (earth) as would be the case for the reprocessing of low energy x rays in the more easily visible parts of the photosphere. The "high" designation refers to the contrary case for high energy x rays where the cosine angle of incidence is deleted.

Column 7 gives the orbital inclination used while column 8 lists which multiple of the fundamental frequency, $f_0 = 0.8078725$ Hz, was used for the calculation. Column 9 gives the number of calculations which were made over different binary phase intervals. The majority of the groups cover 2.91 hour runs centered on the binary phases: 45° , 75° , 90° , 105° , 135° , 165° , 180° , 270° and 285° . Column 10 numbers these sets of calculations for future reference.

Columns 11 and 12 list the values obtained from comparison to the data (Features I and II) for tests utilizing a χ^2 and the product of $-\chi$ with the sine of the orbital phase ($-\chi \cdot \sin \phi$). These will be discussed below in the section relating the orbital inclination to the mass ratio of the system.

In the following discussion the calculations will be referred to by their number and/or by the parameters which differ from the following "standard" set: mass ratio = 2.0, contour = 1.00, $\sin^2 \theta$ vertical profile, prograde rotation, unit duration, 100% instantaneous reprocessing excitation model with both cosine laws included, orbital inclination = 90°

and with frequency $f_0 = 0.8078725$ Hz. Most of the calculations have parameters which differ in only one or two categories from those enumerated above.

3. Results of the Calculations

The calculations will be presented below in an order which is based on the appropriate subject of discussion rather than in the order in which they are listed in Table III (which is roughly chronological)

a. The two profiles

i) The amplitude profile

In order to explain the non-uniformity of optical pulsation in binary phase, (experimental goal number 1), the amplitude results of calculation number 39 for mass ratio $R = 1.65$ are presented in Figure 76. The ratio of optical modulation energy to pulsed x ray energy is plotted on the vertical scale which is marked in thousandths. The lower horizontal scale gives the 1.7-day orbital phase in degrees and the upper scale the same phase in cycles. The zeros correspond to the x ray eclipse phase (as do the 360° and 1.0 points). The parameters other than the mass ratio and the number of phase points calculated all have the default values (i.e. Roche contour, $\sin^2\theta$ vertical profile, prograde rotation, 2.91 hours length, instantaneous 100% efficient reprocessing, $i = 90^\circ$ and the fundamental frequency $f = f_0 = 0.8078725$ Hz).

The profile in amplitude shows a striking two-humped behavior with peaks near orbital phases 0.28 and 0.725. The trough centered on binary phase 0.5 has $\frac{1}{2}$ depth points near 0.33 and 0.67 and a *minimum more than 16 times as small as the peak amplitudes*. The phases between the peaks

and the eclipses also decline in amplitude; the half amplitude points are near 0.2 and 0.8 in binary phase. Phase points below 0.125 and 0.875 were not calculated since the approach toward zero amplitude in these regions is a simple result of the occultation of the illuminated area elements.

This profile is in excellent qualitative agreement with the measured profile plotted in the middle frames of Figures 48 and 50^{*}. The reason why the effect is so extreme can be explained with the help of Figure 72 which shows a Roche lobe for a mass ratio of 1.7.

The limbs of HZ Her at the equator are seen to approach straight lines near the inner Lagrangian point. As long as the reprocessing of the x rays is fast, then for certain directions near 90° and 270° the x ray pulses impinging on the limbs and leaving as optical pulses can satisfy a geometrical condition for spectral reflection: incidence angle = reflection angle. For these directions the pulses from the equatorial regions of the limb of HZ Her will be seen in phase, and hence the signal will be large.

When the system is viewed from orbital phase 0.5, there is no point on the star's lobe which has a surrounding neighborhood which is in phase with the point because of the path differences associated with the backward direction. This is due to the "pointed" shape of the Roche lobe. Any signals which are seen will be the result of almost complete cancellation in phase.

Conversely, if the star should not fill its critical lobe (the one connected to the saddle point) then the end of the star would be blunt and the theoretical levels of pulsations at orbital phase $\phi = 0.5$ would be

*The numerical agreement of the vertical scales is accidental, however, since the theoretical calculations have been normalized to pulsed x ray energy, while the observations have been normalized to the mean maximum optical light from the HZ Her system.

comparable to the levels at $\phi = 0.75$ and $\phi = 0.25$.

Consequently, severe experimental limits on the amount of pulsation from the limb of HZ Her at $\phi = 0.5$ such as those often present in this data (see Figures 39-43, 47-49) near 0.03%, of Groth¹⁹ at the same level and of Cocke et al.⁶⁰ are indicative that HZ Her does, in fact, fill its Roche lobe and is consequently rotating synchronously with the orbital motion (or, at least, its atmosphere is doing so).

Since actual results are regularly detected from the limbs at phases 0.25 and 0.75 (e.g. Figures 54 and 55), it is not possible that the negative results at $\phi = 0.5$ are due to x ray absorption by matter in a disk around (or in a stream flowing toward) Her X-1 since the same impediments would be present at any orbital phase. Furthermore, the detailed observations by Grandi et al.⁶¹ of the 1.7 day light curve indicate that the optical depth for light seen from HZ Her at $\phi = 0.5$ traveling past the Her X-1 region (at the time of secondary minimum) is only 0.2 in the B band. They note the agreement with the estimate of 0.3 of Strittmatter et al.¹⁷ based on mass transfer considerations. Thus the reduction in the pulsation level from HZ Her around $\phi = 0.5$ due to the secondary eclipse would be only about 25% -- and this cannot possibly account for the negative experimental results.

To convert the results of the calculations to pulsed fraction as defined in the data, the results must be multiplied by the ratio of pulsed x ray energy to mean maximum optical energy. This ratio can be derived from the results of Tanabaum et al.², Clark et al.,⁶² Giacconi et al.²⁰ and Ulmer et al.⁶³ for the x rays and the results of Davidsohn et al.,³ Forman et al.,⁴ Bahcall and Bahcall,⁵ Boynton et al.,¹² Petro and Hiltner¹⁰

and Grandi et al.⁶¹ for the optical light. The maximum optical flux is close to 10^{-10} ergs $\text{cm}^{-2} \text{s}^{-1}$ while the observed x ray flux above 1 kev is near $5 \cdot 10^{-9}$ ergs $\text{cm}^{-2} \text{s}^{-1}$. Thus this factor is roughly 50 and the pulsed fraction derived from the theoretical model with the entire limb illuminated could be as high as $0.0016 \cdot 50 = 0.08$ or 8% for the peaks near phases 0.25 and 0.75 and 0.5% for the level of phase $\phi = 0.5$. It is interesting to note that more naive models for the time-of-flight smearing like that of Avni and Bahcall⁶⁴ predict a maximum of 1.6% quoted for orbital phase 0.5. This discrepancy tends to emphasize the importance of considering the geometry in detail.

Recent results of Shulman et al.⁶⁵ indicating a substantial soft x ray flux of $3 \cdot 10^{-9}$ ergs $\text{cm}^{-2} \text{s}^{-1}$ (but only 5-20% pulsed) in a narrow band from 0.18-0.28 kev will tend to modify the x ray to optical energy ratio. In any case the numbers which are derived by ignoring the x ray-to-optical conversion efficiency (for pulsations) will tend to be too high by almost two orders of magnitude. A more detailed discussion will be presented below after the parameters for the HZ Her - Her X-1 system have been carefully established by the calculations to be discussed.

ii) The frequency profile

In order to test the frequency behavior of calculation number 39, Figure 77 has been constructed to include this behavior and also show the events of Feature I and II as plotted previously on Figure 51. The horizontal scales are the 1.7 day phase in degrees and cycles exactly as in Figure 76. The left vertical scale shows the relative velocity derived from the frequency f , $v = (f - f^0) \cdot c / f^0$ where $f^0 = 0.8078735$ Hz, and where

f has been scaled up so that the f_0 for the system is also 0.8078735 Hz. The right vertical scale plots f .

The frequency behavior of the calculation (shown by the line) is definitely not sinusoidal. "Kinks" are prominent near orbital phases 0.2 and 0.8. The reason for these "kinks" in the motion of the "centroid of pulsation" is due to the rapid gain/loss of illuminated area to the line of sight to the earth at low/high orbital phases. The centroids keep changing (if not quite as severely) since the extreme of the velocities are found near phases 0.37 and 0.63 instead of the usual phases of 0.25 and 0.75 respectively.

The data (shown by the parallelograms) are in qualitative agreement for the mass ratio of 1.65. The relatively large uncertainties in the slopes of the parallelograms and the presence of the "kinks" near both regions of interest tend to degrade the ultimate precision to which the agreement can be measured.

b. The effects of varying some parameters

i) Effects of mass ratio

To test exactly how accurately the mass ratio of the system can be determined Figure 78 has been constructed to show the frequency tracking for the mass ratios of 1.0, 1.5, 2.0 and 2.5 (calculations 16, 4, 2 and 5) against the data parallelograms. Only calculations for orbital phases 0.750 and 0.7916 have been done for the phase region 0.5-1.0. Figure 78 is otherwise similar to Figure 76.

The Figure shows that the best matches to the data must lie between the mass ratios 1.5 and 2.0. Mass ratios near 1.0 are seen to give a

velocity behavior for the center of pulsation which is in antiphase with Her X-1 (this is definitely not seen in the data).

Figure 79 shows the corresponding plot of pulsation amplitude versus orbital phase. The axes are as described for Figure 77. The higher mass ratios produce the higher amplitudes, otherwise the profiles are all qualitatively the same. Although the Roche surfaces for the lower mass ratios have more illuminated area (that is, bigger), they are farther from Her X-1 so that they actually subtend less solid angle. A more important effect may be that the size of the wavelength of the optical pulsations ($3.71 \cdot 10^{10}$ cm) relative to the dimensions of the Roche lobe has decreased and therefore the size of the areas which are capable of coherence decreases (at least in terms of solid angle subtended from the x ray source).

Figure 80 shows the frequency tracking in orbital phase for mass ratios 1.6 and 1.7 (Calculations 7 and 22) plotted against the data parallelograms. The closest match to the data would appear to be for a mass ratio value between 1.6 and 1.7. Extrapolating from the difference of the two curves, it appears that ± 0.1 may be an upper limit to the errors associated with the determination of the mass ratio.

ii) Prograde versus retrograde pulsar rotation

The above match of the calculations with the data depended in part upon the use of prograde pulsar rotation to match the asymmetry between the velocities of Features I and II. A sidereal effect on the pulse frequency will decrease that frequency by the relative amount $\delta f/f \cong (fP)^{-1}$ where f is the pulsar frequency in Hertz and P is the orbital period in

seconds. This effect is $8.4 \cdot 10^{-6}$ for the Her X-1 parameters.

The magnitude of the asymmetry which exists in the data can be measured by taking the average of Features I and II. This average is $-3.65 \pm 0.6 \text{ km s}^{-1}$ with reference to, an average f of 0.8078735 Hz .^{*} Figure 81 has been constructed to illustrate the velocity asymmetry between the prograde and retrograde pulsar models (calculations 39 and 19 for mass ratio 1.65). The measured amplitude of difference for orbital phase intervals 0.21-0.25 and 0.75-0.79 is -3.55 km s^{-1} in good agreement with the

^{*}The frequencies derived from the calculations were scaled up to 0.8078735 Hz in order to match the plots and numerical comparisons which use the same frequency for the zero-velocity reference. This probably represents a better average than 0.8078725 Hz for the two years of observations, July 1972-September 1974.

The experimental data of Giacconi⁵⁷ which overlaps part of the observing period may be found conveniently plotted in Brecher and Wasserman.⁶⁶ The average of the frequencies from the period ranging from 09 August 1972 through 03 March 1973 is, in fact, 0.8078744 Hz . This would indicate a revision of the zero-velocity reference for determining the velocity asymmetry in the experimental data of 0.34 km s^{-1} in such a way as to revise the asymmetry from -3.65 to -3.99 km s^{-1} . Any further increase in the frequency from March 1973 to September 1974 would similarly tend to strengthen the experimental constraint on the alignment of the rotation axis of the pulsar.

As for the effect of such changes on the mass ratio, Figure 80 shows that a velocity change of 5 km s^{-1} is produced for a mass ratio change of 0.1. Thus the systematic change in mass ratio introduced by the "spinning up" of the pulsar by $1.125 \cdot 10^{-6}$ would be about 0.007 parts in 1.65 and therefore completely negligible.

experimental evidence.

The theoretical asymmetry of the velocities of Features I and II for the prograde pulsar rotation of $-3.55 \text{ km s}^{-1} = 1.18 \cdot 10^{-5}$ parts is higher than that predicted from the naive sidereal argument because the centroid of pulsation moves over the Roche lobe in such a way as to amplify the effect by 40%.

Calculations 25 and 33 were constructed to test the effects of skewing the Roche lobe off the mass axis as an alternate method of generating velocity asymmetry. Calculation 33, with a 3° angle between the mass axis and the Roche axis with the lobe lagging with respect to the orbital advance, succeeded in producing approximately the same velocity asymmetry as the prograde model has produced. However, it is not obvious how such a lag could be physically induced. Moreover, such models have the side-effect of producing an unacceptably large asymmetry in their amplitude profiles (of the wrong sign from any asymmetry which could be construed as existing in the amplitude profile data).

Figure 82 shows the amplitude profiles for both the prograde and retrograde pulsar beams. The profiles are mirror-images of one another about $\phi = 0.5$ and differ appreciably only near the peaks.

iii) The effects of the contour chosen

The calculations presented so far have all assumed that the reflecting-reprocessing surface that HZ Her occupied was the critical Roche equipotential which connects to the L_1 saddle point on the mass axis. To test the reflection effect on other contours, calculations 3 and 9 were generated for contours lying with the Roche lobe. The measure used to label the

calculations is the distance from the HZ Her center of mass to the intersection of the contour with the mass axis which is between the two stars. This distance is normalized by dividing by the distance from the c.m. of HZ Her to L1. The frequency profiles for calculations 3 and 9 for the 85% and 95% "filled" lobes respectively are displayed with the profile of calculation number 2 (for a 100% "filled" lobe, $R = 2$) in Figure 83. The less filled lobes produce noticeably smaller velocity curves. This does not rule out the underfilled lobes as candidates to match the optical pulsation data since an equally acceptable fit can be obtained with an upward adjustment of the mass ratio (upward, that is, from 1.65). However, the amplitude profiles displayed in Figure 84 place severe limits on the extent of underfilling which the data can allow. The curve for the 95% filled lobe drops to only 40% of its peak value from $\phi = 0.25$ to $\phi = 0.5$. This is totally inconsistent with the data (c.f. Figures 37, 38, 43, 44 and 45 vs. Figures 40 and 41, Figures 47-50 less so because of contamination by the anomalous Feature). Figure 85 has been constructed to measure more accurately the profile from the limb of HZ Her. The mean pulsed fraction near $\phi = 0.5$ is definitely less than 40% of the peak. Other upper limits to the optical pulsation at $\phi = 0.5$ such as those of Cocke et al.⁵¹ at 0.01% can be used to extend the limit on the "filling" of the Roche lobe. The results of Groth and Nelson⁶⁷ also contain some good negative observations at orbital phase 0.5.*

The amplitude profiles of the underfilled lobes are easily explained in

* In fact, the article concludes that HZ Her did not pulse because none of the observations contained positive detections. However, the favored binary phases for Features I, II and III were not sampled in any of the observations.

relation to the geometrical interpretations given previously for the filled lobe profiles. This is that the underfilled contours have rounded ends on the mass axis which provide a flat neighborhood where the reprocessed optical response can be seen to be pulsing coherently in the direction of the earth at $\theta = 0.5$. To provide a flux equal to the filled-contour maximum of 0.16% of the x ray energy, it is only necessary that the region subtend a solid angle of $4\pi \times 0.0016 = 0.02$ steradians. Thus the radius of the contributing region need only subtend 0.08 radians; this is relatively easy for the tips of the lobes to satisfy since the tips are closest to the x ray source.

Since Features I and II are not seen over the entire 35-day cycle, it is of some question whether every negative result at $\theta = 0.5$ has significance in terms of the geometry of the system. However, arguments already presented which discussed the results of Grandi et al.⁶¹ and Strittmatter et al.¹³ indicate that some results do have significance.

iv) The effects of the frequency multiple

To evaluate the dependence of the reprocessing-reflection model on the frequency of the x ray pulses, calculations 6 and 20 have been made for $f = 2.0f_0$ and $f = 0.5f_0$ for a mass ratio of 2.0 to compare with the results of calculation number 2. The results for the velocity behavior is shown in Figure 86; the higher frequency seems to have a greater amplitude and a more exaggerated "kink" than the lower frequency. The "sidereal effect" or the difference of the prograde pulsar beam sweep from perfect anti-symmetry with respect to the horizontal line at $v = 0$ and $\theta = 0.5$ is seen to vary *inversely* with frequency as per the

preceding discussion (this is best seen by observing the location of the three points at $\theta = 0.5$ with respect to the $v = 0$ dotted line).

Figure 87 shows the amplitude profiles for the three frequencies. Several striking observations may be made. First, the response near the maximum of the reprocessing-reflection model is linearly proportional to the period of pulsation. This would correspond to the so-called "first order" effect of Avni and Bahcall.⁶⁴ Second, the response at $\theta = 0.5$ is seen to vary at least as the square of the periods. This would be characteristic of a "second order" or interference-limited surface. Finally, the response at near grazing incidence at $\theta = 0.125$ is seen to vary less than linearly with the pulse period; perhaps it is closer to a $P^{1/2}$ law.

A few conclusions can be drawn from these calculations. First, although Figure 86 shows that the higher harmonics of the pulsation frequency produces a first order change in the velocity behavior of the reprocessing-reflection effect, Figure 87 shows that the same higher harmonics are suppressed in amplitude. This implies that the higher harmonics in the Her X-1 x ray pulse profile will only make a second order difference in the velocity profile and therefore cannot seriously affect the mass ratio determinations based on a model which assumes a sinusoidal pulse profile. Since higher harmonics are not present in great strength in the x ray pulse profile (see e.g. Avni et al.⁶⁸ for the hard x rays and Shulman et al.⁶⁵ for the soft x rays) then the sinusoidal pulse profile approximation is a fortiori appropriate.

The surface of HZ Her would also tend to amplify (relatively) any subharmonics present in the incident pulsed x rays. The lack of strong evidence for subharmonics in Features I and II thus strongly disfavors

any revision of the fundamental x ray period.

v) Effects of orbital inclination

Calculations such as 19, 26, 29, 32 and 39 for $R = 1.65$ and $i = 45^\circ$, 75° , 60° , 30° , and 90° can be assembled to investigate the effects of varying the orbital inclination while keeping the other parameters fixed. The *projected* separation of Her X-1 from the system c.m. is constrained to remain constant at $3.95 \cdot 10^{11}$ cm when the inclination is varied. The profiles in frequency for the five inclinations are shown in Figure 88. The cases for $i = 90^\circ$ and $i = 75^\circ$ are seen to be nearly identical. By $i = 30^\circ$, however, the tracking loses the "kink" near $\phi = 0.2$ (and near $\phi = 0.8$) and has a lower amplitude than the tracks of the higher inclinations except for $\phi = 0.21$.

Since the center of pulsed light for the illuminated surface on a Roche lobe cannot shift rapidly when observed from above the orbital plane, the frequency tracks of the lower orbital inclinations would be expected to lack the "kink". The specific causes for the changes in the frequency tracks for the lower inclinations are threefold. First, the actual linear dimensions of the Roche lobes increase so that the interference patterns are shifted. Second, the Roche lobe is not symmetric about the mass axis, but is more flattened *above* the orbital plane than it is in the orbital plane. This increases the obliquity factor slightly more severely for the area elements further from L1.* Third, the sidereal effect is slightly changed because of the first two changes.

* The angle which the nozzle of the Roche lobe makes in the x-z plane is $\tan^{-1} 3/2 = 56.3^\circ$ for $R = 0$ or $q = 1/R = \infty$ and a few degrees less for finite ratios. The corresponding angles in the orbital plane are 60° at q or $R = 0$ and a little less in between (see, e.g. Lubow and Shu).⁶⁹

Figure 89 shows the amplitude profiles for the five inclinations. Little difference exists between the $i = 90^\circ$ and $i = 75^\circ$ cases, but by $i = 30^\circ$ the amplitude is lower by about a factor of 2 from the $i = 90^\circ$ profile near $\phi = 0.29$. The reason for the decline in pulsed amplitude for the lower inclinations is basically the decrease of solid angle which is able to pulse coherently from the surface because of the increase of the size of the lobe relative to the wavelength of the optical pulsations ($3.71 \cdot 10^5$ km). As in Figure 87, the reduction in amplitude is roughly linear with the increase in size. The amplitude near $\phi = 0.5$ increases for low inclinations because the "point" of the Roche lobe is not seen from along the mass axis; the situation is slowly changing to resemble the $\phi = 0.25$ and $\phi = 0.75$ $i = 90^\circ$ case at all orbital phases, hence the relative profile will completely flatten out by $i = 0^\circ$ (at zero amplitude because the size has become infinite).

c. Some early conclusions from the models

i) The mass ratio vs. inclination from the velocities

In order to determine how the observations constrain the orbital inclination and mass ratio, calculations 7, 8, 22, 27-28, 30-31, 40-41 and 47 were done to augment nos. 19, 26, 29, 32 and 39. The velocity-tracking results were first anti-symmetrized about the value for orbital phase 0.5 to fill in the large gaps in orbital phase. This is an accurate process because of the high degree of anti-symmetry inherent in the models (only the small effect of the pulsar beam sweep can destroy the anti-symmetry). The quality of the anti-symmetry is evident from the results of calculations 39, 7 and 22 as seen in Figures 77 and 80. The calculations mentioned above and others which were similarly augmented in orbital phase coverage were used in an interpolative fashion for a comparison with the velocities and accelerations of the 21 results of Features I and II. Table III contains the results of two tests made for the velocities. The first of these is the more or less ordinary χ^2 test

$$\chi^2 = \sum_{i=1}^N [(v_i - v_{\text{interp}}) / \delta v_i]^2 \quad (9)$$

where v_i is the velocity associated with the i th data point of Features I and II, δv_i is the associated error and v_{interp} is the velocity interpolated from the discrete values of the calculation for the orbital phase, ϕ_i , of the data element. These values are given in column 11 of Table III (when available).

The second test calculates the value

$$A = \sum_{i=1}^N (v_i - v_{\text{interp}}) (\delta v_i)^{-1} \cdot -\sin \theta_i \quad (10)$$

for each model; these values are listed in column 12 when available. The value of A tends to be positive when computed for a model with a larger than optimum mass ratio and vice-versa. Inspection of the two sets of test values obtained for close sequences of mass ratios as in calculations 51, 7, 39, 22 and 50 or in calculations 40, 26 and 27 indicates that the approach of the χ^2 test to a minimum closely follows the approach of A to 0. Accordingly, the second test could be used to determine the mass ratio and its errors for a given model when the χ^2 test could not (i.e., when only two points exist). The mass ratio is then found with a linear interpolation to a 0 A-test value, and in extreme cases, an extrapolation from just one point to a hypothetical 0 A-test value using a slope which has been established from the results of similar models (the behavior of the A values is observed to be fairly linear with the mass ratio).

The more usual procedure for determining the mass ratio for a given model is to calculate the minimum for the parabola (χ^2 for the velocities as a function of mass ratio) which passes through the three nearest models. The error is estimated by the deviation from the minimum χ^2 mass ratio which is necessary to increase the χ^2 in the ratio $(N + 1)/N$ where N is the number of data points for Features I and II. This procedure is identical to scaling up the errors on the data points of Feature I and II so that the χ^2 measure gives N at minimum and then asking what deflection in mass ratio increases the χ^2 value by unity.

The results of the above procedures applied for five values of the

orbital inclination are:

$$i = 90 \quad R = 1.647 \pm 0.027$$

$$i = 75 \quad R = 1.646 \pm 0.030$$

$$i = 60 \quad R = 1.648 \pm 0.027$$

$$i = 45 \quad R = 1.650 \pm 0.025$$

$$i = 30^\circ \quad R = 1.670 \pm 0.025$$

Another constraint on R vs. i may be obtained in the more usual manner from the duration of the x ray eclipse which has been firmly established in Giacconi et al.²⁰ at $0.24 \pm 0.01d$. With the knowledge from above that the Roche lobe must be filled, the mass ratio can be calculated as a unique function of the inclination angle.*

The curves from the limits on the eclipse are shown in Figure 90 along with the allowable region derived from the data. From the intersection of the two regions, the mass ratio is 1.65 ± 0.027 and the orbital inclination $i \geq 84^\circ$.

If these parameters are used with the mass function for the Her X-1 system from Tananbaum et al.² the results are:

$$M_{HZ} = 0.85M \cdot \sin^3 i \cdot (1 + 1/R)^2 = 2.20 \quad \begin{matrix} +0.06 \\ -0.04 \end{matrix} M_\odot$$

$$M_{\text{Her X-1}} = M_{HZ}/R = 1.34 \quad \begin{matrix} +0.06 \\ -0.04 \end{matrix} M_\odot$$

* The values obtained approach an asymptote for infinite mass ratio at $i = 30.0^\circ$ and differ significantly from those of Leach and Ruffini⁷⁰ for $i < 70^\circ$. Van den Heuvel and Heise⁵³ have also taken similar liberties with the Roche model, since they give a curve indicating finite masses for $i < 30^\circ$. The error might have resulted from the improper use of a well known work by Plavec⁷¹ which gives mean radii of the Roche lobe vs. mass ratio. Kopal⁷² also errs in assuming the asymptote is determined by nozzle at $i = 33.7^\circ$ rather than by the limiting form of equipotential for $q = 0$.

These values indicate a fairly heavy neutron star (greater than $1.2M_{\odot}$) but are consistent with the calculations of V. Pandaripande⁷³ which give an upper limit of $1.5 M_{\odot}$ by including the effects of nuclear interactions.

Other possible modifications of these values are discussed below in the sections dealing with the different models for the atmospheric reprocessing, the x ray beam intensity, the shading from an accreting disk of material and the Tidal lobe.

ii) The size of coherent areas

By eliminating parts of the illuminated surface the contribution of that surface area or of the remaining area to the total pulsed amplitude may be measured. In areas which are more or less pulsing in coherence, the resultant amplitude should be proportional to the illuminated area. In other areas where interference is the rule, the amplitude vs. illuminated area relationship may not be monotonic or linear.

In order to tell how extensive a coherent region exists for optical pulsations in the direction normal to the equatorial plane, an equatorial shadow of 0.1 radians in full width as seen from Her X-1 was used in calculation number 14. This particular shadow geometry is also an important plausible modification of the model, since any disk of material which surrounds Her X-1 would tend to shadow an equatorial region of HZ Her. Calculation number 23 was performed to gauge the coherent size along the equatorial region of HZ Her; all area elements with mass axis coordinates closer to L1 than 375,000 km have been excluded from consideration (in effect, painted black).

The amplitude profiles for the above models have been plotted with

that of calculation number 2 for comparison in Figure 91. The results for the equatorial shadow peak near $\beta = 0.25$ at roughly 55% of the unocculted value. Since the dimensions of the shadowed equatorial region are near $3.95 \cdot 10^6 \cdot 0.1 \text{ km} = 4 \cdot 10^5 \text{ km}$, it is apparent that the dimension of coherence normal to the equatorial plane is near $8 \cdot 10^5 \text{ km}$ or roughly 2.5 light seconds.

For the $3.75 \cdot 10^5 \text{ km}$ "chop" or "nose clip" geometry, the maximum results which occur near $\beta = 0.25$ rise to about 70% of the comparison. Since the angle which an equatorial slice of Roche lobe makes with the mass axis is near 60° for any mass ratio (see, e.g. Lubow and Shu⁶⁹) the cutting off of $3.75 \cdot 10^5 \text{ km}$ as measured along the mass axis has, in fact, excluded almost $3.75 \cdot 10^5 / \cos 60^\circ \text{ km} = 7.5 \cdot 10^5 \text{ km}$ of surface along the equator. The fact that pulsed radiation is still seen to the order of 70% of the original amplitude after the exclusion of such a large dimension (about 2.5 light seconds or 2.0 wavelengths) is a remarkable result indeed. Allowing for the diminishing height of the cross section of the excluded area near L1, it is safe to conclude that the actual region which contributes significantly toward the pulsed optical radiation is at least 4 to 5 light seconds in extent along the equatorial direction.

Combining the above two estimates, the area of the region which can contribute coherently to the production of optical pulsations must be at least 10 square light seconds. This result has a direct bearing on those of DM² which show that a region with one tenth of the above size could easily produce the emission observed in the He II 4686 Å and N III 4634-4640 Å features. The extra order-of-magnitude gives an important constraint on

future detailed models.* This constraint will be discussed in more detail later.

Before leaving Figure 91, it is worth noting that both of the shadowed geometries have significantly more pulsation at $\theta = 0.5$ than the unshadowed Roche lobe model. The explanation for this effect is that the region near L1 is interference-limited, any shadow which interrupts the symmetry which leads to cancellation is bound to produce more pulsation. The amplitude at $\theta = 0.5$ for the 0.1 radian equatorial shadow (0.1 RES) is almost 20% of the maximum and near 45% of the maximum for the "nose clip" model. The latter case involves reradiation from an annular boundary (with no neighboring annulus on one side to help cancel its effect); the former case has two boundaries above and below the equator which also lack their interfering neighboring regions.

It is now clear that the more severe upper limits on optical pulsation at $\theta = 0.5$ also place constraints on the extent of any kind of shadow. In fact, the 0.1 RES model becomes only marginally plausible when the current data is considered with some limits down to 0.01% pulsed fraction.

iii) The "apparent" location of the pulsations

Since the reprocessing-reflection effect is now understood as an averaging process rather than a discrete one, the location of the pulsed emission regions from HZ Her under the assumption of discreteness and rigid corotation must be subject to systematic errors. Rather than correct the data in a model-dependent way (e.g. by assuming a mass ratio and other

* These may include a more collimated pulsar beam and/or more limb-darkening for the emission features. The parameters of the atmosphere could also be made less favorable to the emission.

parameters) it is better to indicate where the "ideal" data $\Delta \tau$ be located with reference to the Roche lobe of HZ Her L_1 ; the model in question was correct. Then the locations of the real data may be compared to the locations of the "ideal" model-generated data to determine the validity of the model.

To determine where corotation would place "ideal" data according to a particular model, it is only necessary to take the predicted frequencies which are used to form the velocity profiles and the numerical derivatives of these frequencies and apply the usual linear conversions as described in sub-sections E3 and F4 of the Data chapter (Section IV).

The results of this process for the normal model for a mass ratio of 1.65 is shown in Figure 92. The axes are as described previously for Figures 53 and 63-73. The lobe is drawn for the mass ratio of 1.65. The points are plotted as tiny squares with arrows protruding to indicate the appropriate orbital phase (and the direction to the earth). The squares are seen to begin with the Roche lobe at 60° orbital phase, definitely in the forward half of the lobe. Then they recede in a uniform fashion to the far half of the lobe when a change of direction at orbital phase 120° is marked by a cusp. From this cusp within the far half of the Roche lobe, the squares proceed in a forward direction (toward the earth) with increasing orbital phase until another cusp is reached close to the inner Lagrangian point, L_1 , when $\theta = 0.5 = 180^\circ$. The boxes then back away to another cusp into the "new" far half of the Roche lobe at orbital phase 240° , before at last proceeding "forward" with increasing orbital phase to 300° . The cusps at 120° and 240° are not quite symmetrically located about the mass axis because a prograde pulsar rotation was used in the computation for the Figure (calculation number 39).

d. Systematics of illumination affecting the determinations of the mass ratio

i) The equatorial shadow

Since most models of accretion in a binary system involve the formation of some disk of material about the x ray source (which probably remains in the plane because of angular momentum conservation), it is natural to investigate how the x ray shadow due to such a disk will effect the results discussed above concerning the mass ratio of the binary system. The complementary case of x ray illumination only near the equatorial plane is also of interest not only because of what it may lend to the case of the equatorial shadow, but also because narrow pencil beams are frequently discussed with free precession of the neutron star as a possible mechanism to produce the 35-day hard x ray modulation.

Calculations 35-38, 42-46, 48-49, 52, 65-78, 84-87 and 90-97 were made to explore the effects of equatorial x ray shadows and equatorial x rays bands of illumination on various Roche surfaces. The results are too numerous to discuss in detail; a summary will be given instead.

The effects of the equatorial shadow on the amplitude vs orbital phase plot can be seen in Figure 91 by following the squares. The minimum amplitude has been seen to be about 20% of the maximum amplitude. Moreover, the peak amplitude drops to about $\frac{1}{2}$ of the unocculted value. This factor will be relevant to the discussion below concerning the conversion efficiency of x rays in the atmosphere of HZ Her.

Such a shadow will affect the determination of the mass ratio because the center of pulsation moves slightly away from the x ray source due to the loss of the region near L1; thus a higher mass ratio is needed to agree

with the observations. The best mass ratio for the model with 0.05 radians of equatorial shadow (0.1 RES) at an inclination of 90° is 1.66 ± 0.01 . For an inclination of 75° the 0.1 RES model requires a ratio of 1.71 ± 0.01 .

A larger equatorial shadow of 0.1 radians extent (0.2 RES in column 1) as in calculations 35, 44, 48, 69, 90 and 92 can be ruled out because the amplitude profile drops only to 40% of the maximum value (which is 5% of the maximum value achieved by the 0.1 RES model). Figure 93 shows amplitude profiles for the $K = 1.70$ 0.2 RES model with the 0.1 RES and regular models for mass ratio = 1.65.

ii) The equatorial band

The complementary case to the equatorial shadow -- the equatorial band -- is of interest because it is probably a typical result of a highly collimated x ray beam. Calculations 36, 38, 42, 45, 49, 66-68, 71-76 and 91 involve the use of a band of some size with the two inclinations, 90° and 75° , various mass ratios and atmospheric reprocessing models. The sizes of the bands varied from 0.1 radians full width to 0.4 radians full width.

The optimum mass ratio for a 90° inclination and a 0.1 REB model is 1.656 ± 0.029 with a χ^2 of 147.7 for the fit to the velocities. Figure 94 shows the amplitude profiles of the regular model, the 0.1 REB and the 0.2 REB models all with a mass ratio of 1.65. The amplitude profile for the 0.1 REB model (squares) falls only to a minimum of 20% of the maximum values near orbital phases 0.3 and 0.7.

The optimum mass ratio for 90° inclination and a 0.2 REB full width

full width is 1.691 ± 0.015 with a 21-point χ^2 of 133.8. The peaks in amplitude have risen from 1.28×10^{-3} for the 0.1 REB to above 2.0×10^{-3} for the 0.2 REB case--even higher than the maximum amplitudes for similar calculations with the entire lobe illuminated. This indicates that the band boundaries are now cutting out areas from the lobe which would otherwise interfere with the pulsation. Accordingly, the 0.2 REB can be taken as including practically all of the area which is important in producing the pulsation for $i = 90^\circ$. Moreover, the amplitude maxima for progressively larger EB models can be expected to approach the maxima of the completely illuminated model in an oscillatory fashion. Calculations 72, 73, 75 and 76 give an indication of a similar oscillatory convergence for the optimum mass ratio to the value for the fully illuminated model.

The 0.2 REB model cannot be ruled out of possibility on the basis of its amplitude profile, since its minimum at $\theta = 0.5$ is 10 times smaller than the maxima. It is less acceptable, however, because its maxima are nearer to $\theta = 0.1$ and $\theta = 0.7$ (as are the maxima for the 0.1 REB model) and thereby worsen the condition of agreement with the amplitude profile of the real data as shown in Figure 85.

c. The low and high energy x ray reprocessing limits and their effect on the mass ratio determination

i) The physical reprocessing of x rays into optical light

One effect which might help to shift the peaks of the amplitude profiles outward from orbital phases 0.3 and 0.7 to a better agreement with the data profile shown in Figure 85 has been suggested by the Lambs (D.Q. and F.K., private communication, 1974). The suggestion is that if the

x rays photoionize in a region of the atmosphere which is optically thin in the visible region of the spectrum, then the reprocessing will be observed as a volume effect. Thus, as in the example of ring nebulae, the cosine projection factor would be expected to weaken or vanish.

Conversely, in the high energy x ray limit, the x rays pass deep into the atmosphere without any substantial attenuation from photoionization in the visible regions (photosphere). Thus under each area element dA there exists a reprocessing region which is visible down to an approximately constant slant depth. In this case, the obliquity factor between the normal to dA and the line to the x ray source, Her X-1, $\cos \psi$, would drop out of the model calculations.

The validity of such calculations depends on a few good assumptions. The first of these is that Compton scattering of the x rays is not important. In fact, the photoionization cross section for He II is $6.85 \cdot 10^{-22} \text{ cm}^2 \text{ (keV)}^{-1}$ while the Thompson cross section is $6.65 \cdot 10^{-25} \text{ cm}^2$. With other assumptions about the physical properties of the atmosphere such as those made in DM^2 of an isothermal exponential atmosphere with $T \approx 20,000^\circ \text{K}$, the density where the 4686 Å free-free optical depth reaches unity is $6 \cdot 10^{14} \text{ cm}^{-3}$, the hydrogen is ionized to all but 1 part in 4000 and the helium is mostly He II with a few percent He I.⁷⁴ Thus the atmosphere is a plasma, where the only opacity process in the visible region is free-free self absorption. This opacity becomes unimportant for quanta with energies higher than the visible;⁷⁴ the abundance of He for the purposes of x ray photoabsorption is 0.1.

The second assumption is that the reprocessing is instantaneous. From DM^2 , it is seen that the recombination coefficients for the production of the lines observed in the pulsing narrow band data, He II 4686 Å and

N III $\lambda\lambda$ 4634 - 4640 Å , are both high enough so that the 1/3 of the optical pulsation coming from them is indeed instantaneous at the electron density of $6 \cdot 10^{14} \text{ cm}^{-3}$. The He II recombines after about 25 ms while the N IV recombines after only 1 ms. Moreover, the remaining 67% of the optical pulsation may be inferred to be instantaneous because the 33% amplitude of the two lines may be subtracted from the broadband amplitude on the complex plane to leave a vector which has half the power of the broadband result and an essentially unchanged complex phase. The new phase agreement between the broadband-minus-narrowband part and the narrowband result is still a good one and thus indicates (with better than 90% confidence of the two results) that the remaining 67% of the optical pulsation is also due to rapid reprocessing.

Continuum radiation from the deeper layers of the atmosphere can be eliminated from consideration because the calculations by Basko and Sunyaev³⁴ indicate that the cooling time for HZ Her is near 10 seconds. This cooling is coupled to a gravitational relaxation of the atmosphere whose timescale of relaxation has also been calculated by Dahab⁷⁵ at 10 seconds.

Before describing the various modifications of the calculations due to the volume effect reprocessing, it is worthwhile to examine the reprocessing mechanisms in detail for a 1 keV quantum. This quantum penetrates into the atmosphere until it is destroyed in a photoionization. At 1 keV, unit optical depth for the He II photoionization is reached when the total density of nuclei, n , is $5 \cdot 10^{13} \text{ cm}^{-3}$ (this is obtained by using the scale height $H \approx 5 \cdot 10^8 \text{ cm}$ at $T = 2 \cdot 10^4 \text{ }^\circ\text{K}$ and photoionization cross section of $4 \cdot 10^{-22} \text{ cm}^2$ at 1 keV). When x rays of higher energy penetrate deeper into the atmosphere to higher densities, neutral hydrogen, H I, may

compete with He II and He I for the photons if the HII/HI ionization temperature is nearer to 10^4 °K (at this temperature, the ratio HII/HI is near unity and thus the ratio of HI/HeI eventually goes up to 10 which almost makes up the $\times 16$ higher cross section of the He, the hydrogen then becomes competitive for the more deeply penetrating x rays with energies near 2 keV and up).

Once the photoelectron is emitted, it has a velocity near 2×10^8 cm s^{-1} and undergoes electron coulomb collisions at a rate near 10^{14} s^{-1} (also coulomb collisions with nuclei at the same rate but these do not transfer energy). From considerations of the coulomb scattering energy transfer, the electron thermalizes with the crowd of other electrons in 10^{-6} seconds at $n = 10^{14}$ cm^{-3} in a way which increases nearly linearly with the density of electrons. By considering the possibilities for collisional ionization, perhaps another atom of He II is ionized to He III by the original electron or one of its more energetic secondaries (the probability is about 5%).

Meanwhile, the He II ion (s) is recombining at a rate of roughly 8×10^{14} cm^3 s^{-1} with the electrons. At an electron density of 5×10^{14} cm^{-3} this corresponds to an average 250 ms wait; at higher densities the recombination time drops. If the luminosity of Her X-1 of about 10^{42} ergs s^{-1} were put into 1 keV x rays, an average flux of 3×10^{23} keV cm^{-2} s^{-1} is seen at the surface of HZ Her (this flux can also be obtained by scaling the observed x ray flux of 10^{-8} ergs cm^{-2} s^{-1} for a distance of 3-4 kpc). This is enough to ionize every He atom to a depth where the total electron density is 10^{14} cm^{-3} once a second. Thus the possibility exists that the heavy elements in the upper atmosphere are kept at a sustained ionization by the x ray flux and cannot participate in the x ray pulse reprocessing.

The increasing recombination capability of the He II and He III with increasing density prevents this from happening at the modest densities (at $5 \cdot 10^{11} \text{ cm}^{-3}$ electron density each He atom can destroy 10^{10} x ray quanta per second).

The energy of the photoelectron which has thermalized with other electrons does not catastrophically raise the overall electron temperature since this temperature will adjust so that the free-free reradiation rate will roughly balance the time-averaged x ray input. The time needed for the electrons to come into thermal equilibrium with the nuclei via coulomb interaction is roughly $5 \cdot 10^{-4} - 5 \cdot 10^{-3}$ seconds at $n = 10^{14} \text{ cm}^{-3}$ (about 500 times as long as the $e^- - e^-$ transfer time). Thus it is likely that the transfer of the x ray's energy into thermal energy of the stellar atmosphere can occur on timescales shorter than the pulse period for the layers of modest density. The radiative cooling times for these layers of the photosphere will be discussed in more detail below.

Once the He III ion recombines into He II, the electron cascades rapidly down to the ground state, emitting various lines with most of the energy being lost in the final few ultraviolet transitions. Meanwhile the Paschen alpha (4 - 3) quantum at 4686 Å is usually created following all recombinations onto levels higher than $n = 4$. Half of such visible photons will suffer annihilation by the free-free self absorption process since they will be emitted toward HZ Her. The other half of the emitted quanta will usually escape if the electron density of their birthplace is less than $6 \cdot 10^{14} \text{ cm}^{-3}$ (i.e., $\tau = 1$ from DM²). As the density increases into the atmosphere and the hydrogen population becomes less ionized, the free-free opacity must decrease in proportion to other opacities associated

with line emission and absorption. The presence of the Balmer absorption lines in all spectra of HZ Her tends to support the existence of areas on the Roche lobe where HI exists in the denser layers of the photosphere. In the areas where heating from the x ray flux is taking place, the Balmer absorption may be enhanced by the increase of the $n = 2$ population level in the HI caused by the heating or by electron pumping from the ultraviolet from the recombination processes of the heavier elements.

ii) Calculations of the reprocessing limb-darkening law

To attempt a computation of the reprocessing-reflection limb-darkening law, let $\vartheta(\infty) = L_x \cdot (4\pi r_x^2)^{-1}$ represent the x ray flux seen above the atmosphere of HZ Her at a distance r_x from an x ray point source of luminosity, L_x . Furthermore, let σ_{PI} represent the photoionization cross section of the ion species in question (e.g., He II), α the numerical abundance of this ion species (assumed to be constant), H the scale height of an exponential atmosphere, and ψ the angle between the normal to the surface element dA and the vector to the x ray source. Neglecting photoionization losses due to other species, the flux which penetrates from ∞ to height h is

$$\vartheta(h) = \vartheta(\infty) \cdot e^{-\alpha \sigma_{PI} H \sec \psi n_0} e^{-h/H} \quad (11)$$

where $n_0 e^{-h/H}$ is the numerical density of electrons and/or Hydrogen nuclei at the height h .

The power in visible radiation emitted in a slab of dimensions $dA \cdot dh$ is proportional to $\vartheta(h)$ and is given by

$$dP(h) = \alpha \sigma_{PI} n_0 e^{-h/H} dA dh \vartheta(h) E_\lambda / E_x \quad (12)$$

where E_{λ}/E_x is the ratio of energy produced in the visible line at wavelength λ to the energy of the x ray annihilated in the photoionization.

The half of this converted power which is directed upward has to fight the free-free absorption process to get out. Letting θ represent the angle of emergence (relative to the normal of dA), $\Sigma_{ff} = k_{ff}/n^2$ represent the linear free-free absorption coefficient with the density dependence removed (units of cm^{-1} -- it will be called the free-free absorption "hypersection"), the radiated flux which is seen above the atmosphere is given by

$$d\mathcal{G}(h, \theta) = dE(h) (4\pi r^2)^{-1} \cdot e^{-\left(\Sigma_{ff} n_0^2 H/2 + \sec \theta\right)} \cdot e^{-2h/H} \quad (13)$$

where r is the distance from the volume slab $dA = dh$.

To get the contribution of all slabs beneath the surface element dA , the integral over h is computed (while θ is taken to be large enough to be constant):

$$d\mathcal{G}(\theta) = L_x / (4\pi r_x^2) \cdot E_{\lambda}/E_x \cdot (4\pi r^2)^{-1} a \, dA \, n_0 \sigma_{PI} H^{\frac{1}{2}} \frac{\alpha}{\beta} e^{-\alpha^2/4\beta} \text{erfc}\left(\frac{\alpha}{\sqrt{\beta}}\right) \quad (14)$$

where $\alpha = n_0 \sigma_{PI} H a \sec \psi$, $\beta = n_0^2 \cdot \Sigma_{ff} H/2 \sec \theta$, and $\text{erfc} = 1 - \text{erf}$ is the error function complement. In the limit when $\frac{\alpha}{\sqrt{\beta}}$ is large, i.e. where $\sigma_{PI} \cdot H \sec \psi \gg 2\sqrt{\Sigma_{ff} H \cos \theta}$ this becomes

$$d\mathcal{G}(\theta) = L_x \cdot (4\pi r_x^2)^{-1} E_{\lambda}/E_x \, dA \cdot (4\pi r^2)^{-1} \cdot \cos \psi \quad (15)$$

This is the typical limit for the "volume" effect reprocessing which occurs for low energy x rays.

When $a_{PI} H \sec \psi \ll 2\sqrt{L_{ff} H \cos \theta}$ or in the high energy x ray limit, the flux becomes

$$d\theta(\theta) = L_x (4\pi r_x^2)^{-1} E_\lambda / E_x \cdot dA (4\pi \rho^2)^{-1} \cdot a_{PI} \sqrt{\pi H \cos \theta} / \sqrt{2 L_{ff}} \quad (16)$$

This case is not too important since the flux is small in proportion to $\frac{\alpha}{\sqrt{\beta}}$;

$$d\theta(\theta) = L_x (4\pi r_x^2)^{-1} E_\lambda / E_x \cdot dA (4\pi \rho^2)^{-1} \sqrt{\pi} (\alpha/2\sqrt{\beta}) \cos \psi \quad (17)$$

Such model calculations ignore several important effects. The first effect is the presence of other species such as N III with a different abundance (10^{-4}), a different cross section ($10^2 - 10^3$ higher than that of He II at the same energies) and a different recombination coefficient ($1.6 \cdot 10^{-12} \text{ cm}^3 \text{ s}^{-1}$ for N IV to N III -- about 20 times that for He III). Moreover, emission lines such as H α (6562 Å) encounter a much greater free-free opacity than blue-green lines like N III λ 4634-4640 Å and He II λ 4686 Å. All of these effects will tend to make the broadband optical pulsation from HZ Her more complicated than the above model. Even if the reprocessing-reflection limb-darkening law were to be measured only for one emission line, the number of x ray photons destroyed by photoionization with the other species would still affect the results.

It is interesting to note that the radiation law of equation 16 goes

like $\sqrt{\cos \theta}$ as opposed to the usual $\cos \theta$ of the Black Body law. This is because the free-free opacity dependence upon the square of the density favors a longer, more oblique path through lower density than a shorter, more direct path into higher density.

For the purpose of convenience, the ordinary $\cos \theta$ law was used in the calculations to represent the high energy x ray limit. The low energy x ray limit, which is insensitive to the various sources of inaccuracy quoted above, is represented in the calculation with just the $\cos \psi$ law.

The numbers for the calculations of the high energy x ray reprocessing limit have not been attenuated to compensate for the fact that most of the converted energy is not seen from the earth. Again, a small fraction of the "unseen" converted energy below one optical depth will be rapidly re-radiated thermally by the sudden but slight increase in the temperature of the deeper layers. Although this optically thick "deep" reradiation is distinct from the "high energy" reprocessing, the use of the $\cos \theta$ law is probably a reasonable representation for both processes. The rapid "deep" process may, however, include some dependence on $\cos \psi$.

Calculations 59-62, 81-82, and 84-87 have been performed to measure the optimum mass ratios and amplitude profile changes for inclinations 90° and 75° for the low energy x ray limit (the boundary of this limit would seem to be near 2 kev which reaches a unit optical depth for He II at $4 \cdot 10^{14} \text{ cm}^{-3}$ electron density). The optimum mass ratio for a 90° inclination is 1.678 ± 0.029 with a 21 point χ^2 of 145.7 for the velocity profile. The mass ratio at $i=75^\circ$ is revised from 1.646 to 1.682 ± 0.029 with a 21 point χ^2 of 148.4 for the velocity profile.

For the high energy model, calculations 57-58, 63-64, 79-80 and 94-95

have been performed to help determine the optimum mass ratios. The revisions in the mass ratios are to 1.666 ± 0.0253 , $\chi^2 = 103.2$ and 1.676 ± 0.030 , $\chi^2 = 110.9$ for $i = 90^\circ$ and $i = 75^\circ$ respectively. Of course all of the errors mentioned have been derived from the process referred to above where the data errors have been scaled up so that the actual χ^2 values go from 20-21 when the 1σ limits are reached. The χ^2 values for the high energy model are lower than those for the low energy model. However, later detailed calculations will show that the low efficiency associated with the high energy x ray reprocessing makes the model difficult to reconcile with the level of the observed optical pulsations.

The frequency profiles for both the high and low energy x ray reprocessing models differ very little from those of the ordinary models and thus no Figure is shown.

The amplitude profiles for calculations 39, 59, 80, 85 and 95 (1.65 regular, 1.65 low energy, 1.65 high energy, 1.70 0.1 RES low and 1.71 0.1 RES high) for inclination 90° are shown in Figure 95. The profiles for high energy x ray limits (squares and hourglasses) are seen to tower over the regular model; in the high energy case the artificial nature of the amplitude has already been discussed. The maxima of both of these models have moved closer to $\phi = 0.25$ and further away from $\phi = 0.3$. For similar cases with the 0.1 RES model added, a striking difference appears between the low and high energy cases (the horizontally split squares and the vertically split squares respectively). The high energy profile appears to be acceptable to the data of Figure 85, with a maximum almost exactly at $\phi = 0.25$ (and another which lies to the right of $\phi = 0.75$). The biggest weakness of the high energy 0.1 RES case is the value of the

amplitude at $\phi = 0.5$ where it attains 20% of the maximum value. On the other hand, the low energy 0.1 RES case has a profile which actually flattens out from $\phi = 0.125$ to $\phi = 0.2083$ (and whose maximum is no further shifted toward $\phi = 0.25$ than the regular low energy model). A qualitative explanation of this phenomenon is that the boundary on the surface of the lobe which separates the part which can be "seen" from the earth from that which is not visible is a finite discontinuity for the low energy reprocessing. Such a "raw" boundary then acts to interfere with the usual coherent areas to different degrees at different orbital phases. Apparently, the effect is not pronounced until a disk shadow eliminates some of the usual coherent area.

A similar "raw" boundary exist in the high energy x ray model, except that the $1/r_x^2$ dependence softens the effect so that it is only detected when the lobe is seen end on at $\phi = 0.5$ when the boundary appears as approximate annulus. For orbital phases other than $\phi = 0.5$ the plane of the annulus is tilted so that the boundary can no longer act in coherence.

Figure 96 has been constructed with models for $i = 75^\circ$ for mass ratio 1.65 regular, low and high energy, and mass ratio 1.75 0.1 RES for the low energy and 1.70 0.1 RES for the high energy x ray reprocessing models (calculations 26, 61, 63, 87 and 97). The flattening effect of the 0.1 RES low energy model is more pronounced at this inclination. The 0.1 RES high energy model also shows some degree of flattening. The amplitudes of the models for $i = 75^\circ$ with the 0.1 RES are also only 40% as great as those of the unocculted models for the high and low energy x ray reprocessing limits. The corresponding ratio for the $i = 90^\circ$ cases is 60%. The reason for this unexpected decline (since the 0.1 RES would not have been expected

to affect the $i = 75^\circ$ cases as strongly as the $i = 90^\circ$ case) is that it is due to the destruction of the symmetry about the orbital plane which exists in the $i = 90^\circ$ cases and doubles the area which contributes to the coherence of the pulsations.

The unacceptable flatness of the shadowed profiles for the low energy x ray limit may eventually be used to constrain the size of any shadow of an accreting disk about the x ray source. The results of DM² which indicate that 1/3 of the optical pulsation from the limb of HZ Her is associated with two emission lines help to indicate the appropriateness of the low energy x ray reprocessing models while the results of Shulman et al.⁶⁵ help to confirm the existence of an enormous low energy flux from Her X-1 ($3 \cdot 10^{39}$ ergs cm^{-2} s^{-1} in a 100 ev band from 180 ev to 280 ev).

If, instead of a disk shadow, the shadow of an accreting stream of gas as sketched in Pines et al.²⁴ and computed by Prendergast and Taam⁷⁶ and more extensively computed by Lubow and Shu⁶⁹ were used, then an asymmetry in the amplitude profiles between $\phi = 0.125$ and $\phi = 0.875$ could be produced which is similar to the asymmetry in the amplitude profile which is seen in Figure 85. This is due to the asymmetric nature of such a "following" stream. If such is the case, then the shadow of the stream would perhaps be more important than any disk shadow; otherwise the presence of the stream could not make such a crucial difference.

The application of the shadowed low energy x ray model to account for the data suffers from the drawback that its main effect is to reduce the amplitude at $\phi = 0.21$ much more than it enhance the amplitude at $\phi = 0.125$. Since the real limb darkening law is probably more complicated than the simple models already calculated, the exact profile of the pulsation

amplitude can probably be reproduced by suitable revisions in the model calculations (which may ultimately be too expensive to handle because of the added complication).

f. Tidal lobes

Since the data which has previously been discussed cannot a priori rule out the possibility that HZ Her may not be exactly corotating with the binary system, Tidal lobe calculations 98-103 were done to determine the mass ratio for the $i = 90^\circ$ and $i = 75^\circ$ cases. In practice, the torque from Her X-1 on just 0.1% of HZ Her's mass located at opposite ends of the star and just 3° off the mass axis would be enough to speed up all of the mass of HZ Her to corotation in only 5 months. Since the outer atmosphere of HZ Her is not likely to be strongly coupled by viscous or convective effects, it would be accelerated to corotation all the more quickly -- a non-corotating interior for HZ Her is then irrelevant to the accretion process. However, the tidal calculations will serve as the second of the two points needed in any interpolation or extrapolation near the Roche lobe models.

The mass ratios derived from optimum fits of the velocity data to the tidal model for inclinations 90° and 75° are $1.503 \pm 0.027 \chi^2 = 183.5$ and $1.507 \pm 0.028 \chi^2 = 187.3$, both are for the low energy reprocessing limit. Thus the full Tidal model gives a lower mass ratio by about 0.175 than the Roche model, mainly because the critical tidal lobe is closer to the compact source for any given mass ratio. This difference indicates that if HZ Her were not corotating by 5% (so as to complete 1 more rotation in a 14.58 day interval) then the resulting computed mass ratio change would

be less than 0.01 out of 1.68 -- about 1/3 of the usual quoted errors.

The amplitude profiles produced by the tidal model for mass ratio 1.50 at $i = 90^\circ$ and $i = 75^\circ$ both in the low energy x ray reprocessing limit are shown in Figure 97. The profiles are peaked much more sharply than their counterparts of the Roche model (the squares of Figures 95 and 96) and both are peaked more toward $\phi = 0.25$ than $\phi = 0.3$.

The critical tidal lobe for mass ratio 1.50 is plotted in Figure 98 in a fashion similar to the Roche lobe of Figure 92, including the boxes denoting where the corotation assumption would locate the frequency tracking of the model. The lobe of HZ Her has been abbreviated by the boundary of the plotting frame.

The tidal lobe of Figure 98 shows a few qualitative differences from the Roche lobe of Figure 92. The first difference is that the equipotential about Her X-1 is significantly smaller and more flattened normal to the mass axis than its counterpart about HZ Her. Moreover, a nozzle protrudes from the Her X-1 lobe to the inner saddle point. In fact, a similar nozzle protruding from the HZ Her lobe does exist; however, it is too small for the plotting grid to reveal (this grid is about 3.6° as seen from the center of each lobe).

On the other hand, the shapes of both lobes about HZ Her for Figures 92 and 98 are remarkably similar. Thus it is not obvious why the amplitude profiles of Figure 97 are qualitatively different from their counterparts in Figures 95 and 96; the difference must be due to the finer differences in the geometries.

4. Quantitative Experimental Considerations and the Model Calculations

a. Reflection heating and pulsed efficiency

The reflection effect due to the heating of the surface layers of Her by the x rays has been calculated in a similar (but much simpler) way to the calculating of the pulsed reflection. The radiation which is seen to come from an area element dA of the surface is assumed to be proportional to the entering flux $\cos \psi / (4\pi r_x^2)$ and also proportional to the obliquity factor $\cos \theta$ to the observer.

The profile obtained for a mass ratio of 1.6 is shown in the top section of Figure 99. The result is similar to that of Milne⁷⁷ and as modified by Sen⁷⁸ which is displayed by Wilson¹³ in an argument for a highly convective envelope because of the reflection model's failure to reproduce the real light curve such as that of Davidsen et al.³ The similarity of the upper part of Figure 99 to the upper part of Wilson's Figure 1 is notable. Other authors have argued that the actual light curve is the result of this reflection law coupled with an extended luminous source extending to and/or around Her X-1.

The maximum of this heating effect is just about 3% of the x ray luminosity at orbital phase $\phi = 0.5$. However, by phases $\phi = 0.25$ and $\phi = 0.75$ the amplitude has fallen to 0.86% -- about $\frac{1}{4}$ of the maximum. When the amount of pulsed modulation (using the normal model at $R = 1.60$) is normalized to this profile to calculate an efficiency of conversion, the resulting curve is shown in the middle of Figure 99. The efficiency starts at 0% about $\phi = 0.5$ and doesn't rise until below $\phi = 0.35$ and above $\phi = 0.65$. Near $\phi = 0.20 - 0.25$ and $0.75 - 0.80$ the efficiency curve is flat at about 20%. Shortward of phase 0.15 and above $\phi = 0.85$ the efficiency rises quickly

to 100% (beyond these phases the efficiency is no longer defined because the heating-reflection has gone to 0).

Thus the amount of modulation of heating energy which can be pulsed at orbital phases near 0.25 and 0.75 is 20% with the original reprocessing-reflection model for a mass ratio of 1.60 and an inclination of 90° . *This is a purely geometrical result and it should be valid at every wavelength of the visible spectrum.* In particular, it should apply to the He II λ 4686 Å and N III $\lambda\lambda$ 4634-4640 Å emission lines. However, the results of DM² indicate that these lines are at least 25% pulsed and probably 60% pulsed. The discrepancy between the modulation fraction of the emission lines allowed by the calculations and the amount observed worsens if the low energy x ray reprocessing limit is used because of the greater relative gain in emission volume for the surfaces which are above and below the equatorial plane and which do not pulse coherently as the more dominant equatorial regions do. The d.c. emission profile for the low energy x ray limit is shown in Figure 100 which plots the reflection effect vs orbital phase in a fashion analogous to the top of Figure 99. The major features of this profile are a broad, flat maximum of 4.2% from orbital phase 0.475 to 0.625 which drops abruptly until slight "shoulders" are encountered at phases 0.25 and 0.75 with the reflection near 2.1%. The profile again drops abruptly to 0 near orbital phases 0.125 and 0.875.

This profile can be compared to the dashed line at the top of Figure 3 in Crampton and Hutchings⁷⁹ which gives the average absolute emission intensity of the N III $\lambda\lambda$ 4634 4640 Å line. The profile of Figure 100 compares equally favorably with that of Figure 99 to partially match the emission data profile. The emission data is thus not yet good enough to

to determine whether the N III emission follows a $\cos \theta$ law for limb darkening or not.

When the profile of Figure 100 is compared with the pulsed reflection efficiency of Figure 95 (squares), the ratio of ac reprocessing to d.c. reprocessing has dropped to 12%.

Calculations indicate that if the illuminated region of the star were restricted as in the 0.2 REB model, the d.c. reflection would drop to 0.63 - 0.66% for the low energy reprocessing model between $\theta = 0.21$ and $\theta = 0.25$. Thus the pulsed modulation fraction would increase to 35-45%. This is the *minimum* restriction that the results of DM^2 impose on the x ray beam -- unless a much higher percentage of the continuum pulses between 4616 Å and 4716 Å.

b. The efficiency limits of pulsed emission reprocessing

As mentioned above, the maximum amplitudes given by the various model calculations are critical to the explanation of the degree of pulsation. This section will investigate the quantitative amount of optical pulsation observed and compare it to the amount of pulsation expected from the x ray spectrum.

The mean maximum magnitude of HZ Her which is used as a pulse counting reference for the EMI 9658R photomultiplier tube and its S20 photocathode is 13.0 ± 0.1 as judged from the works of Davidson et al.³ and others^{4,5,10-12} and is consistent with the comparison measurement of other standards. An integration of the S20 - EMI 9658R quantum efficiency curve of Figure 6 yields a 2700 Å (3000 Å - 6000 Å) average response of 21% which is also the quantum efficiency near 4800 Å. A more restricted average or a more inclusive integration which is normalized to a 2500 Å bandwidth may produce

an average of 25% which is typical for the value at 4400 Å. This is in the center of the B band (3900 - 4900 Å fwhm, op. cit. p 201⁷³) where 13.0 magnitudes corresponds to the specific flux near 4400 Å of $4.27 \cdot 10^{-11}$ ergs $\text{cm}^{-2} \text{ s}^{-1} \text{ Å}^{-1}$. If this is scaled up to a bandwidth of 1500 Å, the integrated flux used in pulse counting normalization for the EMI tube is approximated by

$$\phi_{\text{opt}} = 1.0 \cdot 10^{-10} \text{ ergs cm}^{-2} \text{ s}^{-1}. \quad (18)$$

This approximation is affected somewhat by the U - B of -0.8 characteristic of HZ Her at maximum light since the U - B of T = ∞ approaches -1.33 and this limit gives a photon number law which varies inversely as the wavelength cubed (the Rayleigh limit). Correcting proportionately from a U - B of -1.33 corresponding to a quantum number density in the U of $(4400/3600)^3 = 1.82$ times the B number density to a U - B = -0.8 one finds that the number of photons per Å at 3600 Å is only 10% more than the number per Å at 4400 Å. Moreover, the HZ Her B - V = 0.1¹⁰ also tends to similarly flatten the number density spectrum out to the V band (B - V = 0.67 roughly indicates the same energy fluxes for the B and V bands). In conclusion, a number fraction for photons near the center of the B band detected by the EMI 9658R S20 is likely to be equivalent to the energy fraction for the same photons.

The bolometric correction of -2.2 magnitudes for HZ Her at maximum which was appropriate for the energy balance arguments of McClintok et al.²⁷ is inappropriate to apply to the ϕ_{opt} defined above since the black body energy maximum per Å is located near 1500 Å (the -2.2 magnitudes would, in

fact, make the above ϕ_{opt} of $1 \cdot 10^{10}$ ergs $\text{cm}^{-2} \text{s}^{-1}$ consistent with the McClintok et al.²⁷ value of $9 \cdot 10^{10}$ ergs $\text{cm}^{-2} \text{s}^{-1}$). The amount of energy seen to be pulsing is therefore

$$\phi_{\text{opt pulse}} = 1.6 \cdot 10^{-3} \phi_{\text{opt}} = 1.6 \cdot 10^{13} \text{ ergs cm}^{-2} \text{ s}^{-1}$$

(19)

The x ray flux available to produce such optical pulsations is highly uncertain, since the exact nature of the spectrum of the pulsing flux below 2 kev is not known. However, from the results of Shulman et al.,⁶⁵ the pulsing flux of $3 \cdot 10^{10}$ ergs $\text{cm}^{-2} \text{s}^{-1}$ in the 0.18 - 0.28 kev band and its similarity in pulsed fraction to the flux in the 0.5 - 1.0 kev band justify its extrapolation into the 0.18 - 2 kev band giving a total pulsed flux of $5 \cdot 10^9$ ergs $\text{cm}^{-2} \text{s}^{-1}$ with an average efficiency of 1.3 kev^{-1} in the conversion of x ray energy into emission line energy (calculated by integrating over the quantum density for an assumed flat spectrum and normalizing to the bandwidth -- $e = \int_{E_1}^{E_2} dE/E \cdot (E_2 - E_1)^{-1}$). The pulsed flux available from 2 - 30 kev¹⁸ of $5 \cdot 10^9$ ergs $\text{cm}^{-2} \text{s}^{-1}$ converts to emission lines (most of which are not seen) only with an average efficiency of 0.1 kev^{-1} ; however, this flux is much more important in the deeper heating of the star.

Assuming an average energy of 2.5 ev per emission photon, similar to the energies of the He I λ and N III emission lines, and taking the fraction of x ray energy resulting from interference over the whole Roche surface to be $2.5 \cdot 10^{-3}$ for the low energy x ray limit, the optical pulsed energy which should be visible from earth is

$$\begin{aligned}\phi_{\text{pulsed opt theory}} &= 2.5 \cdot 10^{-3} \cdot 5 \cdot 10^{-9} \cdot 1.3 \cdot 2.5 \cdot 10^{-3} \\ &= 4 \cdot 10^{-14} \text{ erg cm}^{-2} \text{ s}^{-1}\end{aligned}\quad (20)$$

This is only 25% of the value obtained from experiment. Secondary ionization from photoelectrons and ultraviolet quanta of the He and N recombination will tend to increase this number somewhat. The He II recombinations which emit λ 4686 Å also emit a 40.8 volt quantum which will subsequently ionize a He I atom. When this He II recombines, optical lines are occasionally produced and another quantum of 20 volt energy will be emitted. This quantum will either escape outward or travel inward to ionize an H I atom. Subsequent recombinations will yield an H α quantum.

The N IV recombinations which emit $\lambda\lambda$ 4634-4640 Å quanta must also decay by one of the three modes $3p^2P^o \rightarrow 2p^2D\lambda$ 691.3 Å, $3p^2P^o \rightarrow 2p^2S\lambda$ 871.9 Å or $3p^2P^o \rightarrow 3p^2P^o \lambda$ 1002.6 Å (the $3p^2P^o \rightarrow 3s^2S\lambda$ 4100 Å is unlikely). The first two decays will emit quanta which, if they do not escape, will ionize an H I atom and an H α quantum will be subsequently emitted. Thus an increase of 50% in the optical pulsation emission efficiency is reasonable from secondary ionization due to the associated UV quanta. However, only the secondary *collisional* ionization is effective in increasing the above value to the 33% of $1.6 \cdot 10^{-13}$ ergs $\text{cm}^{-2} \text{s}^{-1}$ found to be associated with the He II and N III emission by DM¹.

If the remaining 67% of optical pulsations are due to line emission processes, either a substantial pulsing x ray flux below 0.18 kev must be present or the secondary ionization processes must be much more efficient or the x ray flux between 0.18 and 2 kev must be more than 10% pulsed at the times when pulsations are observed from HZ Her. The last possibility

follows automatically when the He and N lines are observed to be highly modulated since it would be impossible for an x ray beam to excite a secondary emission feature which was more modulated than the original beam itself.

The remainder of the optical pulsation could be due to the "heating" reprocessing in the photosphere. Otherwise, it would have to come from a very large number of emission features with small equivalent widths (since none other than the He II $\lambda 4686$ Å, N III $\lambda \lambda 4634-4640$ Å and N V $\lambda 4604$ Å have been detected).

It is difficult to reconcile significant Balmer emission with the observed low level of He II and N III emission because the ability of the latter ions to photoabsorb and recombine in the optically thin regions of the photosphere is much higher than for the hydrogen. In addition, the 2nd level population would be strongly modulated by the thermal changes accompanying the x ray pulses so that, in fact, *more absorption* would occur in step with the pulses. This process would tend to cancel the emission pulsation from the N III and He II emission lines. If this were true, then the *continuum* pulsation could be as high as the full $1.6 \cdot 10^{-13}$ ergs $\text{cm}^{-2} \text{s}^{-1}$ which is observed in the broadband.

c. The pulsed heating reprocessing

To investigate the nature of the heating reprocessing, the equation for the balance of energy flow through an area element dA of the surface down to regions where the density is n protons cm^{-3} with the scale height set at H cm is given by

$$dA \cdot (F_0 \cos w_0 t = T' \cdot \alpha \cdot k \cdot n \cdot H + \sigma T^4 - \epsilon - F_0) \quad (20)$$

where the incident x ray flux is $F_0 (1 + \cos w_0 t)$ ergs $\text{cm}^{-2} \text{s}^{-1}$, S is the outward flux of energy from the normal "F" star process in HZ Her, w_0 is the Her X-1 pulsar frequency and $\alpha \cdot k$ is the heat capacity in ergs/ $^\circ\text{K}$ with α some constant of order 1 - 10 and k is the Boltzman constant. T represents the response of the "effective" temperature in time and the layers are assumed to be close to isothermal out to the photosphere where $\tau \leq 1$.

Letting $T = T_0 + T_1(t)$, the steady state equation for T_0 is

$$\sigma T_0^4 = S + F_0 \cong F_0 \quad (21)$$

A reasonable number for F_0 results in scaling up 10^{-6} ergs $\text{cm}^{-2} \text{s}^{-1}$ from 10^{22} cm to $4 \cdot 10^{11}$ cm which gives roughly 10^{13} ergs $\text{cm}^{-2} \text{s}^{-1}$. Considering the surface flux from a star of $2.5 M_\odot$ gives $S = 10^{11}$ ergs $\text{cm}^{-2} \text{s}^{-1}$ which is only 1% of F_0 . T_0 from equation 21 is about 20,500 $^\circ\text{K}$. Letting $T_1 = T_{10} e^{i w_0 t} + T_{11}(t)$ assuming that $T_{10} \gg T_{11}$, the first order time-dependent equation becomes (in complex convention)

$$F_0 = i w_0 T_{10} \cdot \alpha k n H + 4 \sigma T_0^3 T_{10} \quad (22)$$

Equation 22 also assumes that $T_0 \gg |T_1|$ in removing the non-linear terms.

The solution for T_{10} using Equation 21 to eliminate F_0 is

$$T_{10} = T_0 \cdot (4 + i\omega_0 \alpha k n H T_0 \cdot (\sigma T_0^h)^{-1})^{-1} \quad (23)$$

Letting $\alpha = 3$, $n = 6 \cdot 10^{14} \text{ cm}^{-3}$ and $H = 5 \cdot 10^8 \text{ cm}$ gives $T_{10} = T_0 / (4 + 1.5i)$ or $T_{10} = 5000^\circ \text{K}$ -- not such a small result for the photosphere. Deeper layers with exponential increases in n will give smaller relative modulation.

For the photosphere itself, Equation 20 could have been written for thin, free-free reradiation yielding

$$F_0' \cos \omega_0 t = T' \alpha k n H + 1/3 \alpha_{ff} n^2 H / 2 \cdot T^h - F_0' - S \quad (24)$$

where $1/3 \alpha_{ff} = 0.475 \cdot 10^{-27} \text{ erg cm}^3 \text{ s}^{-1}$ is the coefficient for free-free radiation into $4\pi/3$ steradians and F_0' is the x ray flux absorbed in the layer. For both Equations 20 and 24 the respective cooling times are useful for indicating how the different layers respond to the sinusoidal pulsing. The cooling times are calculated by dividing the thermal energy constant of the gas by the respective energy loss rates

$$t_{cBB} = \alpha k n H T_0 (\sigma T_0^h)^{-1}; t_{cFF} = 6 \alpha k T_0^h \alpha_{ff}^{-1} n^{-1} \quad (25)$$

where the free-free cooling time has been calculated for radiation outward into $4\pi/6$ steradians. The values obtained for the cooling times are

$$t_{cBB} = 0.3 \text{ sec}; t_{cFF} = 0.6 \text{ sec} \quad (26)$$

These two cooling times are consistent with that mentioned by Basko and Sunyaev³² for their photosphere boundary, r_a . As deeper layers are

considered, only t_{cBB} is correct, and this scales up with the density, n .

Therefore T_{10} can indeed be significant in comparison to T_0 in the photosphere since both the Black Body and free-free models indicate that the temperature can respond quickly to the x ray pulsing.

The radiative terms of both Equations 20 and 24 are non-linear and should produce higher harmonics in the response of the solution, $T(t)$. These higher harmonics would tend to be suppressed by the averaging process over the Roche lobe. Neither Equation 20 nor Equation 24 produced any subharmonics when numerically integrated, and only Equation 20 produced a higher harmonic with amplitude greater than 10% of the fundamental (this was the $2f_0$ term which had an amplitude equal to about 12.5% of the fundamental).

It is known that the Mathieu functions⁸⁰ or solutions to the equation

$$T'' + [a - 2q \cos(2wt)] \cdot T = 0 \quad (27)$$

will produce subharmonics for certain characteristic values $a(q)$. The second derivative term can only arise from a non-dissipative energy exchange such as gravitational relaxation. Dahab⁷⁵ has estimated this relaxation time to be on the order of 10 seconds; thus subharmonic response would not tend to be favored in HZ Her because the frequency of excitation is too high, in spite of the countervailing tendency of the large size of HZ Her to favor the longer period oscillations.

When the deeper layers of the star are considered, n increases and with it the required cooling times. Also the imaginary term in the denominator of the right hand side of Equation 23 begins to dominate the real term. Thus as the T_{10} solution becomes smaller for the deeper layers,

it also lags the rapid reprocessing pulses by 90° in phase. Since the hard and soft x ray pulses have been observed to differ in phase by 180° (Shulman et al.),⁶⁵ the phase of the optical pulses resulting from the deep-penetrating hard x ray flux leads the phase of the pulses produced by the rapidly-reprocessed soft x rays by 90° .

Since DM² have obtained excellent agreement between the broadband optical pulsations and the narrow band optical pulsations associated with the N III and He II emission lines, the 67% of optical pulsation *not* associated with the emission cannot be due to the deeper heating-reprocessing mechanism.

On the other hand, *if* the phase shifting process which produced the results of Shulman et al.⁶⁵ is a gradual *advancing* from the low energies to the high energies, *then* it is possible that the extra path delay for the harder x rays into and out of the deeper layers helps to partially cancel the phase differences between a large spectral range of x ray energies. For normal incidence and emergence the depth required for total cancellation of the 90° advance would be $4.7 \cdot 10^9$ cm or about 10 scale heights. However, before the 10-30 kev x rays could get in that far, Compton scattering would complicate the processes.

In addition to raising the efficiency of the optical pulse processes with the "heating" reprocessing, a soft x ray flux below 180 ev could also be invoked. The drawbacks with more soft x ray flux is that it would produce more emission in the N III and He II lines than is observed-- this is the same argument which was given above against the arbitrary increase of the soft x ray flux above 180 ev.

Academically, the amount of flux needed from 180-24 ev to explain the balance of optical pulsation would be $1.5 \cdot 10^{-9}$ ergs $\text{cm}^{-2} \text{ s}^{-1}$ as seen

above the earth with an average conversion efficiency of 12 keV^{-1} assuming a flat $dI/d\omega$ spectrum. This is consistent with the soft x ray source of $\sim 2 \cdot 10^{-8} \text{ ergs cm}^{-2} \text{ s}^{-1}$ at $kT = 90 \text{ eV}$ postulated by McClintok et al.²⁷ to explain HZ Her's continuing 1.7 day light variations during the 24 day "quiescent" interval when hard x rays aren't observed. However, it is also possible that Her X-1 simply turns the hard x ray flux away from earth for the 24 day "off". Moreover, recent soft x ray observations by Shulman (private communication, 1975) of Her X-1 during the 24 day "off" interval on Sept. 5, 1974 set an upper limit on the flux visible from earth at roughly $3 \cdot 10^{-10} \text{ ergs cm}^{-2} \text{ s}^{-1}$ between 180 and 280 eV -- an order of magnitude below the flux previously observed during an "on" interval of April 10, 1973. Apparently, Her X-1 is also capable of turning its soft x ray flux away from the earth. In either case, detailed speculations like those of McClintok et al.²⁷ on persistent soft x ray sources have no basis in observations.

A final comment may be made regarding the "heating" reprocessing model and Equations 20 and 23. Although the phase of the deep heating reprocessing might differ from that of the emission pulses by 90° , it is not obviously impossible that the two effects could act in coherence to produce some net result since the optical pulsation may be the result of a complicated averaging process (the two limits also have different surface efficiency laws for emission and absorption which give the two models different apparent frequencies for the same lobe). However, a check of the complex phases of calculations 80 and 81 (which did not include any intrinsic 90° difference) for $i = 90^\circ$ and $R = 1.65$ indicate that the major components of the peaks features of the same frequency at orbital phases $\phi = 0.21$ and

$\theta = 0.25$ do not differ in complex phase. Thus any net result produced by both models would introduce frequency, but not phase shifts. Therefore, the phase agreement found by DM² (measured at the 20% frequency) cannot be the result of some "mixture" model of both thin and deep heating-reprocessing, but indicates that only the thin heating-reprocessing is the possible source of the remaining 67% of optical pulsation.

5. Summary of Results of the Analysis of Features I and II

We are now in a position to summarize what has been learned from the study of features I and II of optical pulsation from HZ Her. These results are:

1. The anisotropy of the optical pulsation amplitude in orbital phase is caused by the time-of-flight interference over the critical Roche lobe.
2. HZ Her does indeed fill its critical Roche lobe or comes very close to filling it (with the sub-x ray point closer to L1 than 5% of the HZ Her c.m. - L1 separation).
3. The asymmetry in velocities between Feature I and Feature II indicates that the pulsar rotation (i.e. x ray beam sweep) is prograde in the binary system. When the formal errors on the average velocities of Feature I and II are doubled, there is a 68% confidence that the pulsar axis is aligned to within 46° of the orbital angular momentum vector. There is a 95% confidence that the alignment is within 67° .
4. The mass ratio of the system is determined to be 1.69 ± 0.05 with the probable systematic effects included. The "revised"

formal errors on the fits and the systematic errors are all on the order of 0.03 or less.

5. The orbital inclination is determined to be above 84° with the help of the x ray eclipse duration. A 2 σ limit on this duration would put $i > 80^\circ$.
6. The masses of the components of the system are determined to be $M_{\text{He II}} = 2.16^{+0.08}_{-0.06} M_\odot$ and $M_x = 1.28^{+0.08}_{-0.07} M_\odot$ from the mass ratio and inclination quoted above.
7. No "disk" shadow on the lobe may exceed ± 0.05 radians about the observer's equator (the same as the normal equator at $i = 90^\circ$).
8. The x ray profile across the lobe must be at least ± 0.05 radians in width. For $i = 90^\circ$ effectively all of the optical pulsations occur within ± 0.1 radians of the equator.
9. The x ray profile across the lobe can be at most ± 0.1 radians in width in order that the He II and N III emission appear $\geq 25\%$ pulsed.
10. The size of the pulsing region, at the time of maximum optical pulsation, which is self-coherent can be at least 5 and possibly 10 square light seconds of area.
11. Ordinary surface-effect reprocessing with both cosine laws to the earth and to Her X-1 fits the pulsation profile in orbital phase less suitably than either a volume effect reprocessing with no cosine law to earth or a deep reprocessing effect with no cosine law to Her X-1. Both of the latter have similar profiles; however, the volume effect emission law is the only possible choice because of quantitative considerations and because of the evidence of rapid, in-phase reprocessing in DM².

12. The 2/3rds of the optical pulsation not due to either N III or He II emission lines may be caused by either heating-reprocessing in the photosphere, or to many smaller emission features. Balmer emission pulses are less likely on grounds of quantitatively smaller 1: photoionization cross section, 2: recombination rate and 3: unionized population level and in addition may be cancelled by pulses in Balmer *absorption*.
13. The observed x ray flux down to 180 ev (mostly 10% pulsed) can account for roughly 25% of the optical pulsations by emission processes (and hence almost all of the emission pulsation observed by DM²). However, the flux which excites pulsation at the surface of HZ Her must be mostly pulsed.
14. Higher harmonics in the optical pulsations are suppressed as w_o/w in amplitude near the maxima and as $(w_o/w)^2$ near the minimum ($\phi = 0.5$) due to the time-of-flight smearing over the lobe.
15. Only higher harmonics in the optical pulsations can be created by the non-linear temperature-dependent term in the radiation balance equation. The creation of subharmonics may be possible via gravitational oscillation; unlike the higher harmonics, these would not be suppressed by averaging over the Roche lobe. However, subharmonics are only infrequently observed, if ever.
16. A small degree of non-corotation for the outer atmosphere of HZ Her, e.g. 5%, implies a revision of the mass ratio by less than 0.01 out of 1.65.

Table IV lists the models and their respective mass ratios derived from a three point parabolic fit to the χ^2 profiles. Column 1 gives the

orbital inclination, column 2 describes the details of the model used, column 3 gives the mass ratio derived, column 4 gives the formal "revised" error (the change in the mass ratio needed to increase the χ^2 by 1/20 of its value) and column 5 gives the χ^2 at the minimum of the parabola of the fit.

Table V lists the models for which only two points were available in mass ratio. Columns 1 - 3 are the same as in Table IV, but column 4 gives the χ^2 of each particular calculation and column 5 gives the sinusoidal moment of χ in orbital phase. The determination of the "best" mass ratio is left to discretion -- in all cases this should be between the two values.

B. Feature III

1. The Stream-Disk Interpretation

The profile of Her X-1-like pulsed amplitude vs orbital phase is shown in the middle frame of Figure 47 (and Figure 49). It is insignificant everywhere except from orbital phase 0.79 through orbital phase 0.92. The two points between $\phi = 0.50$ and $\phi = 0.58$ are slightly high as a result of some contamination from the previously discussed "anomalous" Feature. The termination of Feature III at $\phi = 0.92$ may be due to the onset of the eclipse (whose $\frac{1}{2}$ width is 0.07 cycles).

The velocity and acceleration characteristics of the six strong events of Feature III are shown by Figure 51 to be following the sinusoidal track of Her X-1. Figure 53 shows the same parameters converted into spatial locations. Only one box is observed to be 2σ off location to its velocity and only one is 2σ off due to its acceleration. The average velocity residual from the Her X-1 track of the six points is $-2.13 \pm 1.23 \text{ km s}^{-1}$ (ie, receding from the earth slightly faster than Her X-1) with a χ^2 of 4.5 for

5 degrees of freedom. The average acceleration residual is $-188 \pm 86 \text{ cm s}^{-1}$ (i.e., accelerating less along the direction to earth) with a χ^2 of 6.9 for 5 degrees of freedom. The maximum projected acceleration of Her X-1 is 724 cm s^{-1} .

The spatial origin of Feature III is thus almost consistent with Her X-1; however, it is more consistent with a region following the source by roughly $10^6 \pm 5 \cdot 10^5 \text{ km}$ and almost directly along the line of sight from the earth to the source. This suggests that Feature III originates in some sort of cloud which follows Her X-1 in its orbit and crosses the line of sight near orbital phase 0.85.

When the 35-day behavior of Feature III is examined in Figure 57, it is seen to appear abruptly at the x ray "turn-on" and disappear approximately $\frac{1}{2}$ way through the active x ray period near cycle 104. An event is also seen near cycle 20.85. The good correlation of this Feature to the 35-day (or 17-day) phase suggests its cause is related to the cause of the 11 day x ray "on" interval.

The possibility that Feature III is sometimes subject to a rather lopsided 17 day modulation has support from the Copernicus satellite which detected significant hard x ray flux from Her X-1 near the middle of the "off" interval (cycle 1.5) in the 35-day interval immediately following the one optical detection near cycle 20.85 (that is, 29 April-- 05 June 1973 optical detection to x ray detection).

Examination of Figures 66-68 does not indicate a good correlation of the orbital phases of the five results shown with the backward-marching dips of Giacconi et al.²⁰ However, since both the dips and the optical pulsation of Feature III indicate the presence of intervening material at

approximately the same orbital phases, different sections of the same cloud are probably responsible for the two phenomena.

The intersection of an accreting stream with a disk has been suggested by Pines et al.²⁴ to account for the x ray "dips". The disk-stream picture has been established previously as a model of the process of the so-called "cataclysmic variables"^{B1-B3} like WZ Sagittae where accretion is occurring onto a white dwarf member of a binary system.

The backward march of the dips through orbital phase is supposedly²⁴ accomplished by the backward motion of the stream-disk interaction spot due to shrinkage of the disk as the "on" progresses.

The calculations of Lubow and Shu⁶⁹ and Prendergast and Taam⁷⁷ give good evidence for the existence of an accretion stream which is hydrodynamically fixed in the corotating system. Figure 101 taken from Figure 4 of Lubow and Shu⁶⁹ shows the sequence of the formation of the disk by the shock which is formed where the stream collides with its returning member. Part 'C' of the Figure labels the geometrical aspects of the disk at equilibrium (when the tangential component of the stream velocity is equal to the Keplerian orbital velocity at the radius of contact).⁶⁹

The anisotropy of Feature III in orbital phase, the indication of its spatial origin being located between Her X-1 and the earth together with the x ray evidence of absorption "dips" and the similarity orbital phases of the two phenomena all strongly indicate that both are caused by the same cloud of gas. Furthermore, the necessary location of this cloud is seen to be close to the stream-disk intersection of several theoretical models. The dimensions and other physical characteristics that this cloud must have in order to produce optical pulsations will now be

investigated.

a. The geometrical shape of the cloud

The anisotropy of Feature III could be explained if the optical pulsations are the result of transmission and reprocessing of x rays through a substantial depth of cloud so that the pulse phase relationships within the cloud tend to favor forward propagation over the reverse direction.

The pulsed optical intensity produced by uniform x ray reprocessing in a column of gas of depth, d , observed at an angle, θ , from the axis of the column is given by

$$a(\theta) = a(0) \cdot \sin[\pi d/\lambda \cdot (1-\cos\theta)] \cdot [\pi d/\lambda \cdot (1-\cos\theta)]^{-1} \quad (28)$$

where λ is the wavelength of the pulsations, c/f_0 , $f_0 = 0.8078$ Hz, and is roughly equal to $3.71 \cdot 10^5$ km. Also, $a(0)$ is the forward amplitude and is obtained by an integration over the reprocessing efficiency through the depth of the cloud.

The amount of forward-backward attenuation which is needed to explain the data as seen in the middle frame of Figure 49 by comparing the $\theta = 0.79 - 0.92$ interval with the $\theta = 0.29 - 0.42$ interval is roughly a factor of five. Figure 102 has been constructed to show the profile of Feature III more precisely by following the Her X-1 track more closely. The level of average pulsation near $\theta = 0.29 - 0.42$ is seen to be small (20% of III) even when compared to the relatively low level of Feature III (III averages $5.5 \cdot 10^{-4}$ at its top while I and II average $9 \cdot 10^{-4}$ and $7 \cdot 10^{-4}$ respectively).

To achieve this forward-backward anisotropy (a factor of ~ 5) the depth of the cloud must be at least $1.5 \cdot 10^{10}$ cm; a depth of $\lambda/2 = 1.85 \cdot 10^{10}$ cm would be optimum since the cloud would produce no "backward" pulsations to the degree that it is a perfectly homogeneous reprocessor of x rays. Equation 28 also indicates that a secondary maximum occurs near $d = 3/4 \lambda$ when the backward amplitude rises to about 22% of the forward amplitude.

Realistically, there is always some absorption or scattering process which may remove more quanta from the far end of the cloud than from the end nearer to earth and thus cause the volume elements of the cloud to interfere less critically. To investigate this possibility, a $\langle \pi \rangle / a(0)$ is calculated as a function of optical depth, τ , through the column:

$$a(\pi)/a(0) = e^{2\pi id/\lambda} (1 - e^{-4\pi id/\lambda - \tau}) (4\pi id/\lambda + \tau)^{-1} \cdot [e^{-2\pi id/\lambda} (1 - e^{-J})/\tau]^{-1} \quad (29a)$$

$$= e^{i\text{phase}} \{ e^{-\tau} 4 \sin^2 2\pi d/\lambda + (1 - e^{-J})^2 \}^{1/2} \tau \cdot (16\pi^2 d^2 \lambda^{-2} + \tau^2)^{-1/2} \cdot (1 - e^{-\tau})^{-1} \quad (29b)$$

where "phase" is the net phase factor from the various terms.

Equation 29 shows that the ratio changes very slowly for $\tau \leq 1$ when $d = 3/4\lambda$ and is still $\leq .157$ for $\tau = 1$ when $d = \lambda/2$. Thus small optical depths do not effect the forward-backward transmission reprocessing ratio.

The situation may be further complicated if the amount of x ray reprocessing is attenuated outward from the x ray source. In this case let τ_x represent the optical depth for the x ray reprocessing mechanism (this may be less than the depth for the x ray quanta if the gas is almost completely ionized due to the high x ray luminosity). Let τ_0 represent the

optical depth for the visible quanta. The ratio of backward to forward pulsation amplitudes is now

$$a(\pi)/a(0) = e^{i\text{phase}} [e^{-(\tau_x + \tau_o)} 4 \sin^2 2\pi d/\lambda + (1 - e^{-(J_x + J_o)^2})^{\frac{1}{2}} \\ \cdot [16\pi^2 d^2 \lambda^{-2} + (\tau_x + \tau_o)^2]^{-\frac{1}{2}} \cdot (\tau_x - \tau_o) \cdot (e^{-\tau_o} - e^{-\tau_x})^{-1}] \quad (30)$$

Again, this ratio is not affected very much unless τ_x or τ_o exceeds 1.

Another complication which can result is that the reprocessing involves a certain delay in response, Δt . If Δt is constant throughout the cloud, the results are affected only by a delay of the optical response with respect to the x ray transmission. The situation is *not* analogous to the modification of group velocity through a refractive medium because the "secondary" wave through the cloud is weak in comparison to the primary flux of x rays.

A detailed treatment of the optically thick case for x ray sources has been done by Tarter and Salpeter.⁸⁴ In effect, a strong x ray source will produce a Stromgren sphere for each atomic species in a cloud which surrounds the source. It is then possible for the reprocessing intensity to remain constant until some transition radius is reached, whereupon the reprocessing flux suddenly drops. In this situation (a recombination-limited reradiating volume bounded by a flux proportional reradiating volume) the cloud of gas could be integrated with a step function rather than an exponential to determine the effects of absorption. Unfortunately, the models calculated by Tarter and Salpeter were computed for spherically symmetric and extremely tenuous cases which are probably unlike the

conditions near Her X-1.

b. The width of the cloud

In addition to establishing the depth of the cloud, Feature II can also be used to limit the width of the cloud along the orbital plane perpendicular to the line of sight to the earth. The width of Feature III (as seen, e.g., in Figure 102) is inversely related to the width of the cloud by the same arguments which give the forward-backward anisotropy. Because the eclipse terminates Feature III, the *minimum* width is therefore 0.17 cycles (or 61°). Thus a maximum can be placed on the horizontal extent of the cloud from the slit diffraction formula

$$a(\theta)/a(0) = \sin(\pi w/\lambda \cdot \sin\theta) \cdot (\pi w/\lambda \cdot \sin\theta)^{-1} \quad (31)$$

where w is now the width.

The $\frac{1}{2}$ width is defined when the argument equals 1.89 radians $= \pi \cdot 0.6055$. Thus to achieve $\frac{1}{2}$ height by 30° or more orbital phase deflection from the center of Feature III:

$$w/\lambda \leq 2 \cdot 0.6055 \quad \text{or} \quad w \leq 4.5 \cdot 10^{10} \text{ cm} \quad (32)$$

Such widths are not difficult to achieve with the disk geometry of Lubow and Shu.⁷¹ Given that $T \approx 30,000^\circ\text{K}$ for the gas in the stream and disk near the x ray source, then $c \sim 0.1$ for the HZ Her-Her X-1 system. The equilibrium disk radius which they drive for $M_D/M_C \sim 0.6$ (interpolating) is 0.072 of the orbital separation or about $4.5 \cdot 10^{10}$ cm. Thus the required

width of the cloud is less than one radian of the circumference of a typical disk.

The fact that Feature III lasts some 60° in orbital phase does not indicate that every observation for that orbital phase shows optical pulsations for that duration. On the contrary, the cloud may dissipate or thicken on the timescale of hours in such a way that optical pulsations are no longer produced in any direction.

The alternative explanation of the width of Feature III in orbital phase is that it is the result of secular variations in the optimum phase for the pulsations--perhaps like the "backward-marching" behavior of the x ray dips. Equation 32 would then be inverted so the $4.5 \cdot 10^{10}$ cm. represents a *minimum* for w . Support for this interpretation comes from the fact that if w were occasionally less than $4.5 \cdot 10^{10}$ cm. in width, then optical pulsation would be detected outside of the orbital phase region of Feature III.

It is worth noting that the *onset* of Feature III corresponds exactly to the calculated position of the equilibrium stream-disk intersection (ψ_h from Figure 101 is 251° for $M_D/M_C = 0.6$, which corresponds to orbital phase $540^\circ - 251^\circ = 289^\circ = 0.80$). If the theory of Lubow and Shu⁷¹ is correct, there are two possible ways out of the dilemma. The first possibility is that the radiative shock at the stream-disk intersection strongly modifies the physical conditions of the stream before it collides with the disk. The second possibility is that the disk is actually *larger* than the equilibrium value set by equating the tangential velocity to the Keplerian velocity so that ψ_h is smaller. A disk radius of 10^6 km is more consistent with the location of the data boxes and with the later phase

center of Feature III.

The fact that the stream width of Lubow and Shu⁶⁹ of $\sqrt{2\pi/\gamma} = 0.33 \cdot \theta_D = 2 \cdot 10^{10}$ cm. is in excellent agreement with the depth of $2 \cdot 10^{10}$ cm, derived from the data lends some support to the possibility that the cloud is actually the accretion stream just prior to its collision with the disk. The radiative shock as shown in Figure 101b tends to deflect the stream into a more tangential orientation so that it would correspond to the necessary cloud orientation. The real question remains where does this reorientation start and how much does it affect the stream?

Finally, Figure 103 shows the behavior of Equation 28 for $d = 1.5 \cdot 10^{10}$ cm. and equation 31 for $w = 4.5 \cdot 10^{10}$ cm. together in polar coordinates.

c. The height of the cloud

The extent of the cloud in the direction perpendicular to the orbital plane (z) may be estimated by using the gradient of the potential at the position of the cloud to establish a scale height. Scaling from the surface of a $1 M_{\odot}$ neutron at a 10^6 cm. radius where $g \approx 10^{14}$ cm s⁻² out to $4.5 \cdot 10^{10}$ cm. radius at a height z above the orbital plane gives

$$g(z,r) = 10^{14} (10^6/r)^2 \cdot z/r \text{ cm s}^{-2} \quad (33)$$

and

$$h(z,r) = kT/[mg(z,r)] = 10^{-26} kT r^3/(mz) \quad (34)$$

We now choose the appropriate z which gives an equal scale height

$$z = 10^{-11} \sqrt{kT/m} \cdot r^{3/2} \text{ cm.}$$

Setting $r = 4.5 \cdot 10^{10}$ cm, $T = 5 \cdot 10^4$ °K then $z \approx 2 \cdot 10^3$ cm. The actual thickness would be twice this number counting scale heights above and below

the plane. This number is also in excellent agreement with the estimate of Lubow and Shu⁷¹ for the height of the stream, $\sqrt{2\pi}/w_z = 0.078 \cdot \epsilon D$ with $\epsilon D = 6 \cdot 10^{10}$ gives $\sqrt{2\pi}/w_z = 4 \cdot 10^9$ cm.

Since the occulting angle, z/r , goes as $r^{-1/2}$ thus the disk should be "thinner" (more than proportionately) as r decreases. Thus, even if the disk is much denser than the stream (as assumed by Lubow and Shu⁷¹ and shown to be equivalent to the assumption that segment 2 flows into segment 3 (of Figure 101) without the production of shocks) and is consequently optically thick to the x rays, the higher profile of the stream and the outermost portion of the disk affected by the collision of the stream is readily exposed to the x ray flux.

Because of the particular position in the orbital plane, the cloud would not easily be scattered by radiation pressure, since the line of sight at $\beta = 0.83$ climbs up the side of the maximum which occurs at the trailing Trojan point. This "hump" may tend to trap a gas stream while escape would ultimately be possible in other directions because the contours turn over more quickly. It may be possible that an unusually high accretion flux together with a similar "trapping" effect of the forward Trojan point produced the singular optical pulsation even observed at $\beta = 0.2$ which is not normally associated with Feature III.

d. The reprocessing mechanisms and the physical conditions of the cloud

With the current cloud dimension of $4.5 \cdot 10^{10}$ cm. by $4 \cdot 10^9$ cm, the cloud subtends roughly 0.7% of the solid angle of the x ray source. More importantly, at a distance of $r = 4.5 \cdot 10^{10}$ cm as compared to $1 = 10^{22}$ m at earth where the x ray flux is 10^{-8} ergs $\text{cm}^{-2} \text{ s}^{-1}$ or greater, the x ray

XBL 7410 1907

Fig. 12

For He, C, N, O, Ne, Si and Fe with abundances $8.5 \cdot 10^{-1}$, $3.3 \cdot 10^{-4}$, $9.1 \cdot 10^{-4}$, $6.6 \cdot 10^{-4}$, $8.3 \cdot 10^{-5}$, $3.3 \cdot 10^{-5}$ and $4.0 \cdot 10^{-5}$ respectively,

$$\begin{aligned} n &\geq 2.0 \cdot 10^8 (\text{He}), 4.6 \cdot 10^{11} (\text{C}), 2.26 \cdot 10^{12} (\text{N}), \\ &4.1 \cdot 10^{11} (\text{O}), 5.0 \cdot 10^{12} (\text{Ne}), 2.5 \cdot 10^{13} (\text{Si}) \quad (40) \\ &\text{and } 7.1 \cdot 10^{13} (\text{Fe}) \cdot (1.5 \cdot 10^{10}/d) \text{ cm}^{-3} \end{aligned}$$

The hydrogen and even the helium are likely to be ionized in the cloud, thus the typical densities necessary for a "dip" which attenuates x rays up to 10 keV (Fe has a 9.2 keV absorption edge) as indicated in Figure 4 of Giacconi et al.²⁰ are a few times 10^{13} cm^{-3} .

Another way of removing quanta from the line of sight is Compton scattering. Since the cloud subtends only a small solid angle of $0.007 \cdot 4\pi$ as seen from Her X-1, only 0.7% of the scattered γ rays find their way into the path to the earth. Compton scattering has a double effect since it not only removes x rays from the line of sight but it also removes the reprocessed optical quanta. Thus a Compton depth of 1.0 acts like $\tau_x = \tau_o = 1$, and this should be considered as an upper limit above which the optical pulsations would no longer be as strongly modulated in orbital phase as they are observed to be. From this constraint we know that

$$1 \geq n \cdot 10^{-14} (d/1.5 \cdot 10^{10}) \quad \text{or} \quad n \leq 10^{14} (1.5 \cdot 10^{10}/d) \text{ cm}^{-3} \quad (41)$$

If successive hard x ray spectra were taken near a "dip" and during a "dip", the ratio of the fluxes seen would place a constraint on the cloud's

subtended solid angle and its thickness, since the amount of Compton scattering could easily be gauged by the x ray spectrum above 10 kev, where no other process can remove x ray quanta.

Because the cloud must interact by absorption and recombination to produce the optical pulsation (and the dips), it must be capable of dissipating power through recombination (free-bound radiation) and free-free radiation so that its temperature doesn't rise indefinitely. A rough estimate may be made by setting the column rate of recombination equal to the incident flux expressed in units of quanta whose energy is equal to the absorption edge energy of the element in question:

$$L_x / (4\pi r^2 Z^2 R_y) \geq 3 \cdot 10^{-11} Z^2 T^{-1/2} n^2 a(Z) d \quad (42)$$

where $R_y = 2.176 \cdot 10^{-11}$ ergs and an approximation from Allen⁷⁴ has been used for the recombination coefficient on the left hand side. Ideally this equation is most properly expressed by equating the differential of the spectrum to the differential abundance. Since He and O and a few other elements are most important, it is simpler to perform the calculation for these elements. Given that

$$L_x \approx 4\pi \cdot 10^{36} \text{ ergs s}^{-1}, r = 4.5 \cdot 10^{10} \text{ cm} \quad (43)$$

the constraint on T as a function of n is:

$$T \geq 3.9 \cdot 10^{-52} Z^8 a^2(Z) n^4 (d/1.5 \cdot 10^{10})^2 \text{ eK} \quad (44)$$

The major contributors to $a(Z) \cdot Z^6$ are:

H: 1.00, He: 1.85, O: 7.323, Ne: 0.69, Si: 1.62, S: 1.08

Fe: 331., and Ni: 1.5 (45)

The Fe is perhaps unrealistically large since it is probably never completely ionized. In this way, the heavier, less completely ionized species tend to increase the effective abundance of the lighter elements in the absorption of the softer x rays.

However, it is likely that the ionization states of the lighter elements increases to the He-like or hydrogen-like states since the x rays strip the outer levels until the higher K-shell recombination rates can successfully compete with the photoionization rate. The calculations of Tarter, Tucker and Salpeter⁸⁵ tend to support this possibility since $\xi = L_x/nr^2$ for $n \approx 10^{13} \text{ cm}^{-3}$, $r \approx 4.5 \cdot 10^{10} \text{ cm}$ and $L_x \approx 4\pi \cdot 10^{36} \text{ ergs}^{-1}$ is ≈ 300 . For $\log \xi = 2.5$ in all models the lighter elements are all stripped to their K shells or farther.

Applying the results of Tarter and Salpeter⁸⁴ in their Figure 1 with $\log L_x/L_\odot = 3.4 - 4.0$ and $n = 1-3 \cdot 10^{13} \text{ cm}^{-3}$ gives $\ell_{\text{ion}} = 1-3 \cdot 10^{11}$ and $\gamma = 70 - 350$.

With $R_{\text{Stromgren}} = \ell_{\text{ion}} \gamma^{-1/3}$ we get

$$R_{\text{Stromgren}} = 7 \cdot 10^{10} \rightarrow 1.4 \cdot 10^{10} \text{ cm.} \quad (46)$$

in excellent agreement with the values established previously.

Thus the use of the K shell recombination coefficient for Eqns. 42-44 is accurate for the lighter elements. For oxygen the equation becomes

$$T \geq 2.85 \cdot 10^{-51} n^4 (d/1.5 \cdot 10^{10})^2 \text{ } ^\circ\text{K} \quad (\text{Oxygen}) \quad (47)$$

and at $n = 10^{14} \text{ cm}^{-3}$ we have $T \geq 285,000 \text{ } ^\circ\text{K}$.

Equation 47 gives the temperature to which the cloud would have to rise in order that the K shell recombination should decrease below the impinging x ray flux. The equation serves better as an upper limit for n because of its high power in the equation.

The combination of all other lighter elements together with oxygen will do no more than double the coefficient in Equation 47 -- thereby only decreasing the n intercept by $\sqrt[4]{2} = 1.19$. Equation 42 could be used to put a smaller upper limit on n for any given T since the light, but more abundant elements are significant absorbers at energies above their absorption edges where numerically fewer quanta are required to make up a given energy. Oxygen, for example, has unit optical depth through a hydrogen column density of $1.5 \cdot 10^{10} \text{ cm} \cdot 2 \cdot 10^{13} \text{ cm}^{-3} = 3 \cdot 10^{23} \text{ cm}^{-2}$ at 3.18 kev.

A similar, though less stringent constraint on the intensive state of the cloud can be established by limiting the free-free reradiation to $\frac{1}{4}$ of the incoming energy flux. Thus

$$4\pi \cdot 10^{36} / [4\pi \cdot (4.5 \cdot 10^{10})^2] \cdot \frac{1}{4} \geq 1.435 \cdot 10^{-27} n^2 T^{\frac{1}{2}} d \text{ ergs cm}^{-2} \text{ s}^{-1} \quad (48)$$

or

$$T \leq 1.3 \cdot 10^{62} n^{-4} (d/1.5 \cdot 10^{10})^{-2} \text{ } ^\circ\text{K} \quad (49)$$

Compared to the density limit from the recombination rates, this is not very significant. However, because a free-free spectrum is flat and because the cloud temperature cut-off is probably no more than one order of magnitude greater than the energy of optical quanta, a substantial fraction (5-10%) of the free-free reradiation would be in the optical band.

A constraint may be applied by setting the free-free cooling time equal to 1 second. This cooling time is:

$$1 \geq t_{ff} = 3/2kT/j_{ff} = 3/2kTn/(1.435 \cdot 10^{-27} T^4 n^2) = 1.44 \cdot 10^{11} T^4 n^{-1} \quad (50)$$

$$\text{or:} \quad T \leq 4.82 \cdot 10^{-23} n^2 \text{ } ^\circ\text{K} \quad (51)$$

Some free-free optical reradiation can therefore be expected to contribute to the optical pulsation at realistic densities and temperatures.

The last effect of the free-free process is its opacity. To get directionally well-modulated optical pulses from a cloud with finite depth, it is not unreasonable to require that the optical depth at 4000 Å is, e.g. 0.1:

$$0.1 \geq 3.692 \cdot 10^8 T^{-4} \nu^{-3} n^2 d; \text{ at } \nu = 0.75 \cdot 10^{15} \text{ Hz} \quad (52)$$

$$T \geq 1.7 \cdot 10^{-50} n^4 (d/1.5 \cdot 10^{10})^2 \text{ } ^\circ\text{K} \quad (53)$$

This constraint also cuts through a likely section in the n-T plane.

When optical pulsation through optical recombination lines is considered, the recombination rates to the favorable levels must also be not

longer than 1 second. Since most of the important species are nearly hydrogen-like, the recombination rate to the n th level is like:

$$\alpha_n = 3.26 \cdot 10^{-6} n^{-3} T^{-3/2} Z^2 \quad (54)$$

The optical transitions are centered on a region where

$$2R_Y Z^2 n_o^{-3} \cong R_Y/5 \rightarrow n_o = 2.15 Z^{2/3} \quad (55)$$

Thus the fraction of recombination which produces optical reradiation is roughly

$$\int_{n_o}^{\infty} n^{-3} dn / \sum_1^{\infty} 1/n^3 = 0.09 Z^{-4/3} \quad (56)$$

As n_o gets higher with increasing Z , the number of consecutive transitions which make optical quanta also increases. Using $0.15 R_Y$ as the optical band width:

$$6Z^2 R_Y n_o^{-4} dn_o \cong 0.15 R_Y \text{ or,} \quad (56)$$

$$dn_o = 0.025 n_o^4 Z^{-2} = 0.54 Z^{2/3} \quad (57)$$

Thus the total amount of energy converted into optical recombination lines is

$$\Delta E_o = dn_o \cdot R_Y/5 = 0.11 Z^{2/3} R_Y \quad (58)$$

while the total recombination energy is $\Delta E_x = Z^2 R_y$ so that the efficiency of conversion into optical line radiation by recombination onto hydrogen-like ions is

$$\epsilon \equiv \Delta E_o / \Delta E_x = 0.11 Z^{-4/3} \quad (59)$$

The constraint which may be derived from the optical recombinations is that the recombination lifetimes must be shorter than 1 second otherwise optical pulsation would be difficult to produce at 1.24 s. The equation is

$$1 \text{ sec}^{-1} \leq 3.26 \cdot 10^{-6} Z^2 n T^{-3/2} \cdot 0.09 Z^{-4/3} = 2.9 \cdot 10^{-7} Z^{2/3} n T^{-3/2}, \text{ or} \quad (60)$$

$$T \leq 4.4 \cdot 10^{-5} Z^{4/9} n^{2/3} \quad (61)$$

For various species the coefficients are:

$$\begin{aligned} \text{He: } & 5.98 \cdot 10^{-5}, \quad \text{C: } 9.75 \cdot 10^{-5}, \quad \text{N: } 1.04 \cdot 10^{-4}, \\ \text{O: } & 1.11 \cdot 10^{-4}, \quad \text{Ne: } 1.22 \cdot 10^{-4} \text{ and Fe: } 1.87 \cdot 10^{-4} \end{aligned} \quad (62)$$

The above discussion has ignored the possibility of free-bound transitions in the visible region of the spectrum and in this sense is conservative. With the gas at more than moderate temperatures ($\approx 30,000^\circ \text{K}$), the minimum $3/2kT$ energy released upon recombination already exceeds visible energies.

The only constraint which remains is to try to produce quantitatively

the amount of optical pulsations which are observed from the earth at roughly $1.6 \cdot 10^{-13}$ ergs $\text{cm}^{-2} \text{s}^{-1}$. Since the cloud dimensions are more or less fixed, the two mechanisms, recombination and free-free radiation, need only be employed with the appropriate constraints on the reprocessing times integrated into the equations in some reasonable fashion.

Given a cooling or recombination time, Δt , and the pulse period, p , then an analytical way to describe the amplitude of the optical pulsation response as a function of Δt would be to multiply with the factor $(1 + 4\pi^2 \Delta t^2 p^{-2})^{-1/2}$. This response parameter is *intensive* and does not depend on the "dimensionality" of the cloud (i.e. would not be cubed for a 3-dimensional cloud).

If the expressions for pulsed luminosities for the two reprocessing mechanisms are computed (assuming a 2 ev band width for the free-free reradiation for a volume of $2.76 \cdot 10^{30} \text{ cm}^3$, the results are:

$$L_{\text{opt rec}}(Z) = 1.9 \cdot 10^{12} (1 + 3.0 \cdot 10^{14} Z^{-4} n^{-2} T^3)^{-1/2} Z^{4/3} a(Z) n^2 T^{-3/2} \quad (63a)$$

$$L_{\text{opt f.}} = 0.2 \cdot 10^7 (1 + 5.3 \cdot 10^{23} n^{-2} T)^{-1/2} n^2 T^{-1/2} \text{ ergs s}^{-1} \quad (63b)$$

where α is some response factor ≤ 1 since the temperature of the cloud would not be below a certain minimum level set by the fractional modulation of Her 1 and the constant luminosity of HZ Her.

For typical values for n , T , Z , $a(Z)$ at $2 \cdot 10^{13} \text{ cm}^{-3}$, $5 \cdot 10^4 \text{ K}$, 8 and $6.6 \cdot 10^{-4}$ respectively

$$L_{\text{opt rec}}(Z = 8) = 2.7 \cdot 10^{29} \text{ ergs s}^{-1} \quad (64a)$$

$$L_{\text{opt ff}} = \alpha \cdot 2 \cdot 10^{31} \text{ ergs s}^{-1} \quad (64b)$$

Thus with the cloud at $5 \cdot 10^4$ °K and $n = 2 \cdot 10^{13} \text{ cm}^{-3}$ only the free-free reradiation seems to give a reasonable contribution -- $\alpha 1.6 \cdot 10^{-6} L_x$ or $\alpha \cdot 1.6 \cdot 10^{-14} \text{ ergs cm}^{-2} \text{ s}^{-1}$ pulsed flux at the earth. If the temperature of the gas were $2 \cdot 10^4$ °K where He recombination would enter then

$$L_{\text{opt rec}}(\text{He}) = 3.1 \cdot 10^{31} \text{ ergs s}^{-1} \quad (65a)$$

$$L_{\text{opt ff}} = \alpha 4.9 \cdot 10^{31} \text{ ergs s}^{-1} \quad (65b)$$

in which case the optical pulsed flux seen at earth would be $\alpha 3.9 \cdot 10^{-14} + 2.5 \cdot 10^{-14} \text{ ergs cm}^{-2} \text{ s}^{-1}$ -- close to what is actually observed.

At these conditions, it may be possible for considerable amounts of N IV to recombine dielectronically and emit $\lambda\lambda 4634\text{-}4640 \text{ \AA}$ just as it does in the atmosphere of HZ Her. The 20 times higher recombination coefficient helps somewhat to compensate for the smaller abundance ($\cdot 10^{-3}$) relative to helium. Future experiments may decide whether these lines actually contribute to the pulsations of Feature III.

The only other way to increase the effective pulsed emission is to increase its height in the z direction. An increase of density quickly loses to Compton scattering and free-free opacity, and an increase of width (along the direction roughly parallel to the circumference of the disk) is physically unrealistic. Gravitational relaxations would tend to have periods characteristic of $2\pi r^{3/2} \cdot 10^{-13}$ -- $6 \cdot 10^3$ seconds at $r = 4.5 \cdot 10^{10}$ about the orbital plane and thus cannot contribute to the pulsations.

Relaxation in the other dimension would have characteristic time of 675 seconds. The single, most effective way of increasing optical pulsations is therefore lowering the temperature of the cloud.

A cool cloud is feasible if much of the stream is shaded by the disk as it approaches the point of collision. This is not inconceivable. The density of $2 \cdot 10^{13} \text{ cm}^{-3}$ considered in the above calculations is consistent with the calculations of Lubow and Shu.⁶⁹ Their characteristic density for the system is $\rho \approx \dot{M}/\Omega D^3$, and $\dot{M} > 6 \cdot 10^{40} \text{ s}^{-1}$ implies that $\rho > 5.6 \cdot 10^9 \text{ cm}^{-3}$. When the stream collides with the equilibrium disk its density is

$$\rho_{SD} > 13.9 \cdot \epsilon^{-2} \cdot 5.6 \cdot 10^9 \text{ cm}^{-3} \quad (66)$$

With $\epsilon \leq 0.1$, $\rho_{SD} > 7.7 \cdot 10^{12} \text{ cm}^{-3}$ which is only 2.5 less than $2 \cdot 10^{13} \text{ cm}^{-3}$. Since the mass loss from HZ Her into the stream was computed by assuming 100% capture efficiency onto the neutron star, a more realistic efficiency would tend to increase ρ_{SD} . Moreover, an upper limit was used for the parameter ϵ , and again ρ_{SD} could be revised upward.

The physical constraints of Equation 39 for Fe (unit photoionization depth), Equation 41 (Compton scattering unit depth), Equation 47 (oxygen recombination reradiation capability), Equation 49 (free-free reradiation capability), Equation 51 (free-free cooling time = 1 sec), Equation 53 modified (1.0 optical depth at 4000 Å -- the coefficient becomes $1.7 \cdot 10^{-52}$), Equation 61 for He (1 second recombination lifetime for He III) are plotted in Figure 104 with $\log T$ on the vertical axis from $10^4 \text{ }^\circ\text{K}$ - $10^7 \text{ }^\circ\text{K}$ and $\log n$ on the horizontal axis from 10^{12} - 10^{15} cm^{-3} . The Figure is plotted for the

case of $d = 1.5 \cdot 10^{10}$ cm, $r = 4.5 \cdot 10^{10}$ cm, F_x at earth of 10^{-8} ergs cm^{-2} s^{-1} and a distance to the source of 10^{22} cm. In addition, the nearly horizontal lines which designate where the K shell electrons become unbound for He and O are plotted. The vertical line for unit photoionization depth for Fe at its absorption edge ends where its K shell boundary should be for $n = 7.1 \cdot 10^{13} \text{ cm}^{-3}$ at $T = 6.5 \cdot 10^6$ °K. Solid triangles mark the points where the amount of optical pulsation was computed in Equations 63a and 63b. The constraint of equation 63a without the factor which accounts for the finite recombination lifetimes is also plotted for He and O labelled "He Pulses" and "Oxygen Pulses".

Moving from left to right near the bottom of Figure 104, the first line encountered has a slope of 2/3rds and marks the boundary where the helium K shell recombination lifetime is 1 second. After this comes the line from Equation 63b (with the cooling time factor removed) which rises upward with a slope of 4 (and also assumes $\alpha = 1$). The line which joins this line at the x axis marked $t_{ff} = 1$ sec indicates the boundary where the cooling time factor will decrease the free-free optical pulsations by a factor of 5.2. The relative positions of the two lines indicate that free-free optical pulsation cannot account for the observed flux for any temperature above 10^4 °K. The line for the cooling time factor from Equation 51 slants upward with a slope of +2.

The next line encountered slants upward with a slope of 4/3rds and marks the boundary where all of the helium in the cloud could produce the observed optical pulsation flux if it were involved in recombination and if the recombination lifetime factor is neglected. Since the boundary for the 1 second recombination lifetime is far away to the left, the

cooling time factor in Equation 63a does not have a very strong effect on the location of the "He Pulses" line.

The "Oxygen Recombination Capacity" line which intersects the K shell boundary of He at the same place as the "He Pulses" line, represents the conditions where the continuous recombination to the K shell of the oxygen ions could balance the power input from the x ray source if the source were all in quanta of the oxygen absorption edge (870ev). This line has an upward slope of +4.

The unit photoionization depth vertical line of Fe is reached near $7 \cdot 10^{13} \text{ cm}^{-3}$ and the unit electron scattering optical depth line is reached at $1 \cdot 10^{14} \text{ cm}^{-3}$. A line with slope of +4 marks the boundary of the conditions at which the thermal free-free opacity for 4000 Å reaches unity. This line is parallel to the last line of finite slope representing the Oxygen Reradiation Capacity.

Rising from the $n = 10^{14} \text{ cm}^{-3}$ point on the horizontal axis with a slope of +4/3 is the line which marks the ability of oxygen atoms to produce the observed optical pulsation through the recombination process. The position of this line at the right of the graph indicates that Compton scattering will destroy the coherence of the pulsations long before the requisite oxygen pulsations are produced.

The last steep line, which has a slope of -4, marks the boundary at which the free-free process would radiate more flux than enters the cloud from the x ray source.

The trial density of $2 \cdot 10^{13} \text{ cm}^{-3}$ which was used in the above calculations is also suitable for producing the type of dip whose spectrum is shown in Figure 4 of Giacconi et al..²⁰ The optical depth for Fe

photoionization is roughly 0.3, along with a 0.2 Compton depth would make the attenuation factor $e^{-0.5}$ near 9 kev -- consistent with the observed attenuation factor of 4, particularly when the smaller cross sections of the lighter elements like Ca, Si and O are included. Giacconi et al.²⁰ have estimated that the low energy cut-off was produced by a column density of 10^{23} cm^{-2} ; however, due to the ionization of H and He at the temperature of the cloud, a column depth of $3 \cdot 10^{23}$ as used in the above examples is more realistic.

2. Feature III and the 35-Day Phase

The behavior of Feature III across the 35-day phase is shown in Figure 57. Several reasons can be given for the optical pulsations stopping half way through the "on" period, while the x ray "dips" continue. One suggestion would be thinning of the disk leading to an increase of temperature in the cloud to a point where the optical pulsations are much less favored. This would tend to correlate somewhat to the return of the "superhot" component of light from the system seen by Boynton et al.¹² An increase or decrease in density would also suppress the pulsations. The dynamics of this cloud may be related to the non-periodic "flickering" observed by Moffett et al..⁸⁶

C. The Origin of the Anomalous Feature

Evidence for the existence of a Feature of the optical pulsations with an anomalous or not easily explainable velocity is best seen in Figures 41-43 and 59. It appears in orbital phase between 0.50 and 0.75 with a velocity toward the earth with respect to the c.m. with the magnitude of the

difference being roughly equivalent to the absolute value of the average for Her X-1 in this orbital interval (In effect, the Feature is about as far above the zero-velocity line as Her X-1 is below it). No correlation of this Feature with the 35-day phase is obvious; however, the experimental limits are not as good as those of Features I-III. Judging from Figure 59, the average pulsed fraction for the anomalous Feature is above zero everywhere except during cycles 0-4. No singular events have been observed which are either consistent with a level of pulsation above $2.0 \cdot 10^{-3}$ or very significant.

The explanation of this Feature is qualitatively simple but quantitatively complex. It is a Feature which results from gas moving as a unit (or "clump") toward Her X-1 along the hydrodynamical accretion stream. Optical pulsations are excited in this clump as discussed above for Feature III.

There are two reasons why the anomalous Feature is only seen between orbital phases 0.50 and 0.75. The first is that a flat section in a stream such as the stream of Lubow and Shu⁶⁹ would geometrically modulate its optical pulsations according to its finite width as

$$a(\theta, \psi) = \sin [\pi w \lambda^{-1} \sin \theta - \sin \psi] [\pi w \lambda^{-1} (\sin \theta - \sin \psi)]^{-1} \quad (67)$$

where w is the width of the stream, λ is the wavelength of the optical pulsations, ψ is the angle of incidence of the x rays to the normal of the face of the clump and θ is the angle of observation with respect to the normal to the face of the clump.

If we denote the angle made by the stream to the mass axis by α

(assuming the clump to be a perpendicular cross section of the stream) and the angle from the mass axis to the line from Her X-1 to the clump by β , then from geometry the orbital phase at which the maximum modulation will be seen is

$$\phi_{\max} = 2\alpha + \beta + 0.5 \text{ cycles} \quad (68)$$

and this occurs when $\theta = \psi = \alpha + \beta$. From the nomenclature of Figure 101c and Table 1 of Lubow and Shu⁶⁹ $\phi_{\max}$ starts at $2|\theta_S| + 180^\circ \approx 220^\circ$. As the stream leaves L1, α decreases slowly but the effect on $\phi_{\max}$ is less because β is also increasing. Using the second entry in Table 3 ($M_D/M_C = 0.5$) of Lubow and Shu⁶⁹ to determine α and β halfway along the stream from the variables X , Y , U_X and U_Y we get

$$\alpha_{\frac{1}{2}S(\bar{w}_d)} = 12.8^\circ; \quad \beta_{\frac{1}{2}S(\bar{w}_d)} = 15.2^\circ; \quad \phi_{\max \frac{1}{2}S(\bar{w}_d)} = 221^\circ \quad (69)$$

Finally, repeating the calculations for $M_D/M_C = 0.50$ (the angles α and β change extremely slowly with mass ratio to that 0.50 is close enough to 0.60 for the purposes of the numbers) at $\bar{w}_d$ we get

$$\alpha(\bar{w}_d) = -21^\circ; \quad (\bar{w}_d) = \phi_n - 180^\circ = 72^\circ; \quad \phi_{\max}(\bar{w}_d) = 210^\circ \quad (70)$$

Thus the orbital phase at which the anomalous Feature reaches a maximum is between 221° and 210° . With $\epsilon \approx 0.1$ the width, w , of the stream varies from $5.5 \cdot 10^{10}$ cm at L1 to $4.4 \cdot 10^{10}$ cm at $\frac{1}{2}S(\bar{w}_d)$ and $2.1 \cdot 10^{10}$ cm at $\bar{w}_d$. The half width in orbital phase of the anomalous Feature is thus a function of position along the stream not only because of the dependence

of Equation 67 on $\sin \theta$ and $\sin \psi$, but also because w varies in the same Equation.

The angular half widths can be calculated from the values for α and β , the width of the stream, w , and Equation 67. The results for the three positions are given in Table VI, with ϕ_0 and $\phi_{\frac{1}{2}}$ representing the orbital phases of the maxima and half amplitude points.

Examination of Figures 40-43 shows that the data is experimentally consistent with the anomalous Feature being stronger from $180-210^\circ$ and $210-240^\circ$ than at $240-270^\circ$, but perhaps even marginally present from $150^\circ-180^\circ$. This is in good agreement with the above interpretation of the theoretical results of Lubow and Shu.⁶⁹

Before proceeding farther, the second reason why the anomalous Feature should be seen mostly in the $\phi = 0.50-0.75$ orbital phase interval is that if the inclination is near enough to 90° , then the accretion disk may occult it until $\phi = 0.5$. The weakening of the Feature toward $\phi = 0.75$ is only explained by the first argument.

The doppler shifts of the anomalous Feature may also be computed for the three points on the stream for various orbital phases. The frequency of pulsation as seen by an observer comoving with the stream will be shifted according to the velocity

$$U = (U_X^2 + U_Y^2)^{\frac{1}{2}} \cdot \cos (\alpha + \beta) \Omega D \quad (71)$$

where D is the separation of the two components and Ω is the orbital frequency in radians s^{-1} .

The frequency seen by an observer on the earth will be again shifted

by the velocity of the stream projected along the line of sight,

$$V = (U_X^2 + U_Y^2)^{1/2} \cdot \Omega D \cdot \cos \gamma = (U_X^2 + U_Y^2)^{1/2} \cdot \Omega D \cdot \cos(\phi - (\alpha + \beta) - \pi) \quad (72)$$

and the net result seen will be shifted by the effective "moving mirror" velocity of

$$W = U + V = D(U_X^2 + U_Y^2)^{1/2} (\cos(\alpha + \beta) + \cos(\phi - (\alpha + \beta) + \pi)) \quad (73)$$

Along with this velocity, there must also be added the velocity of corotation of the clump in the orbital system. This velocity can be designated by the equation

$$T = \sin(\phi + \delta) \cdot \Omega D \cdot (X^2 + Y^2)^{1/2} \quad (74)$$

where $\delta = \tan^{-1}[-Y/(X)]$. The exact coordinates X and Y of the various points along the stream for $M_D/M_C = 0.5$ are taken from Table 3 of Lubow and Shu.⁶⁹ The difference in the scale of the results between $M_D/M_C = 0.5$ and $M_D/M_C = 0.6$ is very slight. The maximum variation would be expected at the position of the inner Lagrangian point, L_1 , and this changes only from 0.57 to 0.55^+ in units of the component separation. Tables VI and VII give the parameters involved in the velocities and numbers for

$$Z = T + W \quad (75)$$

at various orbital phases.

The results for the Z values in Table VII indicate that the clumps which contribute to the anomalous Feature are forming half way between $L1$ and $\frac{1}{2}S(\bar{w}_d)$. At this point the orbital phase of the maximum would be near 220° . The density expected for this part of the stream (assuming high efficiency of accretion) would be expected to be low since the dimensions of the cross section have not yet substantially decreased below those of the stream near $L1$ while the $\approx 100 \text{ km s}^{-1}$ gain in velocity after 10^{11} cm of travel has stretched out the fluid elements in the stream.

Optical pulsations would not ordinarily result from such a low density gas. However, radiation from Her X-1 would have a tendency to force the gas into optically thick clumps whose gravitational weight would then overcome the radiation pressure. Once the denser clumps have formed, then both free-free continuum pulsations and recombination pulsations would be possible by the mechanisms discussed above for Feature III.

If the radiation pressure is, in fact, important to the hydrodynamics of the accretion stream, then its velocity and position could be modified to differ from the first order results of Lubow and Shu.⁶⁹

The other way out of the low stream density, as pointed out by S. Lubow (private communication, 1975), is that the $\dot{M}$ may be scaled up by several orders of magnitude from the nominal lower limit of $10^{17} \text{ gm s}^{-1}$ set by the x ray luminosity and the accretion process. In this case the "cloud" which produces the optical pulsations of Feature III would be the uppermost regions along the z dimension of the actual stream so that the density of the "cloud" would remain near $2 \cdot 10^{13} \text{ cm}^{-3}$. This picture requires that the orbital inclination is not exactly 90° . The section which follows the 35-day phase will also produce a similar requirement.

D. The 35-Day Phase and Remaining Mysteries

1. Backward-Marching Aspects

Most of the details of the mechanism which drives the observed 35-day phase remain a mystery. In the data which has been discussed above, there remains the anti-correlation of Features I and II and the disappearance of Feature III half way through the "on" interval. The uniqueness of the "turn-on" phases at either $\phi = 0.23$ or $\phi = 0.68$ ^{20,27} is also difficult to explain.

The understanding of the material configuration about the neutron star and in the disk together with the x ray beam intensity pattern are essential to unravelling the mysteries. The optical pulsation data is not a great barrier to any such understanding since the above discussion has illustrated the multitude of physical conditions which can modify optical pulsations.

As an example, the anti-correlation between Feature I and II might easily be caused by a 0.2 radian shift of the x ray beam between $\phi = 0.25$ and $\phi = 0.75$ in the z direction. The disappearance of Feature III could be because of a decrease in the stream density produced by radiation pressure inhibiting the low through L1.

The x ray "turn-ons" may be caused by nodes in the accretion disk where the z height is smaller than usual. The other parts of the disk may have a high z profile due to material which is ejected upwards and downwards by the radiative shock which results from the collision of the stream with the disk. The material ejected near orbital phase 0.75 would converge again near orbital phase 0.25 to cause another shock (since the orbital times are all nearly the same at the same radius). Thus the two nodes may be easier for the x rays to "see over" and the turn-ons occur at these points in

orbital phase. The catch in the argument is that there is no reason why the x ray source *stays* on afterward, since it should run into the disk's higher z profile later as well as earlier.

A variety of mechanisms can be invoked to help explain the 35-day clock mechanism. Two of these, free-precession and radiation pressure, would naturally be crucial to altering the physical state of the binary system. When free precession is considered, one has a variety of choices for the period. It could be exactly 34.85 days (20.5 cycles) or it could be some beat between 1.700165 days and 34.85 days.

To test the "beat frequency" hypothesis, it is possible to construct figures similar to Figures 47-50, 85 and 102 except that the folding period can be altered to any value desired.

Figure 105 has been constructed using a period, P , equal to $P_0 \cdot 20.5/21.5$, i.e. the *higher* beat frequency between $P_0 = 1.700165$ days and $20.5 \cdot P_0$ (exactly 21.5 new periods are equal to 20.5 old periods). The fiducial date for the folding process is JD 1506.3921 + 2,440,000.5--the middle of an eclipse in the center of an "on" period (cycle 10.00 reckoning with the old P_0 's). The arbitrary choice of the fiducial will lead to an arbitrary shift of any pattern resulting from the folding process but cannot materially change the results.

Figure 105 does in fact show a non-flat optical pulsation profile in the middle frame; the lower frame also shows the boxes grouping in correlation to the middle frame. The patterns are double-humped with most of the boxes and pulsations grouping about 0.00 1.62-day phase and a smaller grouping and pulsation peak about 0.50 1.62-day phase.

On the other hand, when the data are folded in Figure 106 with a period

equal to $P_0 \cdot 20.5/19.5 = 1.787353$ days (the difference frequency) the patterns seen are less unique in both the middle and lower frames. The rms deviation for the mean pulsed fraction is 0.27 scale units whereas the 1.62 day result gave 0.32 scale units (normalized pulsed fraction of $2.7 \cdot 10^{-4}$ and $3.2 \cdot 10^{-4}$ respectively).

The fact that the sum frequency shows more significance than the difference frequency indicates that some mechanism which is tied to the 35-day behavior is progressing in the opposite sense as the orbital motion (Paul Boynton, private communication, 1975). This is apparent because the shorter period leads to a backward-marching pattern when seen with respect to the 1.7 day period. This correlates well with the backward marching behavior of the x ray "dips". On the other hand, it is not obvious how the free precession of a prograde-spinning neutron star could account for the phenomenon, since the wobble of the pulsar's equatorial plane (equatorial with respect to the eigen tensor of inertia closest to the rotation axis) would wobble in a prograde fashion whether the neutron star were oblate or prolate.

In order to test how non-random the patterns of Figure 105 and 106 really are, Figures 107 and 108 have been constructed using folding periods of $P_0 \cdot 20.75/21.25$ and $P_0 \cdot 21.25/20.75$ or 1.660161 days and 1.741133 days respectively. The arrangement of boxes in the lower frames of both figures are now very random. The middle frame showing the mean pulsed fraction for Figure 108 is also random, whereas a four-fold symmetry is seen in the middle frame of Figure 107. This is a consequence of the shifting of the two-fold symmetry of Figure 105 by a period change which corresponds to exactly $\frac{1}{4}$ cycle of smear every 34.85 days. Thus, in all but one accidental respect, Figures 107 and 108 are indeed very random in comparison to Figure 105.

Proceeding farther, it is also appropriate to beat the 1.7 day period with a period equal to $10.25 \cdot P_0 = 17.427$ days. This has been done to construct Figures 109 and 110 which have been folded at 1.5496392 days ($= P \cdot 20.5/22.5$) and 1.8839666 days ($= P \cdot 20.5/18.5$) respectively. Comparison of the two Figures indicates that the sum frequency where results are shown in Figure 109 produces the more distinctive pattern of the two. Its uniqueness seems comparable to that of Figure 105.

The results of Figure 109 should not really be surprising, since most 35-day mechanisms such as free precession of the pulsar would tend to repeat their physical conditions twice in every 35-day period. For the example case, the only difference in the conditions of the two occurrences could be in the z coordinates and the effect of this difference is not easily distinguishable by an observer near the orbital plane of the system.

2. The Her X-1 Mystery and Beam Precession

Much of the mystery of the HZ Her - Her X-1 system can be explained if Her X-1 is given a fan beam about its equator and a slightly non-aligned spin about its axis of symmetry with respect to the orbital plane (this nomenclature is not necessarily the same as given in Basko and Sunyaev³⁴ but just an empirical description of the radiation pattern -- which is assumed to lie about a plane rather than a cone).

The angle of alignment of the pulsar's magnetic axis with respect to its symmetry axis (spin axis) is difficult to establish. Although all theoretical models for the accretion have the material falling onto the magnetic poles, the opacity processes involved in such dense material ($\approx 10^{21} - 10^{22} \text{ cm}^{-3}$) in large magnetic fields are poorly understood.⁸⁷

However, it seems reasonable that a beam in the equatorial regions of the pulsar's body coordinates could be produced under a variety of a priori conditions by adjusting the alignment of the magnetic axes.

Using a nearly-aligned, precessing, equatorial fan beam, the x ray "on" and "off" interval can be understood as the intervals during which the fan intercepts the direction to the earth. If $d\theta$ represents the angular fraction, $d\psi$, of the 35-day phase is equivalent to the equation:

$$(90 - i) = \theta_0 \cdot \cos(d\psi/2) + d\theta/2 \quad (76)$$

On the other hand, the amplitude, θ_0 , must not be so much greater than the $\frac{1}{2}$ width, $d\theta/2$, that the x ray fan is lost in the middle of the "on" interval. This gives the additional constraint that

$$\theta_0 - d\theta/2 \leq (90 - i) \quad (77)$$

Since $d\psi/2$ is not greater than $\pi/2$ even including the obscuring effects of a material swollen disk before the beginning of the "on" interval (this is an empirically established possibility which is not easily explained at this point) and since $d\psi/2$ is no less than 60° , then θ_0 and $d\theta/2$ must lie somewhere in the triangular region shown in Figure 111 on the $d\theta/2 \leftrightarrow \theta_0$ plane bounded by the points $[(90-i), 2(90-i)]$, $[(2-\sqrt{3}) \cdot (90-i)]$, $[(3-\sqrt{3}) \cdot (90-i)]$ and $[(90-i), 0]$. Thus $d\theta/2$ must lie between $2(90-i)$ and 0 in such a way that Equation 76 is satisfied for the particular value of $d\psi/2$.

The limit of $(90-i)$ established by the intersection of the regions allowed by the optical pulsation data and the observed duration of the

x ray eclipse is about 0.1 radians.

An attempt to explain the sequencing of the 35-day pattern may now be made. The factor $(90-i)$ is chosen at its limit of 0.1 radians since an absorbing disk will be invoked near ± 0.1 radians in the orbital plane. For a significant flux to emerge above this disk and still excite the surface of HZ Her a third criterion about the x ray fan parameters may be invoked, namely that when the fan appears above the absorbing disk it shows at least as much width as 0.1 radians or

$$\theta_0 + d\theta/2 \geq \theta_D/2 + 0.1 \leq 2(90-i) \quad (78)$$

This constraint may be represented in Figure 111 by a line which passes through the points $[(90-i), (90-i)]$ and $[1/2(90-i), 3/2(90-i)]$ on two sides of the triangle derived above.

Figure 112 has been constructed to illustrate the sequencing of the 35-day phase. The x ray fan beam is now limited to the action of its two "lips", one showing above the orbital plane (on the same side as the earth-bound observer, denoted by the +'s) and the other showing below the orbital plane. The orbital phase of the polar coordinates of Figure 112 increases right handedly (counter-clockwise) from the bottom, where it starts at $\theta = 0.0$ (eclipse). The orbital phase at which the upper lip points to HZ Her is marked by lines starting at the outside of the inner circle and ending at the small images of Roche lobes with the + signs marked within. The number with the + superscripts which label the lines at their beginnings at the inner circle indicate the cycle number in the 35-day phase for which the orbital location of the upper lip on HZ Her is

appropriate (e.g., the 10^+ indicates that the upper lip of the x ray fan hits the lobe when the eclipse is occurring in the middle of the x ray "on" at cycle 10.00). The 35 day phase of 9.75 has been chosen to correspond to the upper lip pointing toward HZ Her at eclipse mid-center (and consequently the earth sees the x rays at the other orbital phases) and thus represents the chosen center of the "on" state.

The inner circle is similar to the region outside of the circle whose sequencing has been discussed above. The only difference is that it represents effects of the x ray fan's "lower lip" or minus (-) side.

Starting at approximately cycle 0, the upper lip of the x ray fan strikes HZ Her at $\theta = 0.5$ (180°) and the resulting hot spot is easily seen from earth as long as $(90-i) > 0$. None of the fan strikes HZ Her at $\theta = 0.25$ or $\theta = 0.75$ so that the light curve diminishes from a sharp peak at $\theta = 0.5$ into a deep eclipse and no optical pulsations are seen from the limb.

The one time observation of optical pulsations in Feature III near 35-day cycle 20.8 (29 Apr 73) is consistent with the above description since the pulsed optical radiation pattern is broad in the z dimension (relative to the thinness of the disk) and the observation could easily indicate an x ray-gas interaction on the bottom side of the disk. The x ray observation of Copernicus⁵⁸ in this interval near phase 1.5 (circa 05 Jun 73) is more difficult to explain. One possibility is that at the time of maximum "dip" of the pulsar's spin axis with respect to the earth, the radiation from a location on the neutron star which stays nearer to the poles has fluctuated so that it is now visible to the earth. The other possibility is that the width of the fan suddenly increased a great deal. Both of these

possibilities should produce characteristic changes in the 1.7 day light curve which may have been observed at the time.

As the 35-day phase progresses into the 1.0-5.0 cycle region, the lower lip center regresses from hitting HZ Her at orbital phase 334° to 265° . Since the width of such lips are at least $\pm 60^\circ$, the lower lip of the fan will tend to produce optical pulsations at the appropriate orbital phases of $\phi = 0.70-0.85$ as designated by II^- in Figure 112. The lower results in this interval which can be seen with the higher results in Figure 55 all seem to have later orbital phases consistent with the lip passing behind HZ Her too soon after illuminating a portion of the lobe visible from the earth (the lip orientation being too close to the line-of-sight direction).

Meanwhile, starting with cycle 4.0 the upper lip is directed toward HZ Her at orbital phase 101° and the parts of the lip closer to the earth strike the lobe earlier in binary phase and begin to excite substantial optical pulsations as seen in Figure 54 and in Figure 112 designated as I^+ . These pulsations continue through cycle 8.0 at which time the upper lip points toward HZ Her at 31° orbital phase.

By cycle 7.0, however, the "forward" part of the upper lip has regressed so that the x rays can be seen in the direction of the earth. The cause of the "burn-through" effect with the discrete phases of the "turn-ons" (0.23 and 0.68) is still poorly understood.

The "on" progresses smoothly from cycle 7.0 save for the well-understood orbital eclipses and the poorly understood backward-marching "dips". Just prior to cycle 10.0 the lower lip of the x ray fan hits the lower z portions of the lobe of HZ Her at $\phi = 0.5$. However, the resulting hot-spot is not seen for two reasons: First, the path of the resulting light must travel

directly (plus or minus the width of the x ray fan beam) back to the earth and therefore must pass through the accretion disk. Second, the slight value $(90-i) > 0$ worsens the obliquity factor for viewing the hot spot from the earth. A secondary eclipse is the actual result for orbital phase near 180° .

From cycle 10.0 onward to 14.0 the upper lip of the x ray fan regresses until it is pointed at HZ Her at orbital phase 286° . Optical pulsation designated by II^+ in Figure 112 can be seen throughout this interval as shown in Figure 55. The somewhat ambiguous dip during cycle 11 and the anomalously early results near phases 8.8 and 9.8 (this shorthand designation gives the integral cycle number with the fractional orbital phase attached) may be the result of x ray pulse absorption and/or reprocessing in the part of the accretion stream which is close to L1. The II^+ pulses continue until cycle 16.0.

However, starting before cycle 14.0 and perhaps earlier, the lower lip of the x ray fan is striking the lobe of HZ Her at negative z near $\theta \approx 0.25$ to produce the pulsations designated I^- in Figure 112 and shown in Figure 54. These continue until about 18.0

Three assumptions have been essential to the validity of the above discussion. The first is that the factor $(90-i)$ is finite and perhaps larger than 0.05 radians. The second essential is a backward-precessing x ray beam whose characteristics are restricted by the constraints shown in Figure 111. The third is an absorbing disk, whose $\frac{1}{2}$ thickness, $\theta_D/2 < (90-i)$ but which is dense enough so that the transitions of the x ray fan beam through the orbital plane do not reach the lobe of HZ Her to excite optical pulsations. This last restriction can, in principle,

he used to set a lower limit on $\theta_D/2$, but since the x ray "on" is already at least 120° in extent -- about the same as the extents of I^+ and II^+ in Figure 112 -- this lower limit will be close to (90-i) in any case. Further constraints on $\theta_D/2$ could come from calculations indicating the relative strength of I^-/I^+ and II^-/II^+ with varying θ_D and then comparing them to the experimental results of Figure 54 and 55.

However, according to the calculations discussed above with the 0.1RES, 0.2RES, 0.1REB and 0.2REB models, a shift of only 0.1 radian in the centroid of illumination can make a substantial difference in the amount of optical pulsation. This indicates that the conditions discussed where $(90-i) \leq 0.1$ radians are sufficient to cause the observed ratios of amplitudes I^-/I^+ and II^-/II^+ . These ratios may be more sensitive to θ_0 than to θ_D .

An alternative picture to the precessing x ray fan may be found in the slaved oriented disk model as suggested by Roberts³⁷ where the off-axis spin of *HZ Her* leads to a 35-day precession. The accretion disk is slaved to orient perpendicularly to the spin axis of *HZ Her* -- how this occurs or might even be possible is not entirely clear. A small non-trivial value for $(90-i)$ is again invoked together with an unrestricted x ray fan beam ($d\theta/2 = \pi/2$) which doesn't change.

The equation which represents the width of the "on" interval then becomes

$$-\theta_0 \cos d\psi/2 + d\theta/2 = (90-i) \quad (79)$$

where θ_0 and $d\psi/2$ are now the parameters describing the amplitude and half width of the disk. The equation which prevents x rays from being observed

during the "off" interval is:

$$\theta_0 - d\theta/2 \leq (90-i) \quad (80)$$

Figure 113 is analogous to Figure 111 and shows how these new restrictions limit θ_0 and $d\theta/2$. The solutions are restricted to the triangular region bounded by the three points $\{(90-i), 0\}$, $\{(90-i), 2(90-i)\}$ and $\{3(90-i), 4(90-i)\}$. The dimensions required of the disk width $d\theta/2$ are larger than those previously required of the x ray fan beam width or of the old disk width. The allowed variation on the θ_0 of the disk is also twice that allowed the x ray fan width.

The Slaved Disk model has the advantage that the Copernicus results of x rays seen in mid-"off" require only a fluctuation of the disk lower boundary as it is constrained in equation 80. The backward-marching dips are also more naturally explained by a backward-marching feature on the disk -- presumably due to its intersection with the accreting stream of material.

However, the Slaved Disk may not prove to be enough to explain the extreme variation of the 1.7 day light curve over the 35-day phase without also restricting the x ray beam so that upper portions of the lobe of HZ Her are not strongly illuminated during the x ray "on" interval. Moreover, the Slaved Disk model is at a loss to account for the absorption-free x ray "turn-offs", just as the x ray fan model fails to account for the "burn-through" nature of the x ray turn-ons. Perhaps some compromise between the two models (e.g. where the disk is coupled by radiation pressure to a precessing x ray fan beam) could be synthesized to account for

the observations.

Finally, it would seem that both models involve some considerable disk shadow. If so, the experimental evidence is then slightly less effective in restricting HZ Her to the filled Roche lobe. New, more detailed calculations can be done as soon as some model is found to explain enough of the other observations to be worthy of the trouble.

E. Odds and Ends

A few comments will be made here concerning some sensitive points not elaborated in the above discussion. The first of these -- and the most serious -- is the unexplained "scatter" in the velocities of Features I and II. The origin of this scatter is not known but two explanations are plausible.

The first explanation for the scatter in velocities is that it is due to flares which interrupt the smoother Roche surface. Such flaring (of 0.2-0.3 magnitudes in amplitude) has been observed in the photometric results of Davidsen et al.³ The second explanation is that a time-varying shading on the Roche surface causes a variation in the results.

The systematic effects that either of these causes might have on the determination of the mass ratio is unknown and will remain so until the possibilities are studied in detail. Figure 52 gives an indication that the runs resulting from shorter integration times (which is essentially the same as the ones with the larger errors) are responsible for most of the unwarranted scatter in the data.

The method of assigning the velocities is explained fully in the Appendix and is not in question. Moreover, no such unwarranted scatter is observed in the six events which make up Feature III (a χ^2 of 4.5 for

5 degrees of freedom).

The other point, which was raised in the Data section and promptly ignored, is that of the event seen in Figure 27 to the left of the main peak. The main peak has a full significance of some 38 exponential levels (see Table I) and, as shown in the Appendix, will consequently have a nearby side-lobe pattern as high as 22% of its amplitude. The contention here is that the peak seen in the figure is the result of a not-too-unusual noise event accidentally having the same complex phase as the side lobe. Thus a noise vector of amplitude 2.0 (ordinarily yielding a 4.0 exponential level bin in the power spectrum) adds coherently to a side-lobe amplitude of 1.3 and gives a result whose amplitude is 3.3 and whose power is near 11 exponential levels.

Accordingly, the complex phase which the side lobe should have had was compared to the complex phase of the spike. The two phases agreed closely, thus justifying the peak as a true "accident",

F. Summary

Figure 114 shows a sketch of the Lubow and Shu stream and disk superimposed on a Roche lobe system for a mass ratio of 1.7. The handedness of the picture has been inverted so that the earth is seen to be rotating about the system in a left-handed (clockwise) sense.

The limb Feature I and II have already been discussed ad infinitum. The physical conditions of the cloud of gas which gives rise to Feature III pulsations by transmission reprocessing have been established. The width, depth and density derived are consistent with the dimensions of Lubow and Shu⁶⁹ for the dimensions of a hydrodynamical accretion stream. A

reasonable height of $2 \cdot 10^9$ cm is derived from scale height arguments. The temperature required of the cloud to sustain optical pulsations was found to be roughly $2 \cdot 10^4$ K -- perhaps much cooler than possible. Thus either the cloud must cool itself by some unexplained mechanism, or else somehow produce optical pulsations more efficiently than thought possible -- or both.

The cloud has been shown to be capable of producing an x ray "dip" in simultaneity with its role of reprocessing x rays into optical pulsations. The width required of the cloud, $4.5 \cdot 10^{10}$ cm is consistent with the duration of the x ray "dips" taken together with a distance from the x ray source at $4.5 \cdot 10^{10}$ cm. The density of $2 \cdot 10^{13}$ cm⁻² through $1.5 \cdot 10^{10}$ cm of cloud depth was found to be consistent with the production of both optical pulsations and x ray "dips". However, to see Feature III pulsations, it is not necessary to have the cloud interpose between the earth and Her X-1 insofar as the z coordinate is concerned.

There are four reasons favoring a revision of the cloud x ray source distance to 10^6 km from the nominal $4.5 \cdot 10^5$ km:

1. The orbital phase of Feature III is consistent with the stream intersection with the larger but not the smaller (disk) radius.
2. A cloud at 10^6 km would have less difficulty staying as cool as $2 \cdot 10^4$ °K and therefore less difficulty producing optical pulsations.
3. The volume of the cloud would increase because the (isothermal) vertical height increases as $r^{3/2}$ and thus more optical pulsations could be generated (the results of equation 65 are not quite sufficient).

4. The mean acceleration residual of the cloud is slightly more favorable to a separation from Her X-1 of 10^6 km instead of $4.5 \cdot 10^5$ km.

The stream of Lubow and Shu has been found to be ideally oriented to produce the anomalous Feature of optical pulsations by reflection reprocessing. The directionality agreement depends upon a flat "shock" surface remaining roughly perpendicular to the stream direction. The velocity of the anomalous Feature indicates that the appropriate location of the gas clumps (or waves) in the stream would be only $\frac{1}{4}$ the distance from L1 to the point of collision with the disk ($\sim 800,000$ km).

The variations of the optical pulsations, 1.7 day light curve and the flux of x rays over the 35-day period can be fairly well accounted for by a backward-precessing x ray fan beam or an oriented obscuring disk with the same motion. The inclination angle cannot be *exactly* 90° , but $(90-i) \approx 5^\circ$ seems sufficient to cause the asymmetry of the 17-day variations observed in the optical pulsations. The x ray fan model also requires an obscuring disk of dimensions not larger than $(90-i)$. The finer details of the x ray behavior -- the backward marching dips, the difference between the "turn-ons" and the "turn-offs" in the amount of absorption and finally the discrete phasing of the "turn-ons" -- remain mysterious.

VI. CONCLUSIONS

- A. A volume effect reprocessing reflection effect model has been constructed to successfully reproduce the observed profile of optical pulsations of Features I and II (limb) versus orbital phase.
- B. The low energy end of the x ray spectrum dominates the pulse reprocessing. Although this part has been observed to be only 5-20% pulsed⁶⁵ the results of Davidsen, Margon and Middleditch⁵⁹ indicate that this flux is 100% pulsed in certain directions. The observed spectrum can account for 25% of the total optical pulsation by the recombination process; for a 100% pulsed soft source, ten times as much optical pulsation can be produced if the total (pulsed and unpulsed) soft flux constrained to be constant.
- C. The direction of the pulsar "spin" vector is limited by observation of a speed asymmetry between Features I and II to align to within 46° (1σ) and 67° (2σ) of the orbital angular momentum vector. Any increase of the pulsar frequency above 0.8078735 Hz during the data-gathering interval of July 1972 - September 1974 will tend to rapidly align the two vectors more closely (and will slowly make the experimental results a posteriori less likely).
- D. The mass ratio of the system is determined to be 1.69 ± 0.05 from the study of Features I and II; the effects of systematics and the revised statistical errors are individually all 0.03 or smaller.
- E. The orbital inclination is determined to be near 85° with the help of the x ray eclipse duration. A 2σ limit on this eclipse limits the inclination to $>80^\circ$. The simplest models for the origin of the 35-day variations in optical pulsations, 1.7 day light curve and x ray

intensity variations restrict i from reaching 90° and possibly the interval $87.5^\circ - 82.5^\circ$ is sufficient to explain all effects, with 85° as the most probably value.

- F. The masses of the components of the system are determined to be $M_{\text{HZ}} = 2.16^{+0.08}_{-0.06} M_\odot$ and $M_x = 1.28^{+0.08}_{-0.07} M_\odot$. This is the first accurate measurement of the mass of a neutron star -- precision.
- G. The sizes of any band of illumination or shadow from illumination on the Roche lobe which can diminish or augment the full illuminated/shaded intensity to $\frac{1}{2}$ the maximum value are both 0.1 radians *full width* as seen from Her X-1 (or from the c.m. of HZ Her; centered on the observer's equator (this region is some 400,000 km wide). These rapid amplitude changes for relatively small changes in dimension facilitate the construction the 35-day models which account for the II^-/II^+ and I^-/I^+ ratios with only minimal (90-i) values.
- H. The Roche lobe is known to be filled from the absence of optical pulsations near $\phi = 0.5$ from the limb of HZ Her. This conclusion is essential for the determination of the mass ratio and the orbital inclination. If some disk shadow across the lobe exists, the requirement of filling becomes less stringent.
- I. The observations of Davidsen, Margon and Middleditch⁵⁹ restrict the illuminated portion of the Roche lobe to ± 0.1 radians about the observer's equator in order that the He II and N III emission appear $> 35\%$ pulsed. This area could still be as large as 5-10 square light seconds.
- J. The remaining 2/3rds of optical pulsation in Features I and II can either be continuum or widely dispersed emission lines. Balmer

emission is least likely.

- K. The systematic effects on the mass ratio due to a small degree of non-corotation of HZ Her are miniscule, a 5% non-corotation changes 1.69 by less than 0.01.
- L. Higher harmonics of the pulse profile are suppressed by the averaging process over the Roche lobe. Subharmonics are not generated by the likely 1st order non-linear differential equations relating the radiative energy transport to the gas temperature. The gravitational modes of relaxation might be capable of generating subharmonics in the optical pulsations.
- M. Feature III of the optical pulsations has been shown to be consistent with transmission reprocessing through a cloud of gas $1.5 \cdot 10^{10}$ cm thick, $4.5 \cdot 10^{10}$ cm wide and $4 \cdot 10^9$ cm high of density $2 \cdot 10^{13}$ cm⁻³ and $4.5 \cdot 10^{10}$ cm away from the x ray source.
- These dimensions are consistent with the dimensions of the accretion stream of Lubow and Shu⁶⁹ at its point of intersection with the equilibrium disk about Her X-1. The necessary temperatures for optical pulsation are near $2 \cdot 10^4$ °K -- very low indeed. The cloud can account for an x ray "dip" while simultaneously producing transmission reprocessed optical pulsation. The 35-day behavior of Feature III is not well understood, but changes in T and n of the cloud can easily make the optical pulsations go away. There is evidence which indicates the cloud distance and width may be nearer to 10^6 km. If so, the cloud would be easier to keep cool ($\sim 2 \cdot 10^4$ K), and would have a larger scale height thereby producing optical pulsations more readily.

- N. The 35-day behavior of Features I and II, the 1.7 day light curve and the x ray intensity can be related with a model including an x ray fan beam which precesses in a retrograde fashion and a constant-width absorbing disk or to a model of a slaved oriented absorbing disk with retrograde precession and less restricted x ray beaming (in neither case may $(90-i) = 0$).
- O. The anomalous Feature may be related to clumping or "waves" in the accretion stream of Lubow and Shu⁶⁹ some 800,000 km from L1 which causes a reflection-reprocessing effect. The clumps must present, on the average, flat surfaces oriented perpendicularly to the stream motion (radiation pressure may help stabilize the flatness of the surface and the entire "wave" its lf). The orbital phases of the anomalous Feature agrees with this hypothesis and the width of the stream at the indicated position is sufficient to modulate the strength of the anomalous Feature in orbital phase so that its full width is $\approx 90^\circ$.

VII. APPENDIX

A. Fourier Transforms

1. Introduction

The Fourier transform is one of the most fundamental methods of data analysis used in science. It has several advantages which make it easy to use. The first is linearity. The second advantage is its simplicity. Accordingly, Fourier transforms were used extensively as a data analysis tool in this thesis and a natural generalization of the transform technique is used to measure f and df/dt to fine precision. The transform method is particularly suited to data which lacks higher harmonics -- like the optical pulsations from HZ Her -- since all of the power is recovered near the fundamental frequency.

The effect of a transform of random noise will be discussed first. Following a few sections dealing with some details of Fourier transforms, a section is presented which deals with sinusoidal signals. The convolution theorems are briefly discussed and used to derive the finite, discrete Fourier transform from the infinite, continuous transform. There follows a statistical summary of methods useful in dealing with power spectra and finally a section which discusses signals in the actual presence of noise.

The generalization of the transform technique to measure f and df/dt by signal folding is presented in Section B. Finally, Section C will present a few comments on the application of digital methods to fast photometry.

2. Noise

The r th element of the Fourier transform of a uniformly-spaced data string (n_k ; $k = 0, N-1$) is given by

$$A_r = \sum_{k=0}^{N-1} n_k e^{-2\pi i k r / N} \quad (1)$$

As seen in the complex plane, the summing process is no more than a vector addition, with each vector rotated clockwise (left-handedly) by $2\pi r/N$ from the previous vector. Thus each vector contributor to the sum has its direction fixed by the order in which it appears.

If the n_k ensemble is taken to have constant value, $x > 0$ independent of k , the summation process is seen in Figure 1A to approximate r regular, closed polygons, all of which return to very near the origin. The last step must, in fact, return the net vector exactly to the origin. Thus the process of forming the r th frequency element involves a cycling of the data through r "N/r-gons" on the complex plane. If constant data is used, this cycling must result in a vector which is identically zero for all $r > 0$.

Now suppose a real ensemble, n_k , is used with real fluctuations d_k about an average, x . Because the transform is a linear process, the fluctuations may be considered independently from the constant length steps.

Although the complex phase factor in Equation 1 is fixed for any given k , the d_k value may be either positive or negative. Thus the transform process acting on the fluctuations creates a special kind of random walk in the complex plane. The step directions take on special, *almost*-fixed values but the step sizes may vary considerably. An example is shown in Figure 2A.

The statistical properties of this N step random walk may be estimated by considering the distribution of the components projected along an

arbitrary axis in the complex plane. As long as the fluctuations, d_k , are random -- that is, uncorrelated with k and in particular the "n/r-gon" when considering any particular r -- then the distribution of the projected components will have zero for the first moment. The second moment will be given by $\frac{1}{2} \overline{d_k^2}$ with the $\frac{1}{2}$ factor entering from the average of $\sin \theta_k$, where θ_k is $2\pi kr/N$. So long as the number of "N/r-gons" is large, the result of the random walk projected onto an axis may be considered as a sum of a large number of identically distributed random variables. By using the Central Limit Theorem, the distribution of such sums is Gaussian, with a center at $N \cdot 0 = 0$ and with a variance which is given by

$$\sigma^2 = N \cdot \overline{d_k^2} \cdot \frac{1}{2} \quad (2)$$

This result may be applied to both the real and imaginary axes of the complex plane so that the probability of finding a sum between x and $x + dx$ and between y and $y + dy$ is:

$$\text{Pr}(x,y) \, dx dy = \alpha e^{-(x^2+y^2)/(N \overline{d_k^2})} \, dx dy \quad (3)$$

where α is a normalization constant.

When transformed into polar coordinates, with $2\pi r dr$ used as the degenerate area element and with $r^2 = (x^2+y^2)/(N \overline{d_k^2})$ we get

$$\text{Pr}(r) dr = \alpha e^{-r^2} 2\pi r dr \cdot (N \overline{d_k^2}) \quad (4)$$

When the substitution $P = r^2$ is made, the distribution becomes

$$\text{Pr}(P)dP = \alpha e^{-P} \pi dP (N \overline{d_k^2}) \quad (5)$$

To normalize the probability to unity, α is set

$$\alpha = \pi^{-1} \cdot (N \overline{d_k^2})^{-1} \quad (6)$$

so that

$$\text{Pr}(P)dP = e^{-P} dP \quad (7)$$

where P is now the properly normalized power.

If the axes are labelled in units of $\sqrt{P}$, then from Equations 2-6 it can be shown that the σ of the Gaussian profile (projected onto either axis) is given by $2^{-1/2}$ after the coordinates were scaled by a factor of $(N \overline{d_k^2})^{-1/2}$ -- the nominal root mean squared length of the random walk. The plane with units of $\sqrt{P}$ will be designated as the "amplitude plane".

The above discussion should also apply to the small r 's, although it is more difficult to demonstrate the applicability of the Central Limit Theorem.

The distribution in noise power is seen in Equation 7 to be exponential -- this greatly simplifies the analysis of power spectra. The phases of the vectors resulting from noise are completely random.

In practice, the photopulses from the system are summed in successive time windows of duration, $dt = 1/f_{\text{samp}}$, where f_{samp} is the frequency at which the time windows are "flying by". A run lasts for a time, $T = Ndt$ and the quantity, $\overline{d_k^2}$ with units of counts²/time window is related to the

quantity, MSD, from Section II D3 (the mean squared deviation from the average per second) by the formula

$$\text{MSD} = \overline{d_k^2} \cdot f_{\text{samp}} \quad \text{counts}^2 \text{ s}^{-1} \quad (8)$$

The total power of the run is given by

$$P_{\text{tot}} = \text{MSD} \cdot T = N \cdot \overline{d_k^2} \quad \text{counts}^2 \quad (9)$$

By Parseval's Theorem, the total power in the raw data must be equal to the total power in the Fourier power spectrum. This fact will be used to establish a normalization constant which will guarantee that the average of the bins *any* frequency power spectrum will be *exactly* unity. This constant has already been found in the transition from Equation 3 to 4 and it is equal to $\pi^{\frac{1}{2}} \alpha^{\frac{1}{2}} = (N \overline{d_k^2})^{-\frac{1}{2}}$ (Equation 7 guarantees only that the statistical average of P is unity -- the present discussion will provide a different constraint).

a. Parseval's theorem

A proof of Parseval's Theorem for the incremental transform will be given below -- lacking only the appropriate normalization constant.

Starting with equation 1 and using the complex conjugate to form the power we get:

$$P_r' = |A_r|^2 = \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} n_k n_l e^{-2\pi i r(k-l)/N} \quad (10)$$

where P_r' is the unnormalized power. To find out how to normalize so that

the sum of P_r' is N we sum:

$$\sum_{r=0}^{N-1} P_r' = \sum_{r=0}^{N-1} \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} n_k n_l e^{-2\pi i r (k-l)/N} \quad (11)$$

Inverting the order of summation and a few variables in the exponent we get:

$$\sum_{r=0}^{N-1} P_r' = \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} n_k n_l \sum_{r=0}^{N-1} e^{-2\pi i (k-l)r/N} \quad (12)$$

The last sum can be recognized as the quantity $N\delta_{kl}$ and the sum now becomes

$$\sum_{r=0}^{N-1} P_r' = N \sum_{k=0}^{N-1} n_k^2 \quad (13)$$

Discarding P_0' from both sides of equation 13 we get

$$\sum_{r=1}^{N-1} P_r' = N \sum_{k=0}^{N-1} n_k^2 - \left(\sum_{k=0}^{N-1} n_k \right)^2 \quad (14)$$

With a little readjusting, the right hand side of Equation 14 can be converted so that

$$\sum_{r=1}^{N-1} P_r' = N^2 \cdot \left(\sum_{k=0}^{N-1} n_k^2 - \left(\sum_{k=0}^{N-1} n_k \right)^2 \right) = N^2 \overline{d_k^2} \quad (15)$$

Thus to ensure that the sum of the A.C. part of the power spectrum is N, then the normalization constant for equation 1 is exactly $(N \overline{d_k^2})^{-1/2}$. However, in order to get the *average* of the A.C. spectral bins to be exactly

unity, the normalization constant is $[(N-1)/\overline{d_k^2}]^{-1/2} N^{-1}$. For large transforms the difference between these two constants is negligible.

b. Local power level

Although the *mean* A.C. power level of any Fourier transform may be normalized to unity by the process described above, the nature of atmospheric scintillation causes power spectra to vary considerably from flatness below 0.5 Hz as seen in Figure 7. In fact, so much power may be involved in the low frequency spike that the measure of the noise in the data is, a priori, useless in determining the noise power level at any given frequency.

To compensate for this effect, the analysis of *significance* in the power spectrum is accomplished by a running average centered on, but excluding the frequency bin in question and its 4 nearest neighbors. The Local Power Level is defined to be equal to this numerical average when the entire A.C. power spectrum has been normalized to make the sum equal to $N-1$.

The rationale behind this procedure is that, although the *effective* random fluctuations in the data may vary with frequency and cause the power spectrum to be non-flat, the above analysis which shows the random power to be exponentially distributed is still qualitatively correct. The only modification needed is that the transformation made to get the power in Equations 2-6 must use

$$\overline{d_k^2(f)} = \overline{d_k^2} \cdot LL(f) \quad (16)$$

where $LL(f)$ is the local power level as defined above.

The distribution of "local power" is checked every time a Fourier

transform is computed by reduction with the Fortran statement

$$NSN = 100. * \{1.01 - \text{EXP}(-SNSQ)\} \quad (17)$$

where $SNSQ = P/P_{\text{local}}$ and the integer, NSN, truncates the real number to the lower value. A histogram bin of address NSN is then incremented and the resulting histogram is plotted. The results of this process were observed to be flatness -- except when a large signal was present in the data like the Crab pulsar.

In addition, every section of transform which was saved for the HZ Her program, like those displayed in Figures 9-30, was histogrammed according to the Fortran statement:

$$MSN = 2. * SNSQ + 1. \quad (18)$$

The logarithm of the histogram contents were then plotted against power, as in Figures 32-34. The plots showed the expected behavior except for the occasional peak which was associated with the HZ Her-Her X-1 system.

Thus the assumption of locally exponential behavior for the lower was seen to be well grounded in experimental fact.

3. Fourier Transform Morphology

From equation 1 it can be shown that the transform of real data is symmetric between r and $N-r$ by complex conjugation:

$$A_{N-r} = \sum_{k=0}^{N-1} n_k e^{-2\pi i k(N-r)/N} = \sum_{k=0}^{N-1} n_k e^{+2\pi i k r/N} = A_r^* \quad (19)$$

The element which is its own complex conjugate, $A_{N/2}$, corresponds to the Nyquist frequency -- i.e. the complex amplitude of half the sampling frequency -- if N is even.

Both the Nyquist frequency and the zero frequency elements are real and therefore each contains only 1 degree of freedom. The $(N-2)/2$ points between 0 and the Nyquist frequency each contain 2 degrees of freedom so that the total number of degrees of freedom in the transform is $2 + N-2 = N$. The $(N-2)/2$ complex elements above the Nyquist frequency contain no new information.

In practice, the Fourier transform is computed by an algorithm which forms larger transforms by first computing smaller transform and then combining them to form larger and larger transforms. This algorithm is commonly attributed to Cooley and Tukey⁸⁸ who were the first to develop it in greatest generality and to point out its application to high speed computers. However, the algorithm was known previously to the work of Cooley and Tukey.

The basis of the algorithm can be understood with the help of a few simple manipulations. Starting with Equation 1, restricting N to be divisible by 2 and separating the elements with odd indices from the sum to form another sum we get:

$$A_r = \sum_{l=0}^{N/2-1} n_{2l} e^{-2\pi i 2l r / N} + \sum_{l=0}^{N/2-1} n_{2l+1} e^{-2\pi i (2l+1) r / N} \quad (20)$$

Now by factoring out the $e^{-2\pi i r / N}$ from the second sum and then moving the factor of 2 to divide the denominators of the exponents we get

$$A_r = \sum_{l=0}^{N/2-1} n_{2l} e^{-2\pi i l r / N/2} + e^{-2\pi i r / N} \sum_{l=0}^{N/2-1} n_{2l+1} e^{-2\pi i l r / N/2} \quad (21)$$

The sums on the right hand side of Equation 21 can now be recognized as the r th elements of the transforms of two sequences of elements, each sequence being half of the length of the original:

$$A_r(N) = A_r(N/2)_{\text{even}} + e^{-2\pi i r / N} A_r(N/2)_{\text{odd}} \quad (22)$$

If N was originally a power of 2, the process can be continued until pairs of points are combined to form transforms of two complex numbers. In this way, the Fourier transform can be built from the ground up and will require only $a \log_2 N$ operations where a is some fixed constant. The process also works for every integer divisor of N , although other factors would involve substantially more complicated coding.

It is worth noting from Equations 21 and 22 that two r th complex of the smaller transforms combine to form both the r th and the $(N/2+r)$ th elements of the larger transform in a natural way. In addition, if the elements of the two smaller transforms are ordered in frequency and concatenated (placed in consecutive memory arrays), then the computation of the larger transform may be done *in place* since the position of the r th and $(N/2+r)$ th complex elements of the N point transform is the same as the position of the two r th elements in the $N/2$ -point transforms.

Time and energy may also be saved by taking advantage of the fact that every transform in the generating sequence is symmetric by complex conjugation about the Nyquist frequency.

4. The Length of the Fourier Transforms

The above process would *seem* to restrict the length of any Fourier transform to 2^m data samples, where m is an integer. In fact, this is not the case because points can always be filled in at the beginning or end of the data stream.

What kind of points should be used? Those which do not increase the noise power level in the Fourier spectrum. By Parseval's theorem these points must not contribute to $\overline{d_k^2}$ -- thus the points should be set to the average, $\overline{n_k} = x$, or as close as possible to this average.

What this does to the Fourier transform process is to stop the random walk process, as shown in Figure 2A, when there is no more data to analyze. However, the circling motion shown in Figure 1A continues so that the circle stops at 0 at the end of the transform. This process shall be designated as the insertion of "white silence" because it has no effect at all upon any of the frequency bins and to emphasize the difference from the process of adding "white noise" to data -- a process which might be naively considered to be the proper procedure in this circumstance but which, in actuality, only creates a "noise pollution" problem and decreases the sensitivity of the Fourier transform.

The fact that the Fourier transform is longer than the section of real data transformed makes it "over-resolved" in that the power in adjacent frequency bins becomes somewhat correlated. This will be discussed below in greater detail in Section A 9 c of this appendix.

5. Signals

a. Delta function in frequency

The response of the Fourier transform to signals may be estimated by studying the results of a transform of a sinusoidal signal of a certain frequency.

In the following discussion, the continuous variable, t , will be introduced by the equation

$$t = k \cdot dt \quad (23)$$

where dt is the time window width. The frequency, f_m , of the center of the r th frequency bin is represented by

$$f_m = m \cdot f_{\text{samp}}/N; \quad w_m = 2\pi f_m \quad (24)$$

where $f_{\text{samp}} = 1/dt$ is the sampling frequency in Hertz. A cosinusoidal signal is then represented by the equation

$$s_m = a_m \cdot (e^{i w_m t} + e^{-i w_m t})/2 \quad \text{counts/time bin} \quad (25)$$

where a_m is the amplitude in counts/ time bin. The r th element of the Fourier transform of s_m is then given by

$$\begin{aligned} A_r(s_m) &= a_m/2 \sum_{k=0}^{N-1} e^{-i w_r t} (e^{i w_m t} + e^{-i w_m t}) = a_m/2 \sum_{k=0}^{N-1} (e^{-i (w_r - w_m) t} + \\ &+ e^{-i (w_r + w_m) t}) \quad \text{counts/time bin} \end{aligned} \quad (26)$$

If $r \neq m$, then $w_r \neq w_m$ and the first term in the sum on the right hand side of Equation 26 will cause that part of the sum to execute $r-m$ clockwise circles in the complex plane ($r-m$ being integral). Thus this part will contribute nothing to the r th Fourier element. Moreover, the second term in the sum on the right hand side will execute $r+m$ circles and thus always contributes nothing for all r and m s.t. $r+m \neq 0$.

On the other hand, if $r = m$, then the clockwise-rotating Fourier transform phase factor associated with w_r exactly cancels the counter-clockwise-rotating phase factor associated with the w_m signal and the result is that the sum steps off vectors in the complex plane which are all parallel to a certain direction set by the arbitrary phase of the signal at the beginning of the run. This process is shown in Figure 3A where the arbitrary phase is zero so that the signal steps along the real axis.

b. Off-center delta function

Suppose that the signal frequency is changed to $w_{m+\frac{1}{2}}$ which is given by

$$w_{m+\frac{1}{2}} = 2\pi(m+\frac{1}{2}) \cdot f_{\text{samp}}/N ; \quad f_{m+\frac{1}{2}} = (m+\frac{1}{2}) \cdot f_{\text{samp}}/N \quad (27)$$

We want to know the response of the Fourier transform frequency bins m and $m+1$ to this signal. Consider first the m th bin:

The clockwise-rotating phase factor associated with w_m is slower than the counter-clockwise-rotating factor of $w_{m+\frac{1}{2}}$ by just the amount so that the vectors in the Fourier transform sum gain phase in time, t , of:

$$d\phi(t) = \frac{1}{2} \cdot f_{\text{samp}}/N \cdot t \quad (\text{cycles}) \quad (28)$$

This phase (i.e., the direction of the *tangent* to the path of the sum) accumulates uniformly in time until, by the end of the run, the phase gained is

$$d\theta(T) = \frac{1}{2} f_{\text{samp}} / N \cdot T = \frac{1}{2} T / (N \Delta t = T) = \frac{1}{2} \text{ cycles} \quad (29)$$

Thus the path of the vector sum is a semicircle whose center lies on the positive imaginary axis which is tangent to the real axis as shown in Figure 4A. The *amplitude* response for the *m*th bin is thus decreased by a factor of $2/\pi$ from the optimum response (which is represented by the length of the circumference).

The response for the *m*+1st bin will be similar except that the semicircle is now centered on the negative imaginary axis as shown in Figure 5A. Generalizing from these results the response of a frequency bin to a signal which is separated from the center of the frequency bin to a signal which is separated from the center of the frequency bin by an amount, *df*, is given by

$$q e^{i\theta} = q_0 e^{i\theta_0} e^{i\pi df T/2} \cdot \sin(\pi df T/2) / (\pi df T/2) \quad (30)$$

where q_0 and θ_0 are the amplitude and phase response of the frequency bin for $df = 0$ and df is positive for a signal of a higher frequency than the bin center. The factor multiplying $q_0 e^{i\theta_0}$ will be derived later in the discussion of the convolution theorems. In the *u*-normalized transform, q_0 is simply $aN/2$ counts as can be seen in Figure 3A.

If $r \neq m$, then $w_r \neq w_m$ and the first term in the sum on the right hand side of Equation 26 will cause that part of the sum to execute $r-m$ clockwise circles in the complex plane ($r-m$ being integral). Thus this part will contribute nothing the r th Fourier element. Moreover, the second term in the sum on the right hand side will execute $r+m$ circles and thus always contributes nothing for all r and m s.t. $r+m \neq 0$.

On the other hand, if $r = m$, then the clockwise-rotating Fourier transform phase factor associated with w_r exactly cancels the counter-clockwise-rotating phase factor associated with the w_m signal and the result is that the sum steps off vectors in the complex plane which are all parallel to a certain direction set by the arbitrary phase of the signal at the beginning of the run. This process is shown in Figure 3A where the arbitrary phase is zero so that the signal steps along the real axis.

b. Off-center delta function

Suppose that the signal frequency is changed to w_{m+l} which is given by

$$w_{m+l} = 2\pi(m+l) \cdot f_{\text{samp}}/N; \quad f_{m+l} = (m+l) \cdot f_{\text{samp}}/N \quad (27)$$

We want to know the response of the Fourier transform frequency bins m and $m+1$ to this signal. Consider first the m th bin:

The clockwise-rotating phase factor associated with w_m is slower than the counter-clockwise-rotating factor of w_{m+l} by just the amount so that the vectors in the Fourier transform sum gain phase in time, t , of:

$$\Delta\phi(t) = l \cdot f_{\text{samp}}/N \cdot t \quad (\text{cycles}) \quad (28)$$

This phase (i.e., the direction of the *tangent* to the path of the sum) accumulates uniformly in time until, by the end of the run, the phase gained is

$$d\theta(T) = \frac{1}{2} f_{\text{samp}} / N \cdot T = \frac{1}{2} T / (N \Delta t = T) = \frac{1}{2} \text{ cycles} \quad (29)$$

Thus the path of the vector sum is a semicircle whose center lies on the positive imaginary axis which is tangent to the real axis as shown in Figure 4A. The *amplitude* response for the m th bin is thus decreased by a factor of $2/\pi$ from the optimum response (which is represented by the length of the circumference).

The response for the $m+1$ st bin will be similar except that the semicircle is now centered on the negative imaginary axis as shown in Figure 5A. Generalizing from these results the response of a frequency bin to a signal which is separated from the center of the frequency bin to a signal which is separated from the center of the frequency bin by an amount, df , is given by

$$q e^{i\theta} = q_0 e^{i\theta_0} e^{i\pi df T / 2} \cdot \sin(\pi df T / 2) / (\pi df T / 2) \quad (30)$$

where q_0 and θ_0 are the amplitude and phase response of the frequency bin for $df = 0$ and df is positive for a signal of a higher frequency than the bin center. The factor multiplying $q_0 e^{i\theta_0}$ will be derived later in the discussion of the convolution theorems. In the unnormalized transform, q_0 is simply $aN/2$ counts as can be seen in Figure 3A.

c. Non-sinusoidal periodic signals

Given a signal which is periodic, but not sinusoidal, the Fourier transform will produce a train of harmonics whose frequencies are integral multiples of the fundamental frequency. The sum of power in the fundamental frequency bin and its harmonics may be related statistically to a measure of the variance of the light curve by Parseval's Theorem.

The measure of variance which is appropriate to use is that which integrates the light curve for a time, dt --just as a realistic time sampling would behave if the data were being recorded on line. A uniformly dense set of these integrals can be taken over the signal *period* and then the *average* and the second moment can be calculated. If $g(\phi)$ represents the light curve function in counts s^{-1} over the pulse phase, ϕ , in cycles with f representing the frequency of the pulse, then a member of the sample set may be generated by

$$g_{k,n} = \int_{k/n - fdt/2}^{k/n + fdt/2} g(\phi) d\phi / fdt \quad \text{Counts } s^{-1} \quad (31a)$$

where $g_{k,n}$ is centered on the pulse phase, k/n . The average, $\bar{g}$, and the second moment, $\overline{g^2}$, are calculated by

$$\bar{g} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n g_{k,n} ; \quad \overline{g^2} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n g_{k,n}^2 \quad (31b)$$

Finally, the variance, $\overline{g^2} - \bar{g}^2$, may be related to the expectation value for the unnormalized power in the harmonics by

$$(\overline{g^2} - \bar{g}^2) \cdot T^2 = \sum_{l=1}^{\infty} P(lf) - P_{\text{noise}}(lf) \quad \text{counts}^2 \quad (32)$$

where $P_{\text{noise}}(lf)$ is the power at the frequency, $l \cdot f$, due to noise.

Equation 32 is presented as a conjecture which will not be proven in this text. Several points related in some way to this conjective will be discussed later.

6. The Convolution Theorem

The convolution theorems will be presented below, after which the process of converting an infinite, continuous Fourier transform to a finite discrete transform will be discussed.

a. The transform of the convolution

We assume that the two functions, g and h , are convolved with each other and then transformed like

$$F(g \otimes h) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{-i\omega t} \int_{-\infty}^{+\infty} g(s)h(t-s)ds dt \quad (33)$$

Interchanging the order of integration and inserting the pair, $e^{i\omega s}$ and $e^{-i\omega s}$ in the appropriate spots we get

$$\begin{aligned} F(g \otimes h) &= 1/2\pi \cdot \int_{-\infty}^{+\infty} g(s)e^{-i\omega s} ds \int_{-\infty}^{+\infty} h(t-s)e^{-i\omega(t-s)} dt \\ &= F(g) \cdot F(h) \cdot \text{constant} \end{aligned} \quad (34)$$

Thus the transform of the convolution of two functions is the product of the transforms of each function (to within a constant factor).

b. The transform of the product

Transforming the product of g and h we get

$$F(g \cdot h) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{-i\omega t} g(t) h(t) dt \quad (34)$$

Representing $h(t)$ by an integral over a delta function such as

$$h(t) = (2\pi)^{-1} \int_{-\infty}^{+\infty} h(t') e^{i\omega'(t-t')} d\omega' \quad (35)$$

we get

$$F(g \cdot h) = (2\pi)^{-2} \int_{-\infty}^{+\infty} e^{-i\omega t} g(t) dt \int_{-\infty}^{+\infty} h(t') e^{i\omega'(t-t')} d\omega' \quad (37)$$

By interchanging the order of integration and grouping the right factors we get

$$\begin{aligned} F(g \cdot h) &= (2\pi)^{-1} \int_{-\infty}^{+\infty} d\omega' \int_{-\infty}^{+\infty} g(t) e^{-i(\omega-\omega')t} dt (2\pi)^{-1} \int_{-\infty}^{+\infty} h(t') e^{-i\omega' t'} dt' \\ &= F(g) \cdot F(h) \end{aligned} \quad (38)$$

Thus the transform of the product of two functions is the convolution of the transforms of each.

7. The Finite, Discrete Transform Process

The infinite, continuous Fourier transform of a sinusoidal signal of amplitude a counts s^{-1} is given by

$$F(s) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{-i\omega t} a/2 (e^{i\omega_0 t} + e^{-i\omega_0 t}) dt = a/2 \cdot [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \quad (39)$$

counts

where ω_0 is the frequency of the signal.

In practice, continuous data cannot be handled it must be sampled at

certain time intervals, dt apart, by multiplying with the comb function:

$$C = \sum_{n=-\infty}^{n=+\infty} d(t-ndt) \quad (40)$$

where the function d is equal to unity if its argument is 0 and is equal to 0 otherwise. The comb function is shown in Figure 6A.

The Fourier transform of C is given by

$$F(C) = \sum_{m=-\infty}^{m=+\infty} (\omega - 2\pi m/dt) \cdot \text{Constant} \quad (41)$$

and when this is convolved with the result of Equation 39 we get

$$F(C \cdot \cos \omega_0 t) = a/2 \left[\sum_{m=0}^{m=\infty} \delta(\omega - \omega_0 + 2\pi m/dt) \right] \cdot \text{Constant} \quad (42a)$$

where δ is the usual Dirac Delta Function.

A given signal will therefore have a countably infinite number of frequency spikes over an infinite frequency range, spaced according to the sampling frequency, $2\pi/dt$ radians s^{-1} . This process is called "aliasing" because a signal of a given frequency which is not between 0 and f_{samp} Hz will appear in this interval in two places, namely

$$f_{\text{aliased}(0 - f_{\text{samp}})} = f - m/dt \text{ and } f_{\text{aliased}(0-f_{\text{samp}})} = (m+1)/dt - f \quad (42b)$$

for some m value.

Fortunately, since the samples are not instantaneous, but result from the comb function, C , being convolved with a flat function, B , which is

equal to $1/dt$ from $t = 0$ to $t = dt$ but 0 elsewhere (i.e., the data is integrated for the duration of each time window, dt), the result of equation 42a is multiplied by the function

$$F(B) = e^{-i\omega dt/2} \sin(\omega dt/2)/(\omega dt/2) \quad (42c)$$

The absolute value of $F(B)$ for the interval $0 - 2f_{\text{sample}}$ is sketched in Figure 7A. This function is seen to rapidly diminish the response of the Fourier transform to signals outside of the region, $0 - f_{\text{sample}}$.

Another modification of the Fourier transform procedure comes about because the run length only has a finite duration, T . The product of the signal, s , with the comb function, C , convolved with the time sample function, B , is finally multiplied by another plateau function, D , which is equal to unity from 0 to T and equal to 0 elsewhere. Thus the delta function on the frequency-amplitude plane ends up being convolved with the factor

$$F(D) = T e^{-i\omega T/2} \sin(\omega T/2)/(\omega T/2) \quad (43)$$

Thus the frequency response near ω_0 becomes equal to

$$F(D) * (F(B) * F[C * a \cos \omega_0 t]) = aT/2 * e^{-i\omega dt/2} * \sin(\omega dt/2)/(\omega dt/2) * e^{-i(\omega - \omega_0)T/2} \sin[(\omega - \omega_0)T/2]/\{(\omega - \omega_0)T/2\} \text{ counts} \quad (44)$$

The factor in Equation 43 has already been derived from the above discussion on the off-center delta function in frequency; the results were shown in Equation 29.

The sequence of the four functions and their resultant Fourier transforms

are sketched in Figure 8A.

In actual practice, the instantaneous signal rate in counts s^{-1} is meaningless because nature quantizes any real signal in units of photons which the observer detects through the multiplication of photoelectrons. This process takes anywhere between 1 and 100 nanoseconds depending on the phototube and discriminator used.

In a detector open to the broadband, quanta with homogenous bands of 10^{14} Hz will photoeject electrons in a very short time after the arrival of the front end of the wave packet.

It is also worth noting that, from the above section dealing with noise, a photon distribution which does not change its nature for different sampling intervals (like the ordinary Poisson distribution) will produce a noise power spectrum which is flat at all frequencies. This is unlike the case of any signal modulation of the photon arrival probability which must suffer attenuation for higher frequencies of modulation due to the finite time window integration as shown in the third frame of Figure 8A. All Fourier power spectra of ordinary photon noise are (statistically) flat (unless count-rate limits are introduced by the experimenter's detector or pre-scaling apparatus).

8. Statistical Methods for Power Spectra

a. Single bins

It has been shown above that the distribution of values for a properly normalized power spectrum is exponential. The consequence of this is that the average value and average variance of noise spectrum bins are both equal to unity. An event which exceeds 14.37 "exponential" power spectrum levels has the same likelihood of being an accident as does one which

exceeds 5 σ 's of a normal distribution, namely $1:1.7 \times 10^{-6}$. An approximate conversion of P into σ 's of the same likelihood can be derived from the asymptotic form of the error functions or the integrals of the normal distribution given in Abramowitz and Stegun.⁸⁰ The conversion is

$$\sigma = \sqrt{2P - \ln(2P)} \quad \text{and} \quad P = e^{-\sigma^2/2} + \ln(\sqrt{e^{-\sigma^2/2}}) \quad (45)$$

and these are good to a few percent down to the region where $P = 1.0$ and $\sigma = 2.0$.

The distribution of the variance of the power is also of some interest in the case where tests for flatness are applied to power spectra as in the case of Figures 9-30. Letting X represent $P - 1$, the deviation of P from the average, the probability distribution may be derived for X^2 :

$$\Pr(X^2)dX^2 = (2e)^{-1} \left(e^{-\sqrt{X^2}} + \gamma e^{+\sqrt{X^2}} \right) dX^2 / \sqrt{X^2} \quad (46)$$

where γ is unity for $X^2 \leq 1$ and 0 for $X^2 > 1$. The average of this distribution is 1. The second moment is 9. The variance is therefore $9 - 1 = 8$.

The distribution of the sum of n power spectrum bins approaches a gaussian due to the Central Limit Theorem. The gaussian is centered on n and has a σ of $\sqrt{n \cdot 1} = \sqrt{n}$.

The distribution of the sum of *variances* of n power spectrum bins also approaches a gaussian by the Central Limit Theorem. The mean is n as before. However, the σ of the gaussian will now be equal to $\sqrt{8 \cdot n}$.

b. Multiple bin sums

The probability of the sum of two power spectrum bins exceeding a certain value, S , can be calculated by the integral

$$\int_0^S \Pr(S_2 \geq S - S_1) \Pr(S_1) dS_1 + \Pr(S_1 \geq S) = \int_0^S e^{-(S-S_1)} e^{-S_1} dS_1 + e^{-S} = (1+S)e^{-S} \quad (47)$$

where S_1 and S_2 are the power in the arbitrary bins subscripted 1 and 2 respectively. The formula, $(1+S)e^{-S}$ also gives the probability that the Chi-square of four degrees of freedom exceeds $2S$.

The probability that the sum of m bins exceeds S is given by

$$\Pr(>S:m) = \sum_{j=0}^{m-1} S^j / j! e^{-S} \quad (48)$$

and also gives the likelihood of the Chi-square of $2m$ degrees of freedom exceeding $2S$. The differential probabilities are given by

$$\Pr(S:m)dS = S^{m-1} / (m-1)! e^{-S} dS \quad (49)$$

The averages, second moments and the variances of the variances of 2, 3 and 4 bins are given below

$m = 2$	Avg = 2	2nd Mom = 24	Var = 24 - 2 ² = 20	
$m = 3$	Avg = 3	2nd Mom = 45	Var = 45 - 9 = 36	(50)
$m = 4$	Avg = 4	2nd Mom = 72	Var = 72 - 16 = 56	

It is surmised that the variance of the distribution of the variance of the sums of m power spectrum bins is given by

$$\overline{(X_m^2)^2} - (\overline{X_m^2})^2 = 2m' + 6m \quad (51)$$

Therefore the quantity, X^2/m , is represented by its average and variance:

$$X^2/m = 1 + \sqrt{2(1+3/m)} \quad (52)$$

Thus when a sum of the variances of n *identical* *power spectrum bins* is formed, the Central Limit Theorem guarantees that such sums of variances of sums will approach a normal distribution with the mean and σ given by

$$\overline{X(n;m)} = m \cdot n; \quad \sigma(n,m) = m \cdot \sqrt{2n(1+3/m)} \quad (53)$$

c. Fractional bin sums

When an odd mixture of two power spectrum bins is required, equation 47 no longer suffices. The probability that a times the contents of bin 1 added to b times the contents of bin 2 will exceed the sum, S , is given by

$$\Pr(aS_1 + bS_2 \geq S) = \int \Pr(S_2 \geq S/b - aS_1/b) \Pr(S_1) dS_1 + \Pr(S_1 \geq S/a) \quad (54)$$

The lefthand side of equation 54 is easily converted into exponentials:

$$\Pr(aS_1 + bS_2 \geq S) = \int_0^{S/a} e^{-S/b} e^{aS_1/b} e^{-S_1} dS_1 + e^{-S/a} = (ae^{-S/a} - be^{-S/b})/(a-b) \quad (55)$$

Equation 55 has a removable singularity when $a = b$ when it becomes

$$\Pr(aS_1 + bS_2 \geq S)_{a=b} = \left. \frac{d}{dx} (x e^{-S/x}) \right|_{x=a} = (1 + S/a) e^{-S/a} \quad (56)$$

When $a = b = 1$, Equation 56 reduces to Equation 47.

If three bins are added together with weights a , b and c the probability of the sum exceeding S is given by

$$\Pr(aS_1 + bS_2 + cS_3 \geq S) = - \left[\frac{a^2 (b-c) e^{-S/a} + b^2 (c-a) e^{-S/b} + c^2 (a-b) e^{-S/c}}{(a-b)(b-c)(c-a)} \right] \quad (57)$$

This equation has more singularities than Equation 56 and is consequently more treacherous to handle. In the computer, the difficulties of avoiding the singularities involves logic which is so time-consuming that large numbers of fractional sums are avoided in favor of integral bin sums.

9. Real Signals with Noise

a. Statistics

Suppose that a real signal of s^{-1} of frequency, f , exists in the data with a noise level of $MSD = LL(f)$ counts s^{-1} at the frequency, f . The data is taken for a time, T , and then Fourier analyzed. Assuming that f corresponds to the center of some frequency bin and using Equation 26 and Figure 3A, the result of the unnormalized Fourier transform for this signal will be given by the vector

$$\bar{s} = a/2T \bar{e}_x \quad \text{counts} \quad (58)$$

where the phase of the signal has been arbitrarily fixed to lie along the

real axis and $\bar{e}_x$ is the unit vector along this axis.

A noise vector $\bar{n}$, will form at the frequency f due to the action of the Fourier transform on the fluctuations in the data of $MSD \cdot LL(f)$ counts²s⁻¹. This vector will have a completely arbitrary complex phase and will add to $\bar{s}$ to form the net vector result in the amplitude plane;

$$\bar{r} = \bar{s} + \bar{n} \quad (59)$$

The unnormalized power will be given by the square of $\bar{r}$:

$$P = r^2 = s^2 + n^2 + 2\bar{s} \cdot \bar{n} \quad (60)$$

No matter what the exact distribution of the noise vectors is, only the random phase assumption is needed to show that the average of r^2 is

$$\overline{r^2} = s^2 + \bar{n}^2 \quad (61)$$

Thus, on the average, the signal and noise contributions to the power add in a simple fashion.

The random phase assumption is satisfied for noise vectors on the complex plane which result from fluctuation contributions from *both* background and a sinusoidal signal. The reason why the noise vectors should still have random phase even when the signal strengths are large is a subtle one. It is true that the *magnitude* of the fluctuations will be higher

at the times corresponding to the peaks of the sinusoidal waveform. At the times of the troughs of the sinusoidal waveform the magnitude of the fluctuations is correspondingly smaller, however, since the sign of the fluctuations is truly random, the fluctuations at both the times of the peaks and of the troughs affect the same two directions on the complex plane and consequently average out. A similar argument can be applied in detail to all directions in the complex plane where the contributions of the components of the fluctuations are carefully averaged because of the sinusoidal nature of the signal. However, it is only necessary that a given signal profile have a well-defined average value so that the "self-noise" fluctuations are random.

Figure 9A shows a signal vector and a noise vector adding on the complex plane.

The result of the unnormalized Fourier transform on a data noise level of $\text{MSD} \cdot \text{LL}(f) \text{ counts}^2 \text{s}^{-1}$ for time, T , is given by a 2-dimensional gaussian with a σ of $\sqrt{\text{MSD} \cdot \text{LL}(f) \cdot T/2} \text{ counts}$. This distribution is tacked onto the end of the signal vector as sketched in Figure 10A.

The normalized complex plane is derived by scaling the dimensions by $(\text{MSD} \cdot \text{LL}(f) \cdot T/2)^{-1/2} \text{ counts}^{-1}$ so that the σ of the noise gaussian is $1/\sqrt{2}$ and the noise vectors have an average length of unity.

A probability may be assigned to an arbitrary vector, $\vec{r}$, in the amplitude plane by virtue of its distance from the tip of $\vec{s}$ which is given by $|\vec{n}|$. From the law of cosines, with θ shown in Figure 11A, we have:

$$\text{Pr}(\vec{r})dA = 1/\pi e^{-r^2 - s^2 + 2rs\cos\theta} dA \quad (62)$$

In order to calculate a distribution on r^2 , we fix $\bar{r}$ and vary θ :

$$\text{Pr}(|r|)dr = r dr / \pi e^{-r^2 - s^2} \int_0^{2\pi} e^{+2rs \cos \theta} d\theta = 2e^{-r^2 - s^2} I_0(2rs) \quad (63)$$

where I_0 is the modified Bessel function and the area element, dA , is split into the factors $rd\theta$ and dr .

The probability that r is now less than a certain value, r_0 , is given by

$$\text{Pr}(r \leq r_0) = e^{-s^2} \int_0^{r_0} e^{-r^2} I_0(2rs) r dr \quad (64)$$

Using the asymptotic expansion of I_0 :

$$I_0(z) = e^z \cdot (2\pi z)^{-1/2} \quad (65)$$

in Equation 63 to get an approximate distribution for $\text{Pr}(r)dr$:

$$\text{Pr}(r)dr = e^{-(r-s)^2} \cdot (\pi rs)^{-1/2} dr \quad (66)$$

For s and r large enough to be nearly equal equation 66 simplifies into

$$\text{Pr}(r)dr \approx 1/\sqrt{\pi} \cdot e^{-(r-s)^2} dr \quad (67)$$

Thus, as s and r become large, the distribution for r approaches a gaussian with a σ equal to $1/\sqrt{2}$. This result could have been guessed with the help of Figure 10A; only the fluctuations parallel to $\bar{s}$ contribute to

the fluctuation in amplitude, and thus the 1-dimensional σ of $1/\sqrt{2}$ gives the σ of the amplitude.

Given that $r = r_0 \pm 1/\sqrt{2}$ then, for the power,

$$P = P_0 \pm \sqrt{2P_0} \quad (68)$$

This gives the extent of reproducibility which can be expected for measuring the same signal strength in the same frequency bin with similar apparatus more than one time.

The complex phase of $\bar{r}$ is also subject to error in measurement which is given by

$$\theta = \theta_0 \pm 1/\sqrt{2P_0} \quad \text{in radians} \quad (69)$$

b. Deducing the signal strength

To reduce an observed power to the sinusoidal amplitude of signal necessary to produce the event, we first consider the average unnormalized power and its constituent parts:

$$\overline{P_{\text{raw}}} = a^2 T^2 / 4 + \text{MSD} \cdot \text{LL}(f) \cdot T \quad \text{counts}^2 \quad (70)$$

where the first term on the right hand side of Equation 70 is taken from Equations 58 and 61. The second term represents the *average* noise contribution to the power.

When the power is properly normalized we have

$$\bar{P} = a^2 T / [4 \cdot MSD \cdot LL(f)] + 1 \quad (71)$$

The equation holds for the ensemble average of P and its noise contribution; however, when inverting Equation 71 for a , the measured P is interchanged with $\bar{P}$ to get an average a :

$$a = \sqrt{4 \cdot MSD \cdot LL(f) / T} \cdot \sqrt{(P-1) \pm \sqrt{2P}} \quad \text{counts s}^{-1} \quad (72)$$

If a transparency or efficiency factor, e , has been measured for the particular observation, then the observer-independent amplitude can be given by

$$a' = a/e = \sqrt{4 \cdot MSD \cdot LL(f) \cdot (P-1) / T} / e \quad (73)$$

c. Systematics on the deduced amplitude

If the signal frequency, f , does not happen to correspond to the center of some frequency bin, m/T Hz, then the response of the power in the bin goes like the square of Equations 29 and 43:

$$P = P_0 \cdot \sin^2(\pi dfT) / (\pi df \cdot T)^2 \quad (74)$$

where df is the difference between f and m/T and P_0 is the power derived for an on-center signal (with no noise) of the same amplitude.

If the next nearest frequency bin is included in the calculation of the "excess power", i.e. the quantity,

$$P_{e,n} = \sum_{i=1}^n P_i^{-n}, \quad \text{for } n=2 \quad (75)$$

which is inserted into Equations 72 and 73 in place of the ordinary $P = 1$, then a greater fraction of the total excess power is recovered. The amount of power recovered using the two nearest bins is given by

$$P_{e,2} = P_0 \cdot \sin^2(\pi dfT) \cdot \pi^{-2} \cdot [(df \cdot T)^{-2} + (1-df \cdot T)^{-2}] \quad (76)$$

where $0 < df \cdot T < 1$. The lowest value for this function occurs when the signal is half way in between the two bins when it becomes

$$P_{e,2}(1/2) = P_0 \cdot 8/\pi^2 = 81\% P_0 \quad (77)$$

Thus the worst systematic effect which can occur to the estimation of the amplitude of the signal, a , is that it may be 10% too low. The average systematic effect due to this "off-center" phenomenon makes the estimation of a about 5% lower than it should be.

Of course, if the experimenter should include many bins in the calculation of the excess power, the sum becomes

$$P_e = P_0 \cdot \sin^2(\pi dfT) \pi^{-2} \sum_{n=-\infty}^{n=+\infty} 1/(df \cdot T - n)^2 = P_0 \quad (78)$$

and no power is lost.

However, the noise introduced by n bins with a total power of P can be approximated by

$$dP = \sqrt{2(P-n)} + n = \sqrt{2P_{\phi} + n} \quad (79)$$

and as n gets large, so does the noise. In practice, it is severely uneconomical in terms of signal to noise ratio to include more than the contents of two bins in measuring the excess power due to a narrow signal.

Another systematic may enter in the determination of pulsed amplitude when a stretch of "white silence" has been tacked onto the data. If the MSD used in Equation 73 is taken as the average counts² s⁻¹ over the whole data string in the transform, then the square of the estimated amplitude, a^2 , will be proportionately too low. If T is taken as the time of the whole transform, a^2 will again be proportionately low. A remedy for the misestimation may be achieved by scaling e down in the proportion of real data time to transform time.

A more insidious systematic occurs for the artificially extended runs whose excess power is estimated with more than one frequency bin. If the real data containing a uniform signal lasts for a fraction, $g < 1$, of the transform time, then the sidelobe pattern becomes

$$P(f) = P_0 \sin^2\{\pi(f-f_0)Tg\}/[\pi(f-f_0)Tg]^2 \quad (80)$$

where f_0 is the frequency of the signal. Letting f take on the whole range of discrete values, m/T Hz, and summing over all bins for the power in the signal we get

$$P_e = P_0 \sum_{n=-\infty}^{n=+\infty} \sin^2(\pi g(x-n))/(\pi g(x-n))^2 = P_0/g \quad (81)$$

where $x = \min(m-f_0 T)$, the frequency separation of the bin closest to f_0 .

Equation 81 gives the side lobe pattern for a transform which is a factor of g overresolved. The estimated excess power is a factor of $1/g$

too great. This is not a very serious systematic effect since a finite bin sum tends to underestimate P_e and thus acts to cancel the overestimation. In addition, the factor of $1/q$ enters beneath the square root sign.

B. Signal Folding

1. Relation to Fourier Transform

If the contents of the time windows are folded over at a given period and used to augment a relatively small set of bins ($\emptyset_j$), fixed with respect to the pulse phase, then the contents of the phase bins will make up the *pulse profile* (or light curve) of the frequency. Starting with the unnormalized transform sum for the r th frequency bin.

$$A_r = \sum_{k=0}^{N-1} n_k e^{-2\pi i r k / N} \quad (82)$$

a *pulse* phase, $\emptyset_j(k)$, may be assigned to every k for the frequency, $r f_{\text{samp}}/N$, where $\emptyset_j = 2\pi(j-1)/n$ radians for some i out of the set population, n . The sum of equation 82 can then be transformed into

$$A_r = \sum_{k=0}^{N-1} n_k e^{-i \emptyset_j(k)} \quad (83)$$

As long as the set, $(\emptyset_j)$, is dense enough (ie. n is large), the error involved in equation 83 will be small. With (C_j) representing the set of contents of the phase bins $(\emptyset_j)$ of the pulse profile, the elements of the sum in equation 83 may be separated by phase bin (i) and then recombined to form A_r again:

$$A_r = \sum_{j=1}^n C_j e^{-i\theta_j} = \sum_{j=1}^n C_j \cos\theta_j - i \sum_{j=1}^n C_j \sin\theta_j \quad (84)$$

where

$$C_j = \sum_{k=0}^{N-1} n_k \delta_{jk} \text{ counts} \quad (85)$$

with $\delta_{jk} = 1$ if $2\pi \text{ mod}(rk/N, 1)$ is within the phase region $2\pi(j - \frac{1}{2} \pm \frac{1}{2}/N)$ radians and $\delta_{jk} = 0$ otherwise.

Thus the r th element of the raw Fourier transform may be constructed by signal folding. The advantages of this technique over the ordinary Fourier transform are fourfold: 1. One frequency element may be calculated at a time. 2. The resolved pulse profile is obtained simultaneously. 3. The frequency, rf_{samp}/N , may be chosen for any value of r . and 4. Any length of real data can be economically folded.

If MSD is known from the data handling involved in the signal folding and $LL(rf_{\text{samp}}/N)$ is known from the ordinary Fourier transform process, then the properly normalized r th frequency element may be found:

$$\bar{r}_r = [\text{MSD} \cdot LL(rf_{\text{samp}}/N) \cdot T]^{-1} \left(\sum_{j=1}^n C_j \cos\theta_j - i \sum_{j=1}^n C_j \sin\theta_j \right) = R + iI \quad (86)$$

The properly normalized power is given by

$$P = R^2 + I^2 \quad (87)$$

In practice, the signal folding is accomplished by a program which weighs the overlap of the data time window with the phase window and then augments the phase window by the contents of the time window scaled by the ratio of the overlap width to the time window width. It is likely that such a procedure is the most accurate way of folding.

2. Measuring the Frequency

a. The A-B measure

If some sinusoidal signal with exact frequency, f^0 , is present in the data with a fair degree of significance, it is possible to measure f^0 to a much greater precision than the $\pm 1/2T$ Hz resolution of the Fourier transform.

To see how this might be accomplished, we consider the equation for the response in power to a nearby frequency:

$$P(x) = P(0) \cdot \sin^2 \pi x / (\pi x)^2 \quad (88)$$

where $x = (f - f^0) T$. Expanding the function of the right hand side of Equation 88 in terms of πx we obtain

$$P(x) = P(0) \cdot (1 - 1/3(\pi x)^2 + (\pi x)^4/22.5) \quad (89)$$

which is quadratic in (πx) to within a few percent for $\pi x \approx 0.7$.

If we now use the Fourier transform to allow us to guess an f_0 which must be nominally close to f^0 we may calculate by signal folding

$$P_0 = P[(f_0 - f^0)T] = P(x_0) \quad (90)$$

In order to sense where f^0 might be with respect to f_0 , we calculate the two quantities by signal folding

$$P_+ = P(x_0 + z) \quad \text{and} \quad P_- = P(x_0 - z) \quad (91)$$

which act as test perturbations about P_0 at an arbitrarily chosen separation of z/T Hz. The z value which was used to reduce the HZ Her data was typically 0.2.

If $f^\circ = f_0$, the the vector tracks which the signal makes in the complex plane during the folding is shown in Figure 12A, the complex phase associated with P_0 has been arbitrarily chosen so that the vector lies along the real axis. The paths associated with P_- and P_+ produce vectors whose lengths are shortened due to the 72° of circular "kink" in the paths.

However, if f° is slightly greater than f_0 , then the path of P_+ will be straighter, that of P_0 will now be slightly curved and the path of P_- will be greatly curved as shown in Figure 13A. The intuitive result of the experiment is that the quantity $(P_+ - P_-)/P_0$ may provide an almost linear measure of the quantity $f^\circ - f_0$ and thus allow us to measure f° by estimating a revision for f_0 .

The quantity $(P_+ - P_-)/P_0$ can be estimated from Equations 90 and 91 as the derivative of the factor in Equation 89 with respect to x :

$$(P_+ - P_-)/P_0 = -4\pi^2 z x / 3 \quad (92)$$

and thus we see that $(P_+ - P_-)/P_0$ does indeed provide (in approximation) a linear measure for x .

The actual plot of $(P_+ - P_-)/P_0$ vs x may be calculated analytically and used as a look-up table -- no matter if the response is not exactly linear. The quantities, P_+/P_0 and P_-/P_0 , are defined as the constants A and B respectively for the sake of convenience in the discussion below.

The precision to which the frequency can be measured will depend upon the amount of random fluctuations which P_+ and P_- may execute about P_0 .

(whereas, the measuring technique uses the *systematic* fluctuations of P_+ and P_- about P_0).

As the size of the perturbation becomes small (e.g. smaller than $0.2/T$ Hz) the paths of P_+ and P_- as shown in Figures 12A and 13A become more identical to the P_0 path (for both the signal and noise components) and consequently both the random and systematic deviations of P_+ and P_- about P_0 become small. As long as the P_+ and P_- paths do not turn a complete circle in the complex plane of Figure 12I, then the P_+ and P_- values will not be completely independent of the P_0 value.

The precision to which f° will be measured will depend on the ratio of the systematic deviations to the random fluctuations and thus should be independent of the size of the perturbation chosen; however, it will be required that the perturbation is small in order that the details remain simple (and calculable).

b. The fluctuations in the f° measurement

As mentioned above, the departure of the P_+ path from P_0 is the result of a 36° rotation and a "kink" which causes the slope of the P^+ path relative to the appropriately rotated P_0 path to vary according to:

$$d\theta(u) = 2\pi z(1-u); \quad u \equiv t/T \quad (93)$$

When the discrete constituents of any path in the complex plane add up to a net vector sum, the sum process in which this vector is formed may be represented as the addition of *components* of the discrete constituents which are parallel or anti-parallel to the net vector direction. This process

can be symbolized by

$$\sqrt{P} = \sum_{k=0}^{N-1} y_k \cos \theta_k \quad (94)$$

where θ_k is the direction (complex phase) associated with the kth data element and y_k is the properly normalized kth data element amplitude as given by

$$y_k = n_k \cdot [\overline{d_k^2} \cdot LL(f) \cdot N]^{-1/2} \quad (95)$$

When the path associated with P_+ is formed by inserting the "kink" of Equation 93 into the θ_k 's of Equation 94 we find

$$\sqrt{P_+} = \sum_{k=0}^{N-1} y_k \cos(\theta_k + d\theta(k/N)) \quad (96)$$

The cosine may be expanded in the usual fashion:

$$\sqrt{P_+} = \sum_{k=0}^{N-1} y_k \cos \theta_k \cos d\theta(k/N) - \sum_{k=0}^{N-1} y_k \sin \theta_k \sin d\theta(k/N) \quad (97)$$

Although we know for the chosen alignment of the real axis to the P_0 path that

$$\sum_{k=0}^{N-1} y_k \sin \theta_k = 0, \quad (98)$$

the inclusion of the $\sin d\theta(k/N)$ factors in the last term in Equation 97 will effectively relax this constraint.

Because ϕ_k and $d\phi(k/N)$ are uncorrelated, the *average* value of the last term of Equation 97 is still zero so that no systematic variations are introduced by the sum of the sine terms. However, it is possible for the sine terms to introduce fluctuations into the sum for $\sqrt{P_+}$. To find the unsquared fluctuation, we square the sum and then average to get

$$\begin{aligned} & \text{(average over many such sums)} \\ & \left\langle \left(\sum_{k=0}^{N-1} y_k \sin \phi_k \sin d\phi(k/N) \right)^2 \right\rangle = \left\langle \sum_{k=0}^{N-1} y_k^2 \sin^2 \phi_k \right\rangle \left\langle \sin^2 d\phi(1/N) \right\rangle_1 \quad (99) \\ & \qquad \qquad \qquad \uparrow \\ & \qquad \qquad \qquad \text{(average over 1)} \end{aligned}$$

where the cross terms must average to zero even if $k=1$ because ϕ_k and $d\phi(1/N)$ are uncorrelated.

From our previous knowledge of properly normalized Fourier transforms the first term on the right hand side of Equation 99 is equal to $\frac{1}{2}$. The second term can be calculated by integrating

$$\left\langle \sin^2 d\phi(1/N) \right\rangle_1 = \int_0^1 4\pi^2 z^2 \left(\frac{1}{4} - u + u^2 \right) du = (\pi z)^2 / 3 \quad (100)$$

where the approximation, $\sin x \approx x$, has been assumed because of the 36° upper limit on $d\phi(1/N)$ from Equation 93 for $z=0.2$.

Thus Equation 99 becomes

$$\left\langle \left(\sum_{k=0}^{N-1} y_k \sin \phi_k \sin d\phi(k/N) \right)^2 \right\rangle = z^2 \pi^2 / 6 \quad (101)$$

Squaring up Equation 97 to get P_+ doubles the errors:

$$P_+ = P_+ \pm 2\pi z \sqrt{P_+ / 6} \quad \text{or} \quad A = A \pm 2\pi z / \sqrt{6P_+} \quad (102)$$

with $P_+ \approx P_0$ assumed for the sake of simplicity.

The values of $d\theta(k/N)$ for the "kink" in the path associated with P_- should be equal and opposite to those affecting $\sqrt{P_+}$ in Equation 97. Thus if the last term on the right hand side of Equation 97 should fluctuate one way when A is formed, an equal and opposite fluctuation occurs in the formation of B. Therefore the total error on the quantity, A-B, is given by

$$d(A-B) = 4\pi z / \sqrt{6P_0} \quad (103)$$

Converting Equation 92 to eliminate x , we see that the revision in frequency is approximately

$$(f^0 - f_0) \cdot T \equiv dx \cdot (A-B) \cdot 3/(4\pi^2 z) \quad (104)$$

Thus the precision to which f^0 can be measured is

$$df^0 = 3/(\pi \sqrt{6P_0} T) \text{ Hz} \quad (105)$$

c. The fluctuations of the cosine terms

Returning our attention to the first term on the right hand side of Equation 97, the $\cos[d\theta(k/N)]$ factors can be understood to decrease the amplitude, $\sqrt{P_0}$, to the amplitude, $\sqrt{P_+}$, when an ensemble average of such values is considered.

The average decrease may be estimated by expanding the cosine and then integrating over u :

$$\cos[d\theta(k/N)] = 1 - d\theta(k/N)^2/2 = 1 - 2\pi^2 z^2 (k - u + u^2) \quad (106)$$

$$\langle \cos d\theta(k/N) \rangle_k = 1 - \pi^2 z^2 / 6 \quad (107)$$

The errors introduced in A or B due to the fluctuations allowed in the first term on the right hand side of Equation 97 can be computed by taking the variance of the term, $d\theta(u)^2/2$, in the middle of Equation 106 and multiplying it by the variance of the terms in the sum of Equation 94. As in the previously discussed constraint of Equation 98, the constraint of Equation 94 does nothing to the variance, so that

$$d^2 \left(\sum_{k=1}^{N-1} \gamma_k \cos \theta_k \right) = 4 \quad (108)$$

where d^2 is a shorthand notation representing the calculation of the variance.

The variance of the $d\theta(u)^2/2$ term may be computed by subtracting the square of the last term in Equation 107 from the second moment of $d\theta(u)^2/2$ which is given by

$$\langle d\theta(u)^4/4 \rangle = 4(\pi z)^4 \int_0^1 (1-u)^4 du = (\pi z)^4/20 \quad (109)$$

The variance of $d\theta(u)^2/2$ is then

$$d^2[d\theta(u)^2/2] = (\pi z)^4/45 \quad (110)$$

and the σ of the first term on the right hand side of Equation 97 is therefore

$$d\sqrt{P_{+\cos}} = (\pi z)^2/\sqrt{90}, \quad dP_{+\cos} = 2(\pi z)^2\sqrt{P_{+}/90} \quad \text{and} \quad dA_{\cos} = 2(\pi z)^2/\sqrt{90P_0} \quad (111)$$

This is approximately 1/9th of the σ due to the sine terms as taken

from the root of Equation 101 for $z = 0.2$. Moreover, the quantity, $A-B$ is not affected by this contribution to the noise since it is symmetric between $d\theta(u)$ and $-d\theta(u)$; all of the fluctuation is removed by the subtraction process.

d. The A+B check

However, the quantity $A+B$ is doubly affected by the fluctuations since they add coherently, thus

$$d(A+B) = (2\pi z)^2 / \sqrt{90P_0} \quad (112)$$

The quantity, $A+B$ is a useful check on whether the signal is consistent with a δ function which is present for the entire run duration, T , or whether either of these two conditions is not satisfied. A run where $A+B$ is not low enough (it should be roughly equal to $2(1-\pi^2 z^2/3)$) could be shortened by seeing if the new $A+B$ is more consistent with the ideal.

e. The difference of complex phases

A final check on this analysis could be made on the complex phase change between the $\sqrt{P_+}$ and $\sqrt{P_-}$ vectors. This average change should be

$$\theta_- - \theta_+ \equiv D\theta = 2\pi z \text{ radians} \quad (113)$$

as is evident from Figure 12A.

From an analysis similar to that performed for $A-B$, the errors on $D\theta$ are

$$d(D\theta) = 2\pi z / \sqrt{6P_0} \text{ radians, or } d(D\theta) \approx 60z\sqrt{6/P_0} \text{ degrees} \quad (114)$$

If $D\theta$ were too small, it would indicate that the signal was present for only part of the run.

f. The pulse phase and its errors

Finally, the pulse phase, θ° , may be inferred from the measurement of f° by interpolating to the f° frequency using f_0 , f_+ , f_- , θ_0 , θ_+ and θ_- . The error in the pulse phase may be taken from equation 105 multiplied by $T/2$ (integrated times tdt/T from 0 to T):

$$d\theta^\circ = 3/(2\pi\sqrt{6P_0}) \text{ cycles} = 3/\sqrt{6P_0} \text{ radians} \quad (115a)$$

However, in any signal folding process where the frequency has a *fixed* value, Equation 69 applied to the errors

$$d\theta = 1/\sqrt{2P_0} \text{ radians} \quad (115b)$$

and the ratio between the two errors is given by

$$d\theta/d\theta^\circ = 1/\sqrt{3} \quad (115c)$$

g. Monte Carlo calculations

The errors of the three quantities, $A-B$, $A+B$ and $D\theta$ as given in Equations 103, 112 and 114 were confirmed by Monte Carlo calculations which produced 10,000 sets of A , B and θ 's for data samples of several hundred elements.

h. The measurement of frequency and the real data

The (A,B) pairs obtained by measuring the frequency centroids of the data from HZ Her are plotted in Figure 14A on the A-B plane. The axes are drawn for $A = 1.0$ and $B = 1.0$ and the grid ticks are spaced in intervals of 0.1.

The spread of the points along the lines of constant $A+B$ depends on the experimenter's ability to guess the correct centroid for pulsation and on the value of z used, which in almost all cases was 0.2. The curve of roughly constant $A+B$ is drawn to indicate the locus of ideal data. The points are labelled according to their run number and coded by symbol according to which pulsation feature they are associated with (I, II, III or other). Run 111 was measured using $z = 0.3$ and thus was omitted from the Figure.

The Figure shows a tendency for some data points to lie above the $A+B$ line, which indicates a decrease in the pulsation amplitude at the beginnings or ends of the runs. The point which lies to the lower left of the ideal $A+B$ curve is probably a statistical accident.

The uniform spread of these points along the $A+B = \text{Constant}$ direction indicates the degree of randomness of the experimenter's initial guesses for the exact frequency. The σ of this process is seen to correspond roughly to $A-B = 0.2$. From Equation 104 with $z = 0.2$ the accuracy of the "guesses" in frequency is roughly $0.08/T$ Hz. However, this is misleading since a really bad guess for f_0 would be abandoned for a subsequent, more acceptable, guess.

On the other hand, the small values of the revisions, dx , for the runs does mollify the systematic effects on the frequency measurements of the few runs whose $A+B$ values place them to the upper right of the ideal

curve. Even if the dx values obtained are systematically small by some fraction for the worst cases (e.g., 40%), the fact that these small dx values are mostly less than or equal to the size of the errors (from Equation 105) indicates that the systematic effects are truly not significant.

1. Measuring f

a. The A-B Measure

A similar perturbation analysis may be used to measure the time derivative of the frequency, f . In this case, the perturbation is obtained by applying an f of magnitude, $z/2T$ Hz s^{-1} while constraining the average frequency to the value, f_0 .

The frequency behavior used to construct the P_+ path is given by

$$f_+ = f_0 + z/2T + zu/T; u \in t/T \quad (116)$$

and this is shown in Figure 15A. The instantaneous slope difference between the P_+ and P_0 paths may be calculated from the integral of Equation 116:

$$(s_+ - s_0)(u) = -\int_0^u z/2T + (2u'-1)Tdu' = z/2 u(1-u) \text{ cycles} \quad (117)$$

The instantaneous complex phase difference between the P_+ and P_0 paths can be computed by integrating the difference in slope:

$$(\theta_+ - \theta_-)(u) = \int_0^u z/2 u'(1-u')du' = z/2 \cdot u^2(1-u/3) \text{ cycles} \quad (118)$$

The typical value of z used in the measurement of the data is unity.

If Equation 118 is evaluated for the phase difference between the endpoints

of the two paths, the result is exactly 30° .

The paths associated with P_+ , P_0 and P_- are shown in Figure 16A for $z = 1$. The maximum absolute value of the slope differences reaches 45° at the midpoint of the time interval, $t = T/2$ ($u=1/2$).

As in the previous discussion, the change from P_0 to P_+ or from P_0 to P_- involves a rotation (of 30°) and a "kink". The equation of the "kink" for the path associated with P_+ is given by

$$d\theta(u) = \pi z(u-u^2 - 1/6) \text{ radians} \quad (119)$$

The measure of $\dot{f}$ through A-B has been calculated numerically and is displayed in Figure 17A which plots $(A-B)/z$ versus the $\dot{f}T^2$ of the signal. Three curves are given, the upper curve for $z = 1$, the lower curve for $z = 2$ and the middle curve for $z = 1$ with the real signal frequency differing from f_0 by $0.2/T$ Hz.

The response of $(A-B)/z$ is seen to be fairly linear for $T^2\dot{f} \leq 2$, is not affected when f_0 differs from f° by $0.2/T$ Hz, and is only affected by 15% when z changes from 1 to 2. The limiting slope for the $z = 1$ line near the origin is equal to 0.21 and increases to 0.22 by $T^2\dot{f} = 1.0$. Thus A-B is related to $\dot{f}$ like

$$\dot{f} = 4.65(A-B)/zT^2 \text{ Hz s}^{-1} \quad (120)$$

The noise in $\sqrt{P_+}$ due to the second term on the right hand side of Equation 97 is given by:

$$\sigma_{\sqrt{P_+}} = 2^{-1/2} \pi z \left(\int_0^1 (u-u^2-1/6)^2 du \right)^{1/2} = \pi z / \sqrt{360} \quad (121)$$

As in the previous case for f , the fluctuations of P_+ and P_- due to the sine terms are equal and opposite so that the error in $A-B$ is given by

$$d(A-B) = 2\pi z / \sqrt{90P} \quad (122)$$

The error in the determination of $\dot{f}$ is therefore

$$d\dot{f} = 9.3\pi / \sqrt{90P} T^2 \text{ Hz s}^{-1}$$

For a signal where P_0 is 14.37 (50) the $\dot{f}$ can be measured to 81% of $1/T^2$.

b. Fluctuations of the cosine terms

The mean decrease in $\sqrt{P_+}$ due to the "kinking" or the cosine terms in Equation 97 can be found by averaging $\langle \delta P(u) \rangle^2$ which has already been done in Equation 121. The result is

$$\sqrt{P_+} = \sqrt{P_0} (1 - \pi^2 z^2 / 360) = 0.9725 \sqrt{P_0} \text{ for } z = 1 \quad (124)$$

and consequently the average of P_+ or P_- should be 94.5% of P_0 for $z = 1$.

The expected behavior of the quantity, $A+B$, has been calculated as a function of $\dot{f}$ and is displayed in Figure 18A. The quantity $[2-(A+B)]/z^2$ is plotted on the vertical axis as a function of $\dot{f} T^2$ on the horizontal axis. The same three conditions are plotted on this Figure as in the previous Figure, $z = 1$, $z = 2$ and $z = 1$ with $f^0 - f_0 = 0.2/T \text{ Hz}$. The difference between the curves is very slight.

The noise introduced from the cosine terms of Equation 97 may be

found by calculating the quantity

$$\sqrt[3]{P_+} \cos \theta = (2)^{-1/2} \overline{(d\theta(u)^2 - d\theta(u)^2)}^{1/2} \quad (125)$$

It is easily shown that this quantity is equal to

$$\sqrt[3]{P_+} \cos \theta = (\pi z)^2 / 60\sqrt{63} \quad (126)$$

For $z = 1$ the value of $\sqrt[3]{P_+} \cos \theta$ is about 1/8th of the value of $\sqrt[3]{P_+}$ from Equation 121.

The error in $A+B$ is given by

$$d(A+B) = \pi^2 z^2 / 15\sqrt{63} P_+ = 0.0829 z^2 / \sqrt{P_+} \quad (127)$$

The quantity, $A+B$ acts as a check on the persistence of the signal during the chosen time interval as in the previous case for the measurement of the frequency.

c. The complex phases

The complex phase difference between the vectors of the paths associated with P_+ and P_- is nominally

$$D\theta \equiv \theta_+ - \theta_- = \pi z/3 \text{ radians} \quad (128)$$

From a noise analysis similar to that done for $A-B$ and $A+B$ the errors in $D\theta$ can be shown to be

$$d(D\theta) = \pi z / \sqrt{10 P_0} \text{ radians, } d(D\theta) = 62 \sqrt{10/F_0} \text{ degrees} \quad (129)$$

d. Monte Carlo calculations

Monte Carlo calculations similar to those performed to check the measurement of the frequency confirm the analysis for $\dot{f}$ and the errors given in calculations 123, 127 and 129.

e. The measurement of $\dot{F}$ and the real data

The (A,B) pairs obtained in the measurement of $\dot{f}$ for the HZ Her data are plotted on the A-B plane in Figure 19A. The axes are drawn for A = 1.0 and B = 1.0, the grid ticks have units of 0.1 and the points are coded exactly as in Figure 14A.

The spread of points along the lines of constant A+B now depends on the range of actual $\dot{f}$'s which the HZ Her optical pulsations have produced and the duration, T, for each of the measurements.

Figure 14A indicates that some points are scattered to both sides of the ideal A+B curve. The points to the upper right could be caused by data whose pulsation amplitude is weaker toward the ends of the time interval. Conversely, data whose amplitude was fainter during the center of the time interval tend to drift systematically to the lower left of the ideal A+B curve.

In spite of such difficulties, it is encouraging that the error bars of half of the 29 points overlap the ideal A+B curve and that as many as 22 are no more than 1 σ away from some overlap.

f. Systematics on the $\dot{f}$ measurement

It is important to arrange T for the $\dot{f}$ measurement so that systematic deviations of $A+B$ from the ideal curve are kept to a minimum. Consider the case where the signal has a duration of $T/2$ but where the analysis assumes a duration of T . The size of the perturbing $\dot{f}$ will be equal to z/T -- a factor of 4 too small with respect to the $T/2^2$ time interval. Moreover, within the $T/2$ time interval the response of any $\dot{f}$ in the signal will be correspondingly a factor of 4 too small. When this smaller response is interpreted in the framework of the z/T units, another systematic factor of 4 will be introduced into the answer. Thus the $\dot{f}$ which is measured is 16 times smaller than the real $\dot{f}$ for the signal.

Thus if a signal should last for only a fraction, a , of the time interval, T , then it is possible that the $\dot{f}$ measured for the signal is small by the factor a^4 .

On the other hand, suppose that the signal amplitude is more-or-less randomly sporadic throughout the time interval, T . This case rapidly becomes indistinguishable from a uniform signal and therefore the $\dot{f}$ measurement would not have any systematic error.

g. The $\dot{F}$ measurement in general

The fiducial $\dot{f}$ used in the scheme described above does not have to be 0 Hz s^{-1} , but can be any desired value. To generate the P_+ and P_- paths one adds or subtracts z/T^2 to the fiducial, always keeping the average frequency the same for all three paths. The situation of high $\dot{f}$ is particularly relevant to the attempted study of the short-period binary pulsar, PSR 1913+16 of Hulse and Taylor.⁸⁹

4. Summary of Measurement Procedures for F and $\dot{F}$

To measure f or $\dot{f}$ accurately it is first necessary to do the discrete Fourier Transform to get a general idea of where the features lie, what their $\dot{f}$ might be and to obtain an exact number for the quantity, $LL(f)$, relating the global noise of the raw data to the power spectrum noise level near the frequency of interest.

To measure f° , one first guesses an f_0 , then folds the data at f_0 , $f_0 + z/T$ and $f_0 - z/T$ where z is typically 0.2. With the quantities P_0 , P_+ , P_- , β_0 , β_+ and β_- determined, the quantities, $A = P_+/P_0$ and $B = P_-/P_0$ are calculated.

The revision needed to go from f_0 to the true frequency, f° is approximately equal to

$$f^\circ - f_0 = dx/T = 3(A-B)/(4\pi^2 zT) \quad (130)$$

Alternatively, dx may be taken from a lookup table. The complex phase associated with the frequency, f° is equal to

$$\theta^\circ = \beta_0 + (f^\circ - f_0)T/2 \quad \text{cycles} \quad (131)$$

The errors on f° and θ° are given by

$$df^\circ = 3/(\pi\sqrt{6P_0}T) \text{ Hz}; \quad d\theta^\circ = 0.195/\sqrt{P_0} \text{ cycles} \quad (132)$$

Two other quantities, $A+B$ and $\beta_- - \beta_+$, may be formed as a check on this analysis. Their nominal values and errors are

$$A+B = 2.0 - 6.25z^2 \pm 4\pi^2 z^2 / \sqrt{90P_0} \text{ and } \theta_- - \theta_+ = 2\pi z (1 \pm (6P_0)^{-1/2}) \text{ radians} \quad (133)$$

A lookup table may also be used to find the exact values of $A+B$ or $\theta_- - \theta_+$, given z and dx . If $A+B$ is systematically large or if $\theta_- - \theta_+$ is systematically small, the measurement may be repeated with a smaller time interval, T .

To measure f it is preferable that the average frequency, f_0 , is defined to be equal to f^0 obtained from the previous analysis. The frequency behavior needed for the signal folding to obtain P_0 , P_+ and P_- is then

$$f_0 = f^0; \quad f_+ = f_0 + z(u-1)/T; \quad f_- = f_0 + z(1-u)/T \quad (134)$$

where $u \equiv t/T$. Once P_0 , P_+ , P_- , θ_0 , θ_+ and θ_- are obtained, the f measured for the signal is approximately

$$f = 4.65(A-B)/zT^2 + 9.3\pi/\sqrt{90P_0} T^2 \text{ Hz s}^{-1} \quad (135)$$

Alternatively, Figure 17A may be used as a lookup table.

The remaining two quantities, $A+B$ and $\theta_+ - \theta_-$ and their errors are given by:

$$A+B = 2 - 0.104z^2 [1 - 0.575(A-B)^2/z^2] \pm 0.0829z^2/\sqrt{P_0}, \text{ and} \quad (136)$$

$$\theta_+ - \theta_- = \pi z/3 \pm \pi z/3\sqrt{10P_0} \text{ radians.}$$

The exact values for $A+B$ may be taken from Figure 18A.

5. Odd Comments

It is interesting to note that $\dot{f}$ may be measured to almost the same precision by measuring f for two *equal* halves of the data interval and then taking the difference divided by $T/2$ (the strength of the signal is assumed to be constant over the whole interval, T). An unequal split of the data measures $\dot{f}$ to poorer precision.

If the run is split up and the pulse phases are connected to measure f for the time interval (using the average of f_1 and f_2 from each of the half-intervals to extrapolate well enough), so that the cycle number *can* be fixed and using Equation 69 for the phase errors for an apriori fixed frequency), the errors are $\sqrt{4/3}$ of the errors in the above analysis. This is due to the fact that signal averaging for only phase has lost the information of the frequency within each run segment. When the frequencies for the smaller half-intervals, f_1 and f_2 , are averaged together and then combined with the phase-connected frequency with weights which vary inversely as the square of the errors, then the net error is the same as in the process described above. The difficulties of this process are that the run must be split exactly in half and that the factors involved in the averaging process -- the average and phase-connected frequencies -- are not obviously independent of one another.

All of the above discussion applies to signals which are sinusoidal. Presumably, if the pulse profile is known, the procedures can be suitably generalized to accomodate any mixture of harmonics. The $\dot{f}$ and f of sharp-pulsed objects like the Crab pulsar in the optical band can, in principle, be measured to greater accuracy because of the extra information contained in the higher harmonics (i.e., due to the "sharpness"

of the detail in the light curve).

C. Digital Photometric Methods

Throughout this entire Thesis we have been discussing the results and interpretations of a digital photometric technique. It is therefore logical to include some discussion of the advantages and disadvantages of this technique and the difficulties encountered in putting it to optimum use.

1. Advantages

One advantage of digital photometry over analogue methods is the ease of analysis of such data by large digital computers such as the CDC 6400, 6600 and 7600, especially when long trains of data are involved. The other advantage afforded the user is the higher resolution of count level, higher than, for example, the measurement of the level of a strip chart recorder pen on a sheet less than 30 cm high. The improved resolution allows a greater sensitivity due to the reduction of random errors in the measurement process.

2. Digitization Noise

The limiting noise that one encounters in a digital apparatus (labelled digitization noise) may be fairly easily calculated. This is due to the fact that the continuous values are rounded to digital values and have thus changed by 0 to $\pm \frac{1}{2}$ in value.

For example, let a random variable, x , with a mean and variance about the mean be amplified (or divided) and then offset in such a way that it

averages 31.5 and has a mean squared deviation from this average also of 31.5 (the term: "mean squared deviation from the average", variance, "central variance" and "variance about the mean" are used interchangeably). Next digitize x into a recording system which can register any number between 0 and 63 (a six bit register). The central variance of the ensemble of numbers coming from the six bit register will now be slightly higher than 31.5 due to the addition in quadrature of the digitization noise. Since we have

$$\bar{x} = 1/N \sum_{i=1}^N x_i = 31.5; \quad 1/N \sum_{i=1}^N (x_i - \bar{x})^2 = 31.5 \quad (137)$$

then,

$$1/N \sum_{i=1}^N (x_i - \bar{x} - a_i)^2 = 1/N \sum_{i=1}^N (x_i - \bar{x})^2 - 2/N \sum_{i=1}^N a_i (x_i - \bar{x}) + 1/N \sum_{i=1}^N a_i^2 \quad (138)$$

So long as the a_i 's are uncorrelated with the signs of $x_i - \bar{x}$, the second term on the right hand side of Equation 138 will average to 0. The ensemble average of the third term on the right hand side (where the quantity, a_i , represents the difference between the random variable, x_i , and the nearest integer) can be calculated as

$$\int_{-1/2}^{+1/2} a^2 da = 1/12 \quad (139)$$

a universal result. Thus the new central variance of our example is now $31.5 + 0.0833 = 31.5833$, a change of 0.26% in central variance and of 0.13% in the standard deviation (σ) of the sample. In this particular case,

one sigma of the sample is about $\sqrt{31.5} = 5.6$ bins -- in some nomenclatures this is a "5.6-bit quantization of sigma" although the word, "bit" is inappropriate. The quantity, σ , is defined here to be only the square root of the central variance -- no Gaussian distribution is implied.

If we preamplified or divided the signal down to where we had a "one bit quantization of sigma" (and still had enough register places so that the two tails of the distribution were not chopped off) then the σ' of the digitized distribution would be given by

$$\sigma' = \sigma \sqrt{1+1/12} = 1.04 \sigma \quad (140)$$

i.e., a "one bit quantization of sigma" is only 4% noisier than the infinitely resolved original distribution. We have, of course, ignored the possibility of effects which come from the possible uneven distribution of round-off errors throughout the distribution of x . Thus the sample would get a little more or less noisy as the center of the x distribution moves with respect to the digit center.

If the digitization truncates, instead of rounding, the only effect is that the new average, x' , is lower than $\bar{x}$ by $1/4$ and the sigma about $\bar{x}'$ is the same as the previous sigma about $\bar{x}$.

In the case of a raw photon source which is already digital thanks to mother nature, nothing is added to the central variance, therefore nothing is sacrificed by digital counting.

In practice, most runs on HZ Her counted around 3500 counts sec^{-1} so that we most often divided by two in order to well-confine the distribution between 0 and 63 for 40 ms integrations (subtracting about 38 per 40 ms

interval to center the distribution). The digital divider was not reset by the 25 Hz clocking signal so that even the relatively few remainder counts were not lost, although their arrival was delayed by 40 ms. This non-resetting divider is similar to the data analysis situation described by Peterson,⁹⁰ only the latter apparatus prescales the count rate very severely. In this case, the digitization noise becomes non-flat with frequency, with the higher frequencies (which have much more severely limited distributions) receiving the greater noise while the lower frequencies remain unaffected (Kalnajs and Peterson).⁹¹

3. The Poisson Minimum

To analyze photon statistics, we first start with the well-known Poisson distribution. The Poisson statistics are a consequence of equal a priori probability of receiving a count in equal intervals of time. Given a time invariant count rate of R counts s^{-1} (also ensemble invariant by the ergodic theorem), the probability of receiving n counts in time interval dt is

$$P_n(Rdt) = (Rdt)^n e^{-Rdt} / n! \quad (141)$$

This distribution has a mean of Rdt and a central variance also of Rdt . Another description would be that the source had an average of R counts s^{-1} and a central variance of R counts²/second (which is also called the "power rate" or "squared noise rate").

The Poisson distribution is important because it represents the *minimum* noise rate of any given photon source detected by any electronics

system so long as the count rate is not so high that a substantial fraction of the counts are being lost to coincidence.

In practice, the counting noise is always higher due to brightness fluctuations of the source and/or sky on various time scales. Given that the sky scintillations affect the same percentage of light from any source for the same aperture size (by throwing light outside the aperture) and that transparency changes are also strictly proportional, then the relative amount of excess noise seen in fainter stars is much less than that seen in brighter stars. Moreover, this ratio generally increases as frequency decreases (measuring the noise ratio as a function of frequency is accomplished by computing the power spectrum) so that photon counting at higher frequencies tends toward the Poissonian limit in noise as shown by the power spectrum of Figure 7.

The only exception to this trend is the so-called 30 Hz component of sky noise, which shows up in the power spectra of brighter stars as a broad, low hump ranging from 30 to 45 Hz in frequency before the trend once again continues downward. Any star as bright as magnitude 7.0 would produce such a hump if it were recorded with the EMI 9658R at the 61 cm telescope at Lick (the typical broadband counting rate would be around 10^6 counts s^{-1}).

4. Rate Limits

When the photon count gets so high that coincidence begins to limit the rate by 10% or more, the noise can actually get lower than the Poisson minimum due to the fluctuations being suppressed. However, long before this effect can be used to increase sensitivity, the loss of information (i.e.,

signal) to coincidence is such that the signal to noise ratio is always less than optimum and decreases monotonically with increasing loss to coincidence.

Returning to the low frequency end of the counting spectrum we see from Figure 7 that even though the Poisson minimum is exceeded for the low frequencies (The power at 12.5 Hz is lower than the power at 1.0 Hz) the spectrum is reasonably flat down to below 0.2 Hz. This is fairly characteristic of the Lick sky on a reasonable night for a count rate of around 3500 counts s^{-1} .

Somewhere below 0.2 Hz it becomes advantageous to use a two channel system with the second channel monitoring sky brightness or a comparison star. The process of subtracting one channel from another in general increases the noise over the Poisson minimum of the first channel because the central variances add. However, when the Poisson contributions to the noise at a particular frequency are negligible as compared to the non-statistical contributions, then sensitivity can be gained by the subtraction process because the "systematic" noise has been subtracted out.

5. Afterpulsing

Unfortunately, there are other ways to exceed the Poisson minimum. One such other way is to have the photomultiplier experience afterpulsing, which the EMI does bountifully. This afterpulsing is generally associated with positive ions drifting back toward the negative high voltage of the photocathode. This picture is plausible because the time scale involved seems to be in the ball-park and the photocathode of the EMI is not only

50mm in diameter but is also geometrically red-sensitized (by multiple reflection) and thus is both easy to hit and easy to excite (due to the possibility of glancing angles and multiple collisions) with the positive ions.

About 10% of the pulses from the EMI are afterpulses, arriving roughly 0.5 to 1.0 microseconds after normal pulses. This holds for a wide range of counting rates. Without correction, this would tend to make the variance about the mean roughly 10% higher than the mean. The true mean count rate would also be about 90% of the actual mean count rate. This effect could be corrected by calibration, but the best way to deal with it is simply to introduce a 1-2 microsecond dead time after every pulse with a 1-2 microsecond one-shot in the system.

It is useful to note that, as long as the statistical distribution of the after-pulse population remains the same with varying countrate, then the ratio of real noise to the Poisson noise should remain constant.

6. Afterbursting

A while after the 10% afterpulsing was eliminated with the microsecond deadtime, evidence of the existence of another effect began to accumulate. During the 1970-71 observing season, runs on the Crab Pulsar were being taken on the Crosley 91 cm reflector and subsequently on the 61 cm reflector of Lick Observatory. Checks on the chi-square of fits to the light curve would occasionally rise to 1.3 or 1.4 times the 300 or so data points fitted to the main pulse. A pulsar search program was also initiated in the Spring of 1971 using electronics which were essentially the same as described in this thesis. Checks on the central

variance of the growing number of runs in the pulsar search program showed that the central variance was sometimes as much as a factor of three higher than the average count rate. When data from both of these programs were carefully examined, it was revealed that the ratio of real noise to expected noise was highest at the lower count rates and decreased toward unity by the time the rate was between 10k and 20k counts s^{-1} . When the pulses from the EMI were examined with an oscilloscope, it became evident that occasional pulses would be followed by a train of a few dozen more within 50 to 100 microseconds. These trains we called afterbursts; the reason why they were more devastating to our sensitivity will be discussed in the following paragraph (it is also a useful theorem to remember).

Suppose we have one set of random variables, x_i , with a mean, $\bar{x}$, and a variance about this mean, σ_x^2 (for example the number of detected bona-fide photons in one 4 ms time window) and another similar set of random variables, y_i with $\bar{y}$ and σ_y^2 (for example the number of afterburst counts per 4 ms time window). We form the sum set, $x_i + y_i$. If the fluctuations of one set about its mean are independent of the fluctuations of the second set about its mean, then the central variance of the sum set is equal to the sum of the central variances of the two component sets.

Or:

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N [\bar{x}_i + \bar{y}_i - (x+y)]^2 &= \frac{1}{N} \sum_{i=1}^N (x_i - \bar{x})^2 + (y_i - \bar{y})^2 + 2(x_i - \bar{x})(y_i - \bar{y}) \\ &= \frac{1}{N} \sum_{i=1}^N (x_i - \bar{x})^2 + (y_i - \bar{y})^2 = \sigma_x^2 + \sigma_y^2 \end{aligned}$$

(142)

Thus a very small number of counts in a distribution with an enormous

central variance are capable of strongly affecting the central variance when added to a more normal distribution with a much higher count rate.

The origin of the afterbursts is unknown. According to Professor J. Wampler, they may be due to the fluorescence of x rays produced in the photomultiplication process when passing through certain types of window glass; two tubes mounted face to face would show these pulses in coincidence. Still unexplained, however, is the strong rate dependence of the proportion of afterbursts. Since most of the pulsar search results had used 4 ms integration times, there was no possibility that they were being hidden at higher counts rates (it was possible, though, that the Crab results did hide some of the effect because of the much shorter 25 μ sec channel widths; neighboring channels would therefore tend to become correlated).

Some non-linear process with count rate may ultimately be invoked to explain the phenomenon. For example, if the afterbursting depends upon the action of the positive ions (whose production will be proportional to the count rate), and if also some ambient electron concentration at the appropriate point of space and time were proportional to the count rate, then the destruction rate of the positive ions would be proportional to the square of the count rate. The real difficulty of this picture lies in pinning down the w and n , where and how.

The afterbursts are probably not coming from the LOEL discriminator-amplifier since the characteristic oscillation time is almost an order of magnitude shorter than the average pulse period in the burst; the pulses also seem to appear more or less randomly in the burst.

The dominance of the afterburst effect at the lower count rates led

to a fairly simple cure. The dead time was set by the observer to lie between 1 and 2% of the average pulse interval. Thus we sacrificed 1 or 2% of the normal Poissonian photon counts in order to discard all of the afterbursting. Subsequently, the Crab runs gave a chi-square of 0.98 to 0.99 per degree of freedom and the central variance checks on the pulsar search runs with lower count rates were all reasonably close to the average count rates.

The length of the pulsar search runs and those on HZ Herculis kept the number of dead time readjustments per night to a minimum. If the EMI tube were to be used in an ordinary photometric situation where 1% accuracy were important, then its anomalies would have to be more carefully studied. Fortunately, for the Crab, pulsar search and HZ Herculis programs, a 10% accuracy is more than sufficient.

REFERENCES

1. Giacconi, R., Gursky, H., Kellogg, E., Schreier, E. and Tananbaum, H. 1971, Ap. J. (Letters), 167, L67.
2. Tananbaum, H., Gursky, H., Kellogg, E. M., Levinson, R., Schreier, E. and Giacconi, R. 1972, Ap. J. (Letters), 174, L143.
3. Davidzen, A., Henry, J. P., Middleditch, J. and Smith, H. E. 1972, Ap. J. (Letters), 177, L97.
4. Forman, W., Jones, C. A. and Liller, W. 1972, Ap. J. (Letters), 177, L 103.
5. Bahcall, J. N. and Bahcall, N. A. 1972, Ap. J. (Letters), 178, L1.
6. Kukarkin, B. V., Parenago, P. P., Efremov, Y. I. and Kholopov, P. N. 1969, General Catalogue of Variable Stars (2nd ed; Moscow: Academy of Sciences).
7. Warner, B. and Robinson, E. L. 1972, Nat. Phys. Sci., 239, 2.
8. Lamb, D. Q. and Sorvari, J. M. 1972, I.A.U. Circ. No. 2422.
9. Groth, E. J. and Nelson, M. R. 1972, Ap. J. (Letters), 178, L111.
10. Petro, L. and Hiltner, W. A. 1973, Ap. J. (letters), 181, L39.
11. Kurochkin, N. E. 1973, Inf. Bull. Var. Stars, No. 753.
12. Boynton, P. E., Catena, R., Crosa, L., Deeter, J. and Gerend, D. 1973, Ap. J., 186, 617.
13. Wilson, R. E. 1973, Ap. J. (Letters), 181, L75.
14. Joss, P. C., Avni, Y. and Bahcall, J. N. 1973, Ap. J., 186, 767.

15. Pringle, J. E. 1973, Nat. Phys. Sci., 243, 90.
16. Crampton, D. and Hutchings, J. B. 1972, Ap. J. (Letters), 178, L65.
17. Strittmatter, P. A., Scott, J., Whelan, J., Wickramasinghe, D. J. and Woolf, N. J. 1973, Astr. and Ap., 25, 275.
18. Middleditch, J. and Nelson, J. 1973, Ap. Letters, 14, 129.
19. Groth, E. J. 1974, Ap. J., 192, 517.
20. Giacconi, R., Gursky, H., Kellogg, E., Levinson, R., Schreier, E. and Tananbaum, H. 1973, Ap. J., 184, 227.
21. Fabian, A. C., Pringle, J. E. and Rees, M. J. 1973, Nature, 244, 212.
22. Sharma, D. P., Jain, A. K., Kasturirangan, K., Jayanthi, U. B. and Rao, U. R. 1973, Nat. Phys. Sci., 246, 107.
23. Chevalier, C. and Ilovaisky, S. A. 1973, Nat. Phys. Sci., 245, 87.
24. Pines, D., Pethick, C. J. and Lamb, F. K. 1972, Talk at the Sixth Texas Symposium on Relativistic Astrophysics (Pub. in the Annals of the New York Academy of Sciences as Part of the Symposium Proceedings).
25. Arons, J. and Bahcall, J. N. 1972, Ap. Letters, 11, 194.
26. Arons, J. 1973, Ap. J., 184, 539.
27. McClintok, J. E., Clark, G. W., Lewin, W. H. G., Schnopper, H. W., Canizares, C. R. and Sprott, G. F. 1974, Ap. J., 188, 159.
28. Davidson, K. and Ostriker, J. P. 1973, Ap. J., 179, 585.
29. Pringle, J. E. and Rees, M. J. 1972, Astron. Ap., 21, 1.

30. Lamb, F. K., Pethick, C. J. and Pines, D. 1973, Ap. J., 184, 271.
31. Brecher, K. 1972, Nature, 239, 325.
32. McCray, R. 1973, Nat. Phys. Sci., 243, 94.
33. Avni, Y., Bahcall, J. N., Joss, P. C., Bahcall, N. A., Lamb, F. K., Pethick, C. J., and Pines, D. 1973, Nat. Phys. Sci., 246, 36.
34. Basko, M. M. and Sunyaev, R. A. 1973, Astrophysics and Space Science, 23, 17.
35. Komberg, B. V. 1973, Preprint of Inst. Appl. Math.
36. Katz, J. I. 1973, Nat. Phys. Sci., 246, 87.
37. Roberts, W. J. 1974, Ap. J., 187, 575.
38. Lamb, F. K., Lamb, D. Q. and Pines, D. 1974, Letter to Seth Shulman, NRL, Wash. D.C., 3rd May, 1974.
39. Rosenbluth, M. N., Ruderman, M., Dyson, F. and Bahcall, J. N. 1973, Ap. J., 184, 907.
40. Brecher, K. and Morrison, P. 1974, Ap. J., 187, 349.
41. Brecher, K. and Morrison, P. 1973, Ap. J. (Letters), 180, L107.
42. Lamb, D. Q. and Van Horn, H. M. 1973, Ap. J., 183, 959.
43. Buff, J. and McCray, R. 1974, Ap. J., 189, 147.
44. Jones, C., Forman, W., Tananbaum, H., Schreier, E., Gursky, H., Kellogg, E. and Giacconi, R. 1973, Ap. J. (Letters), 181, L43.

45. Jones, C. A., Forman, W. and Liller, W. 1973, Ap. J. (Letters), 182, L109.
46. Wenzel, W. and Hessner, M. 1972, Inf. Bull. Var. Stars, No. 733; Mitteil. Verander. Stern. (in press).
47. Whelan, J. 1973, Ap. J. (Letters), 185, L127.
48. Zeipel, H. von. 1924, M. N. R. A. S., 84, 665.
49. Crampton, D. 1974, Ap. J., 187, 345.
50. Anderson, L., 1974 Submitted to Ap. J.
51. Alme, M. L. and Wilson, J. R. 1973, Ap. J., 194, 147.
52. Schreier, E., Levinson, R., Gursky, H., Kellogg, E., Tananbaum, H. and Giacconi, R. 1972, Ap. J. (Letters), 172, L79.
53. Van den Heuvel, E. P. J. and Heise, J. 1972, Nat. Phys. Sci., 239, 67.
54. Someone said it but I can't find the reference.
55. Nelson, J., Hills, R., Cudaback, D. and Wampler, J. 1970, Ap. J. (Letters), 172, L79.
56. Moulton, F. R., Celestial Mechanics, 1947, second revised edition, the MacMillan Co., p. 307.
57. Giacconi, R. 1973, Proceedings of the Sixteenth Annual Solvay Conference on Physics (in Press) (c.f. Brecher, K. and Wasserman, I. 1974, Ap. J. (Letters), 192, L125).
58. Davison, P. J. N. and Fabian, A. C. 1974, M. N. R. A. S., 169, 27p.
59. Davidsen, A., Margon, B. and Middleditch, J., to be published in the

Ap. J. circa June, 1975.

60. Cocke, W. J., Hintzen, P., Scott, J. S. and Worden, S. P. 1973, Nat. Phys. Sci., 244, 137.
61. Grandi, S. A., Hintzen, P. M. N. O., Jensen, E. B., Rydren, A. E., Scott, J. S., Stickney, P. M., Whelan, J. A. J. and Worden, S. P. 1974, Ap. J., 190, 365.
62. Clark, G. W., Bradt, H. V., Lewin, W. H. G., Markert, T. W., Schnopper, H. W. and Sprott, G. F. 1972, Ap. J. (Letters), 177, L109.
63. Ulmer, M. P., Baity, W. A., Wheaton, W. A. and Peterson, L. E. 1972, Ap. J. (Letters), 178, L61.
64. Avni, Y. and Bahcall, J. N. 1974, Ap. J., 191, 221.
65. Shulman, S., Friedman, H., Fritz, G. and Yentis, D. 1975 submitted to the Ap. J. (Letters).
66. Brecher, K. and Wasserman, I. 1974, Ap. J. (Letters), 192, L125.
67. Groth, E. J., and Nelson, M. R. 1972, Ap. J. (Letters), 178, L111.
68. Avni, Y., Bahcall, J. N., Joss, P. C., Lamb, D. Q., Schreier, E. and Tananbaum, H., 1974, Ap. J. (Letters), 188, L35.
69. Lubow, S. and Shu, F. 1975 to be published in the Ap. J. circa June 1975.
70. Leach, R. W. and Ruffini, R. 1973, Ap. J. (Letters), 180, L15.
71. Plavec, M. and Kratochvil, P., B. A. C. (1963), 15, 165.
72. Kopal, Z. (1969) Astrophys. and Sp. Sci., 5, 360.

- Kopal, Z. (1959) Close Binary Systems, Chapman Hall and John Wiley, London, New York.
73. Pandharipande, V. R. 1971, Nuclear Physics, A178, 123.
74. Allen, C. W. Astrophysical Quantities 1973, Third Ed., The Athlone Press (University of London).
75. Dahab, R. E. 1974, Ap. J., 187, 351.
76. Prendergast, K. H. and Taam, R. E. 1974, Ap. J., 189, 125.
77. Milne, E. A. 1926, M.N.R.A.S., 87, 43.
78. Sen, H. K. 1948, Proc. Nat. Acad. Sci., 34, 311.
79. Crampton, D. and Hutchings, J. B. 1974, Ap. J., 191, 483.
80. Abromowitz, M. and Stegun, I. A. 1964, Handbook of Mathematical Functions, National Bureau of Standards, p297.
81. Kraft, R. P. 1963, Adv. Astr. and Ap., 2, 43.
82. Smak, J. 1970, Acta Astronomica, 20, 311.
83. Krezeminski, W. and Smak, J. 1971, Acta Astronomica, 21, 133.
84. Tarter, C. B. and Salpeter, E. E. 1969, Ap. J., 156, p953.
85. Tarter, C. B., Tucker, W. H. and Salpeter, E. E. 1969, Ap. J., 156, p943.
86. Moffett, T. J., Nather, R. E. and Vanden Bout, P. A. 1974, Ap. J. (Letters), 190, L63.

87. Wilson, L. W. 1974, Ap. J., 188, 349.
88. Cooley, J. W. and Tukey, J. W. 1965, Math of Comput., vol. 19, pp297-301, April.
89. Hulse, R. A., and Taylor, J. H. 1975, Ap. J. (Letters), 195, L51.
90. Peterson, B. A. 1974.
91. Kalnajs, A. J. and Peterson, B. A. 1974, prepared for the 8th annual general meeting of ASA, May 22-24, 1974 (Australia).

PARAMETERS OF SIGNIFICANT PULSATION EVENTS

Run	Calendar Date UTC.	Run Center Days (JD-2,440,000.5)	35-Day Binary Phases	Run Length (Days)	TABLE I		Errors In Frequency (Hertz)	Derivative Of Frequency (Hertz/sec)	Errors In Derivative (Hertz/sec)	Power In Feature
					Run Length Cycles	Barycentric Frequency (Hertz)				
7	06AUG72	1535.275369	6.283	.121	.071	.807924	9.35E-06	-3.27E-09	6.85E-09	15.8
20	09SEP72	1569.219398	5.248	.121	.071	.807910	7.75E-06	4.54E-09	6.19E-09	23.0
31	12APR73	1784.410091	10.819	.121	.071	.807457	1.05E-05	1.74E-09	8.37E-09	12.5
36	29APR73	1801.401647	20.813	.187	.110	.807437	6.15E-06	1.02E-09	4.11E-09	17.5
39	10MAY73	1812.346564	6.250	.243	.143	.807917	4.29E-06	3.90E-11	1.72E-09	22.6
44	16MAY73	1818.421647	9.824	.150	.088	.807459	8.95E-06	-6.73E-09	5.36E-09	12.5
46	21MAY73	1823.408029	12.756	.121	.071	.807845	1.03E-05	3.49E-09	4.85E-09	15.0
48	23MAY73	1825.279094	13.857	.121	.071	.807752	1.09E-05	-3.33E-09	8.08E-09	11.6
52	13JUN73	1846.355363	5.254	.232	.137	.807918	3.24E-06	-7.21E-10	1.35E-09	36.4
63	17JUL73	1880.274615	5.204	.121	.071	.807924	5.50E-06	9.91E-09	5.30E-09	32.3
68	25JUL73	1888.263776	9.903	.121	.071	.807608	7.75E-06	1.89E-08	5.82E-09	27.6
70	30JUL73	1893.265412	12.845	.139	.082	.807807	8.85E-06	-7.74E-09	4.89E-09	19.6
71	03AUG73	1897.291211	15.213	.121	.071	.807941	8.95E-06	-1.52E-08	7.26E-09	17.0
72	04AUG73	1898.275599	15.792	.167	.098	.807798	5.80E-06	0.	3.19E-09	21.4
78	20AUG73	1914.281081	4.206	.206	.118	.807944	6.45E-06	8.15E-09	3.72E-09	14.0
79	21AUG73	1915.234644	4.767	.121	.071	.807817	9.20E-06	-5.46E-10	7.45E-09	17.9
82	28AUG73	1922.223894	8.878	.121	.071	.807556	8.95E-06	1.19E-08	6.77E-09	19.2
82	28AUG73	1922.223894	8.878	.121	.071	.807814	1.13E-05	1.38E-08	8.56E-09	12.4
86	02SEP73	1927.220154	11.817	.121	.071	.807840	1.05E-05	-7.45E-09	7.89E-09	13.3
91	19SEP73	1944.200510	1.804	.132	.078	.807819	1.16E-05	-4.15E-09	6.71E-09	11.8
95	21APR74	2158.411583	4.799	.203	.119	.807808	5.20E-06	-3.91E-09	2.41E-09	19.7
109	04JUL74	2232.276404	6.245	.121	.071	.807904	7.80E-06	-6.43E-09	5.85E-09	23.5
110	05JUL74	2233.342988	6.872	.133	.078	.807551	8.60E-06	1.48E-08	6.37E-09	18.1

TABLE I contd.

111	21JUL74	2249.292457	16.253	.185	.109	.807901	8.75E-06	-3.55E-09	2.15E-09	32.4
116	07AUG74	2266.281611	6.246	.121	.071	.807917	6.50E-06	-1.52E-09	4.82E-09	32.8
120	24AUG74	2283.267313	16.236	.196	.116	.807913	5.30E-06	-1.62E-09	2.41E-09	19.2
125	10SEP74	2300.237998	5.218	.148	.087	.807920	4.44E-06	3.48E-09	3.08E-09	38.7

TABLE II

"Turn-on" Epochs

Interval Number	Start of Interval (0.0 Cycles) (JD-2,440,000.5)	"Turn-On" (JD-2,440,000.5)	Turn-on phase (Cycles)	$T^+ - T^0$ (Cycles)	$T^0 - T^-$ (Cycles)
1	1488.89	1501.18x	7.23	21	--
2	1524.59	1535.95x *	6.68	21	21
3	1560.30	1571.65x	6.68	20	21
4	1594.30	1606.59	7.23	20	20
5	1628.30	1639.68	6.68	20	20
6	1662.31	1674.60	7.23	20	20
7	1696.31	1708.60x *	7.23	20	20
8	1730.31	1742.60x	7.23	21	20
9	1766.02	1778.31 *	7.23	21	21
10	1801.72	1814.01 *	7.23	21	21
11	1837.42	1848.78	6.68	20	21
12	1871.43	1883.72	7.23	21	20
13	1907.13	1918.49	6.68	20	21
14	1941.13	1953.43	7.23	21	20
15	1976.84	1988.19	6.68	20	21
16	2010.84	2023.13	7.23	21	20
17	2046.54	2057.90	6.68	20	21
18	2080.55	2092.84	7.23	21	20
19	2116.25	2127.61	6.68	20	21
20	2150.25	2162.54	7.23	21	20
21	2185.96	2197.31	7.23	21	20
22	2221.66	2233.02 *	6.68	20	21
23	2255.66	2267.95	7.23	21	20
24	2291.37	2303.72	6.68	20	21
25	2325.37	2337.66	7.23	--	20

TABLE III

Reflection Model Calculations

θ	Container Chosen	Vertical Beam Profile	Sense of Beam Rotation	Duration of Run *3.51 Hr	Surface Emission Model	Orbital Inclination i	Frequency Multiple $f=2^k$	Number of Phases Calculated	Number	Chi-square	Chi* (1-sig)
1	1.00	sin ⁶	Retrograde	1.00	----	90°	1.00	11	1		
			Prograde					9	2	890.9	+114.0
	0.95	"	"	"	"	"	"	"	1		
1	1.00	"	"	"	"	"	"	"	6	127.2	-50.4
5	"	"	"	"	"	"	"	"	5	2021.4	+221.9
0	"	"	"	"	"	"	2.00	"	6		
✓	"	"	"	"	"	"	1.00	19	7	149.4	-18.1
.7	"	"	"	"	"	15°	"	"	8	120.9	+10.2
.0	0.95	"	"	"	"	90°	"	"	9		
"	1.00	"	"	0.5	"	"	"	"	10		
"	"	"	"	1.0	5-48	"	"	"	11		
.0	"	"	"	"	"	"	"	"	12		
1.7	"	"	"	"	"	"	"	"	13		
2.0	"	"	"	"	0.1885	"	"	"	14		
"	0.95	"	"	"	"	"	"	"	15		
1.0	1.00	"	"	"	----	"	"	"	16	9616.6	-396.1
2.0	"	"	"	0.5	"	"	"	18	17		
1.65	"	"	Retrograde	1.0	"	"	"	19	18	131.0	+24.9
"	"	"	Prograde	"	"	45°	"	"	19	139.0	-2.2
1.0	"	"	"	"	"	90°	0.50	"	20		
"	Constant	Perpendicular	"	"	"	"	1.00	"	21		
1.7	"	sin ⁶	Prograde	"	"	"	"	19	22	149.8	+21.7
2.0	"	"	"	"	0.275 Chry	"	"	"	23		
1.6	"	Constant	Perpendicular	"	3" Roche Advance	"	"	19	25		
1.65	"	sin ⁶	Prograde	"	----	75°	"	"	26	129.1	+2.8
1.70	"	"	"	"	"	"	"	9	27	169.7	+23.4
1.60	"	"	"	"	"	60°	"	"	28	152.5	-17.4
1.65	"	"	"	"	"	"	"	"	29	132.0	-2.2
1.55	"	"	"	"	"	45°	"	"	30	262.6	+62.0
1.60	"	"	"	"	"	30°	"	"	31	154.2	-17.0
1.65	"	"	"	"	"	"	"	"	32	116.1	-7.7
1.60	"	Constant	Perpendicular	"	3" Roche lag	90°	"	19	33		
50.0	0.196	sin ⁶	Prograde	"	----	"	"	9	34		
1.75	1.00	"	"	"	0.2825	"	"	"	35	95.0	+8.5
1.55	"	"	"	"	0.2825	"	"	"	36	246.7	+47.6
1.75	"	"	"	"	0.1825	"	"	"	37	177.0	+29.2
1.55	"	"	"	"	0.1825	"	"	"	38	241.8	8.1
1.65	"	"	"	"	----	"	"	19	39	126.2	+7.2
1.60	"	"	"	"	"	75°	"	"	40	143.5	-18.1
1.70	"	"	"	"	"	50°	"	"	41	155.1	+20.8
1.60	"	"	"	"	0.1825	90°	"	"	42	174.1	-19.8
1.70	"	"	"	"	0.1825	"	"	"	43	132.5	+7.1
"	"	"	"	"	0.2825	"	"	"	44	82.0	+11.8
1.60	"	"	"	"	0.2825	"	"	"	45	109.6	-18.1
1.70	"	"	"	"	0.1825	75°	"	"	46	100.3	-1.6
1.60	"	"	"	"	----	45°	"	"	47	165.0	-21.4
1.70	"	"	"	"	0.2825	75°	"	"	48	93.0	-11.2
1.60	"	"	"	"	0.2825	"	"	"	49	137.4	-6.8
1.75	"	"	"	"	----	90°	"	"	50	210.9	+40.1
1.55	"	"	"	"	"	"	"	"	51	207.2	-16.1
1.65	"	"	"	"	0.1825	75°	"	"	52	132.4	-21.2
2.70	"	"	"	"	RTOR RAY	90°	"	"	53	96.5	-4.1
1.75	"	"	"	"	"	"	"	"	54	91.7	+15.1
1.70	"	"	"	"	RTOR	"	"	"	57	119.1	+14.1

Table III contd.

1.75	1.00	sin ² θ	Prograde	1.00	HIGH	90°	1.00	9	58	157.5	+12.3
1.70	"	"	"	"	LOW	"	"	"	59	148.6	+7.0
1.75	"	"	"	"	"	"	"	"	60	137.4	+24.6
1.65	"	"	"	"	"	75°	"	"	61	156.8	-13.1
1.70	"	"	"	"	"	"	"	"	62	151.1	+7.1
1.65	"	"	"	"	HIGH	"	"	"	63	114.6	+0.1
1.70	"	"	"	"	"	"	"	"	64	114.6	+15.5
1.65	"	"	"	"	0.1000	90°	"	"	65	101.1	-8.6
1.65	"	"	"	"	0.2000	"	"	"	66	148.0	+0.5
1.70	"	"	"	"	0.2000	"	"	"	67	110.3	+10.2
1.65	"	"	"	"	0.3000	75°	"	"	68	151.6	+14.2
1.65	"	"	"	"	0.3000	"	"	"	69	114.4	-27.9
1.75	"	"	"	"	0.1000	"	"	"	70	111.7	+17.0
1.65	"	"	"	"	0.1000	"	"	"	71	166.2	+22.9
1.70	"	"	"	"	0.3000	90°	"	"	72	119.8	+14.3
1.65	"	"	"	"	0.4000	"	"	"	73	139.7	+16.4
"	"	"	"	"	0.2000	"	"	"	74	183.1	-13.3
"	"	"	"	"	0.1000	"	"	"	75	119.0	-4.9
1.60	"	"	"	"	0.4000	"	"	"	76	132.0	-5.6
"	"	"	"	"	0.2000	"	"	"	77	309.6	-30.1
"	"	"	"	"	0.1000	75°	"	"	78	134.1	+5.2
"	"	"	"	"	HIGH	"	"	"	79	145.1	-21.9
1.65	"	"	"	"	"	90°	"	"	80	105.2	-3.9
"	"	"	"	"	LOW	"	"	"	81	152.0	-11.6
1.75	"	"	"	"	"	75°	"	"	82	186.5	+24.7
1.70	"	"	"	"	LOW 0.1000	"	"	"	84	202.2	-32.1
"	"	"	"	"	"	90°	"	"	85	162.6	-3.6
1.72	"	"	"	"	"	"	"	"	86	141.1	+3.6
1.75	"	"	"	"	"	75°	"	"	87	150.4	-14.5
1.75	"	"	"	"	0.2000	"	"	"	90	70.5	+4.8
1.60	"	"	"	"	0.2000	90°	"	"	91	117.3	+1.8
1.72	"	"	"	"	0.2000	"	"	"	92	82.9	-2.2
1.67	"	"	"	"	0.1000	"	"	"	93	97.8	-1.3
1.69	"	"	"	"	HIGH 0.2000	"	"	"	94	126.5	-6.3
1.71	"	"	"	"	"	"	"	"	95	106.1	+5.4
1.65	"	"	"	"	"	75°	"	"	96	143.9	-29.5
1.70	"	"	"	"	"	"	"	"	97	89.5	-8.0
1.60	"	"	"	"	TIDAL LOW	90°	"	"	98	311.6	-48.2
1.50	"	"	"	"	"	"	"	"	99	183.6	-0.8
1.60	"	"	"	"	"	"	"	"	100	247.2	+43.3
1.60	"	"	"	"	"	75°	"	"	101	318.8	+49.0
1.50	"	"	"	"	"	"	"	"	102	187.9	+1.3
1.60	"	"	"	"	"	"	"	"	103	285.5	+20.9

Notes to Table: ** This model was an approximation to the obliquity factors given in the preprint of Aime and Wilson⁸⁶, however, the numbers are no longer accurate.

*** This model ignores all area elements of the lobe which are closer to L1 than 175,000 km as measured in projection along the x axis.

**** This model for the high energy X ray saturated case, where not only is the obliquity from the area element to Mer X-1 ignored, but the $1/r_{\text{ex}}^2$ dependence of X ray flux is also ignored.

TABLE IVDetermined Mass Ratios

Inclination	Model	R	dR	$\chi^2_{\min}$
90	----	1.6474	0.0265	126.1
"	0.1RES	1.6663	0.0188	97.6
"	0.2RES	1.7152	0.0193	82.7
"	0.1REB	1.6564	0.0291	147.7
"	0.2REB	1.6908	0.0147	133.8
"	LOW	1.6780	0.0294	145.7
"	HIGH	1.6660	0.0253	103.2
"	TIDAL LOW	1.5030	0.0269	183.5
75°	----	1.6456	0.0296	129.0
"	0.1RES	1.7119	0.0233	99.1
"	0.2RES	1.7411	0.0207	69.9
"	LOW	1.6819	0.0293	148.4
"	HIGH	1.5757	0.0297	110.9
"	TIDAL LOW	1.5073	0.0279	187.3
60°	----	1.6485	0.0268	132.0
45°	"	1.6502	0.0253	139.0
30°	"	1.6696	0.0252	113.0

TABLE VTwo Point Mass Ratios

Inclination	Model	R Used	χ^2	$\chi \cdot (-\sin \theta)$
90°	LOW 0.1RES	1.70	143.6	-3.4
"	"	1.72	141.1	+3.6
"	HIGH 0.1RES	1.69	124.5	-6.3
"	"	1.71	109.1	+1.4
75°	0.1REB	1.60	134.1	+5.2
"	"	1.65	166.2	+22.9
"	0.2REB	1.60	137.4	-4.8
"	"	1.65	151.8	+14.2
"	LOW 0.1RES	1.70	202.2	-32.1
"	"	1.75	150.4	-14.5
"	HIGH 0.1RES	1.65	143.9	-29.5
"	"	1.70	89.8	-8.0

TABLE VI

Parameters for Geometry $M_D/M_C = 0.5$

Position	α	β	ϕ_0	$\phi_{-}^{1/2}$	$\phi_{+}^{1/2}$
L1	19.7°	0°	220°	204.2°	248.2°
$\frac{1}{2}S(\bar{w}_d)$	12.8°	15.2°	221°	191°	270°
$\bar{w}_d$	-21°	72°	210°	143°	---

TABLE VII

 $M_D/M_C = 0.5$, $\Omega D = 253.4 \text{ km s}^{-1}$

Position	$(U_X^2 + U_Y^2)^{1/2} \alpha + \beta$		$(X^2 + Y^2)^{1/2}$	ϕ_{arb}	δ	U km s^{-1}	V km s^{-1}	T km s^{-1}	Z km s^{-1}
L1	0.0	19.7°	0.237	180°	0	0	0	0	0
"	"	"	"	210°	"	"	"	-30	-30
"	"	"	"	240°	"	"	"	-52	-52
"	"	"	"	270°	"	"	"	-60	-60
$\frac{1}{2}S(\bar{w}_d)$	0.887	28.0°	0.442	180°	8.05°	198.4	198.4	-15.7	321.1
"	"	"	"	210°	"	"	224.6	-69.0	354.0
"	"	"	"	240°	"	"	190.6	-103.9	285.1
"	"	"	"	270°	"	"	105.5	-110.9	193.0
$\bar{w}_d$	2.751	51°	0.648	180°	5.75°	439.0	439.0	-16.4	861.6
"	"	"	"	210°	"	"	651.3	-100.5	989.8
"	"	"	"	240°	"	"	689.0	-149.7	978.3
"	"	"	"	270°	"	"	542.1	-163.3	817.8

FIGURE CAPTIONS

Caption

- Fig. 1 Plot of photometric and photographic B and photometric U-B taken from HZ Her. The data have been folded over with the binary ephemeris taken from Tananbaum et al.² for Her X-1, with orbital phase on the horizontal axis. The 1.7 day x ray modulation of Her X-1 is shown below the two sets of data. The Figure has been taken from Davidsen et al.³ in which this light curve and the optical pulsations observed from HZ Her formed the basis of the optical identification.
- Fig. 2 Shows the Her X-1 intensity data in Figure 2 of Giacconi et al.²⁰ plotted vs time in days on the horizontal axes. Three complete 11-day "on" intervals are shown, with the numerous x ray eclipses in each marked by the vertical lines.
- Fig. 3 Shows the B flux from HZ Her from Boynton et al.¹⁰ vs. orbital phase folded over for different times of the 35-day phase. The numbers on the right hand side of the curves indicate the cycle number with respect to the x ray "turn-on", with cycle 0 corresponding to the "turn-on". This Figure shows the light curve variations for approximately 2/3rds of the 35-day interval (cycles -7 to +7).
- Fig. 4 Shows the photographic magnitudes for HZ Her during its extended "off" interval (1-10 yrs) plotted vs orbital phase as taken from Figure 4 of Jones et al.⁴⁵
- Fig. 5 A schematic of the electronics used for the HZ Her optical

pulsation experiment.

- Fig. 6 The spectral response of the EMI 9658 S20 photocathodes in quantum efficiency (%) vs wavelength in microns as taken from the EMI Manual.
- Fig. 7 A coarsely-averaged power spectrum of HZ Her as taken from Figure 1 of Middleditch and Nelson.¹⁸ The logarithm of the power is plotted on the vertical scale vs frequency in Hz on the horizontal scale. A signal near 0.80786 Hz (corresponding to run 20 -- see Figure 12) is hidden by the coarse averaging process.
- Fig. 8 Approximately $8' \times 10'$ of field near HZ Her as enlarged from the Palomar Sky Survey blue print. HZ Her, the Comparison Star and the Offset Guide Star are marked accordingly.
- Fig. 9 Detail of power spectrum centered on the Her X-1 pulsation frequency.
- 30
Power is plotted vertically, while the upper horizontal scale shows the measured frequency. The lower horizontal scale is derived from the upper by a doppler argument, $v = (f - f_0)c/f_0$ where f_0 is the c.m. frequency of Her X-1 taken to be 0.8078735 Hz. Thus v represents the velocity relative to the center of mass of the binary system projected along the line of sight to the Earth (the sign convention is anti-Hubble). The horizontal dotted line near the bottom scale represents the average power for two much larger regions of power spectrum which begin just beyond the sides of this frame; this defines unity for the left hand vertical power scale.

The circle in the upper center represents both the tracking of the Her X-1 pulsar frequency and the Her X-1 orbital position

as a function of orbital phase. Terrestrial observers look at the orbit circle from the bottom, thus the orbital progression shown is for the left-handed (clockwise) sense of rotation. The track of Her X-1 over the duration of this run is shown by the darkened part of the circle. The companion star to Her X-1 will occupy a roughly tear-shaped region away from Her X-1 to the opposite side of the circle and perhaps farther. Her X-1 is eclipsed when it is tracking near 0° on the orbital circle and consequently lies above the companion.

The three vertical dashed lines near the bottom center of the frame mark the lower limit, center, and upper limit of the Her X-1 pulsation frequency.

The right hand vertical scale gives the log base ten of the probability of any given bin accidentally exceeding a vertical height. Due to the exponential nature of the distribution of bin heights in a power spectrum, this scale is derived from the left hand power scale by a fixed ratio.

Fig. 9 Show optical pulsation events which may be interpreted as
- 11 originating somewhere along the line of sight to Her X-1 by x ray
 excitation of gas in the binary system or from Her X-1 directly.

Fig. 12 Optical pulsation events which appear to be coming from gas
- 18 near the center of mass (c.m.) of the system; this may, in fact,
 be part of the limb of the non-compact element in the binary
 system.

Fig. 19 Near-limb type events which now occur near 0.8 cycles
- 22 instead of 0.2 cycles (0.0 = eclipse).

- Fig. 23 Optical pulsation events which show both limb-type and line-
- 25 of-sight type of frequencies simultaneously.
- Fig. 26 Two more optical pulsation events which occurred simultaneously;
 however, the Her X-1 track falls precisely between the two bins with
 excess power.
- Fig. 27 A run with a very significant event and with another just
 lower in frequency. The less-significant spike is shown to be
 associated with an incident enhanced by the side-lobe of the
 more-significant spike at the end of Section VI.
- Fig. 28 The two parts of run 71 were combined in order to both bring
- 30 up the power and to limit more severely the width of the feature
 in relative velocity.
- Fig. 31 The sum of all the data. Note that the relative proportions
 of the features below the circle would be quantitative only if the
 data coverage over binary phase were uniform; however, non-
 uniformities in the coverage do exist up to a factor of two.
- Fig. 32 A histogram for all of the individual power spectrum bins
 which made up the lower portion of Figure 31 (below $f_0 - 1.2df$,
 where $df = 169.2 \pm 0.8078735/299792.5$ Hz, the amplitude of Her X-1's
 doppler sinewave). The population is plotted vertically on a
 logarithmic scale while the power is plotted horizontally on a
 linear scale. The straight line is a theoretical curve for random
 noise with one free parameter--the y intercept. The figure shows
 that the data are consistent with randomness.
- Fig. 33 Like Figure 32 except that the frequency region which was
 used to select the spectral bins for histogramming is now between

$f_0 - 1.2df$ and $f_0 + 1.2df$. The figure shows a significant excess of population for bins with power above 4.5 local power levels -- most of which are presumably due to optical pulsation activity of HZ Her.

Fig. 34 Like Figures 32 and 33 except that the selection for the histogram has been restricted to the power spectrum bins whose frequency is above $f_0 + 1.2df$. The population is consistent with random noise.

Fig. 35 The sum of each 30° of data over the whole 360° of binary
- 46 phase. Each section is identified by the darkened track on the Her X-1 doppler orbit.

Fig. 47 This figure shows all of the data folded over in binary phase versus binary phase on the horizontal (nominally labelled 10.00 to 11.00 because this is the midpoint of the 20.5 cycles of the long 35-day phase). The top frame shows the data coverage. The lowermost frame shows the power spectrum bins which pop through a statistical screen thrown over all of the spectra in the region between the two horizontal dashed lines (representing twice the extent of Her X-1's doppler amplitude in relative velocity/frequency). The power threshold is 10.0, the expectation value for the number of accidental boxes to appear above the screen is roughly 0.16 for the current data ensemble. There are, in fact, some 33 boxes shown. The track of Her X-1's pulsation frequency in binary phase is shown by the sine wave.

The dots enclosing this sine wave indicate the model which was used to derive the pulsed fraction results which are plotted

in the middle frame. The vertical scale has units of thousandths of the mean broadband optical brightness of Her X-1 for binary phase near 0.3 and 0.7 as extrapolated from a nearby comparison star of similar color and magnitude. The coverage histogram of the top frame has been divided out of the averaging in the middle frame so that the results represent a true mean.

Fig. 48 This figure is the same as 47 except that the statistical screen has been lowered to 8.0 local power levels and the dotted line has been shifted in the lower frame so that the results of the middle frame for mean pulsed fraction now exclude the Her X-1 track and include the events associated with the limb of HZ Her. The expected number of accidental boxes in the lower frame is now 1.35, with 40 boxes actually shown.

Fig. 49 Figure 49 is the same as 47 except that the statistical screen for the lower frame has been lowered to 7.0 local power levels. Some 3.7 boxes are expected from noise; 50 boxes are shown.

Fig. 50 Figure 50 is the same as 48 except that the statistical screen is down to 6.0 local power levels with 10 boxes expected from noise and 66 actually appearing.

Fig. 51 The 29 most significant results plotted on velocity-orbital phase space as in the lower frames of Figures 47-50. The left hand vertical scale gives the relative velocity derived from the frequency of the right hand vertical scale. The horizontal scale gives the binary phase. The results are plotted as parallelograms whose slope is determined from the precisely measured f and

whose vertical location and errors are determined from the precisely measured f .

Fig. 52 The precise frequencies and errors for the 27 most significant events which were associated in the three groups of Figure 51 are plotted vs time on the horizontal axis for two and one half observing seasons. The twelve events plotted above the zero-velocity dotted are associated with Feature I (limb pulsations at orbital phase 0.2). The six points nearest the dotted line are associated with Feature III (x ray line-of-sight pulsations near orbital phase 0.85) and have had the Her X-1 sinewave subtracted from their velocities (and thus represent velocity *residuals*). The remaining nine lowermost points are associated with Feature II (limb pulsations near orbital phase 0.8).

Fig. 53 Figure 53 the location of the optical pulsations in a hypothetical binary system for $m_{\text{HZ}}/m_x = 1.70$. The narrow dimensions of the boxes are derived from $\pm 2\sigma$ of the precise f measurement. The long dimensions of the boxes are derived from $\pm 1\sigma$ of the precise $\dot{f}$ measurement.

The left vertical and lower horizontal scales give the dimensions of the *projected* binary system in millions of kilometers. The + marks indicate the locations of the centers of mass for the system. The system c.m. and the Her X-1 c.m. locations are independent of the mass ratio chosen. The lobes and the HZ Her c.m. positions are dependent upon the assumed mass ratio. The right vertical and upper horizontal scales are used to indicate the scale of the relative velocity and frequency respectively.

The details are plotted in the co-rotating reference frame; the earth and the rest of the universe are seen from this frame to be rotating in a right-handed (counter-clockwise) fashion every 1.7 days (thus the system has been chosen as a left-handed rotator). The arrows at the ends of the boxes indicate the direction to the earth at the time of the observation. The corresponding orbital phase may be read off the directional pointers at the upper right.

Fig. 54 The 35-day systematics on the pulsed fraction of Feature I. The pulsed fraction is plotted on the vertical scale in thousandths. The legend in the upper right of the frame gives the binary phase-frequency region used to select the spectra and regions therein respectively from which to plot the pulsed fractions. The horizontal scale is discontinuous above each tick mark and integer label because the region of binary phase used has been mapped linearly onto the whole region between each pair of consecutive integers. This has been done so that the data do not overlay each other too closely and so that any short-term variability within each of the small binary phase regions could be more easily discerned from the case of non-reproducing data. The x ray "on" region starts at 7.0 ± 0.25 and lasts roughly until 14.

The results for this binary phase-frequency region (Feature I) indicate a correlation to the 35-day phase where two higher levels of optical pulsation fall before and after the x ray "on" interval.

Fig. 55 The same as Figure 54 except that Feature II is plotted. The results are seen to be definitely low for two regions of the

21-cycle phase and ambiguously high for two other regions. No less than four independent points are involved in the anomalous dip in pulsed fraction in cycle 11.

Fig. 56 This Figure shows why the Her X-1 type of pulsation near binary phase 0.2 is not considered as a proper "feature" of optical pulsation. The result of run 10 is seen to be singular.

Fig. 57 The correlation of Feature III in 35-day phase. Activity marks the first half of the x ray "on" interval, but dies out during the second half. One night's results (two points) are seen during cycle 20.

Fig. 58 Again, like Figure 54, the correlation of optical pulsation with the 35-day phase. The binary phase-frequency region covers the "anomalous" Feature. All but six of 31 points have an excess of power.

Fig. 59 Random binary phase-frequency regions which are plotted to
- 61 test correlation to the 35-day phase. Negative results are the rule, not the exception.

Fig. 62 The 1.7 day-35-day optical pulsation template. The orbital phase is split into 12 equal intervals of 30° each and the frequency interval between $f_0 - 2df$ and $f_0 + 2df$ is split into 21 equal intervals of width $0.2df$. The 35-day phase is represented by the repetition of seven frames, each of which represents the integration of 3 cycles of orbital phase into one. The labels on the horizontal axis indicate the appropriate cycles used for the frames (x ray "on" = 7 - 14). The amplitude of pulsation in the small rectangular regions is represented by number of lines

crossing the rectangle lengthwise with the mean pulsed fraction given by: $\text{mpf} \cdot 1000. = 0.2 \cdot (n-4)$ where n is the number of lines. Zero lines represents insufficient phase coverage and 1 line represents all results whose mean pulsed fraction $\cdot 1000.$ is less than 0.0. The darker the region, the higher the mean pulsed fraction.

Fig. 63 The same as Figure 53 except that each Figure in the sequence
- 73 covers data contributions from only 2 cycles out of the 21-cycle interval of the 35-day phase. Figure 63 starts with cycles 0 - 2 and Figure 73 ends with cycle 20-21.

Fig. 74 The 35-day phase coverage histogram. Like the coverage histograms of Figures 47-50 except that each histogram of the 11-fold stack of Figure 74 represent the data coverage for exactly 2 cycles of the 35-day phase (except for the histogram at the top of the stack which covers only cycle 20-21). The lowest histogram in the stack starts with cycles 0-2 etc.

Fig. 75 The geometry of the theoretical optical pulsation calculations.

Fig. 76 The amplitude profile for the theoretical calculation no. 39 for mass ratio of 1.65. The pulsed optical/x ray energy ratio is plotted in thousandths on the vertical scale vs the binary phase on the horizontal scales. The behavior is an excellent qualitative match for the middle frames of Figures 48 and 50.

Fig. 77 The frequency-tracking results of the theoretical optical pulsation calculation number 39 for $R = 1.65$ plotted against the data parallelograms for Features I and II. The vertical scale give the orbital phase. The theoretical curve is the line

connecting the diamonds.

Fig. 78 The frequency-tracking results for four theoretical models with $R = 1.0, 1.5, 2.0$ and 2.5 (calculations 16, 4, 2, and 5) plotted against the data parallelograms for Features I and II. Most of the detail on the right hand side of the frame has not been calculated in deference to the high degree of anti-symmetry about $\beta = 180^\circ$. The Figure indicates that the proper mass ratio lies between $R = 1.5$ and $R = 2.0$.

Fig. 79 The amplitude profiles for calculations 16 ($R = 1.00$, diamonds), 4 ($R = 1.50$, squares), 2 ($R = 2.00$, hourglasses) and 5 ($R = 2.5$, split squares). The higher mass ratios subtend larger solid angles by virtue of their proximity to Her X-1 and thus produce more optical pulsation.

Fig. 80 The frequency tracking for calculations 7 and 22 ($R = 1.60$, diamonds, and $R = 1.70$, squares) plotted against the data for Features I and II.

Fig. 81 The frequency profiles for calculations number 39 and 19 (prograde pulsar spin vs retrograde pulsar spin) for $R = 1.65$. The prograde spin gives a much better match to the data.

Fig. 82 The amplitude profiles for calculations 39 and 19 (Prograde, diamonds and Retrograde, squares). Little difference is seen between the two profiles.

83 The three frequency profiles for calculations 2, 3 and 9 (100% filled, diamonds, 95% filled, squares and 85% filled, hourglasses). A large difference between the three cases is shown.

The amplitude profiles for calculations 2, 3 and 9. The

two are seen to produce unacceptably high results for $\theta = 180^\circ$.

Fig. 85 Like Figure 48 but with the dotted region narrowed to clarify the optical pulsation profile for Features I and II in the middle frame.

Fig. 86 The frequency tracks for calculations 20, 2 and 6 ($f = 0.5f_0$, diamonds; $f = f_0$, squares; and $f = 2.0f_0$, hourglasses) for $R = 2.0$. Changing interference patterns across the Roche lobe account for the differences shown.

Fig. 87 The amplitude profiles for calculations 20, 2 and 6 ($f = 0.5$, 1.0 and $2.0 f_0$ respectively). The higher amplitude for the lower frequency is a result of the difference in the areas of the corresponding coherently pulsing regions on the Roche lobe.

Fig. 88 The frequency profiles of calculations 39, 26, 29, 19 and 32 for $R = 1.65$ and $i = 90^\circ, 75^\circ, 60^\circ, 45^\circ$, and 30° respectively. Not much change is indicated. The projected system separation is constrained to remain at $3.95 \cdot 10^{11}$ cm (from the system c.m. to Her X-1)

Fig. 89 The amplitude profiles for the inclinations of $90^\circ, 75^\circ, 60^\circ, 45^\circ$, and 30° with $R = 1.65$. The decrease of amplitude with decreasing inclination is a consequence of the increased absolute size of the system.

Fig. 90 The x ray eclipse constraint on R vs. i plotted with the theoretical-experimental optical pulsation constraint on R vs. i with R on the vertical axis and i on the horizontal axis. The intersection of the two constraints provides a good measure of R and i .

- Fig. 91 The amplitude profiles for calculations 2, 14 and 23 for $R = 2.00$ with 14 having a ± 0.05 radians width of equatorial shadow and with 23 having the forward 375,000 km tip (as measured parallel to the mass axis) shadowed. The gain in optical at $\phi = 0.50$ pulsation (relative to the maxima near $\phi = 0.25$ and $\phi = 0.75$) sets a limit on the amount of shadow.
- Fig. 92 The apparent locations of the optical pulsation vs. orbital phase for calculation 39 using the analysis which related f and $\dot{f}$ to spatial coordinates as in Figures 53 and 63-73 which depicted the results of the *data*.
- Fig. 93 The amplitude profiles of calculations 39, 65 and 44 ($R = 1.65, 1.65$ and 1.70 and no shadow, 0.1RES and 0.2RES respectively).
- Fig. 94 The amplitude profiles for calculations 39, 66 and 67 ($R = 1.65$ and fully-illuminated, 0.1REB and 0.2REB respectively).
- Fig. 95 The amplitude profiles for various limb-darkening laws (high and low energy $\times$ ray limits) and an occasional 0.1RES as in calculations 39, 59, 80, 85 and 95. Some qualitative changes are seen in the profile near 45° and in the shift of the maxima near $\phi = 0.25$ and $\phi = 0.75$.
- Fig. 96 The amplitude profiles for $i = 75^\circ$ for various limb-darkening laws and an occasional 0.1RES as in calculations 26, 61, 63, 87, and 97. The qualitative changes of the last Figure have become more extreme.
- Fig. 97 The amplitude profile for $R = 1.50$, Tidal model, low energy $\times$ ray limit limb-darkening law for $i = 90^\circ$ and $i = 75^\circ$

(calculations 99 and 102). Little difference is observed. The orbital phases of the maxima do not compare as favorably with the data in Figure 85 as do the low energy x ray limit Roche models.

Fig. 98 The apparent locations of optical pulsation vs binary phase for calculation number 99 (Tidal, Low). Otherwise the same as Figure 92. The left hand side of the frame cuts off the lobe of HZ Her.

Fig. 99 The theoretical d.c. reflection law with the same model parameters as calculation 7). Efficiency is plotted against orbital phase in the top frame. The middle frame gives the theoretical x ray-induced ratio of d.c. optical emission to pulsed optical emission. The lower frame gives the pulsed fraction normalized to the instantaneous brightness of HZ Her -- a quantity irrelevant to most of the discussion because of the way the data models are normalized.

Fig. 100 The d.c. reflection efficiency plotted vs. orbital phase for a model whose parameters are like calculation 81 ($R = 1.65$, Low energy limit limb-darkening). The broad top is a result of the lack of an obliquity factor from HZ Her to the observer in this limit (when the same portion of the star is visible at some angle for a long phase interval).

Fig. 101 The geometry of a semi-detached system and its stream-disk interaction as taken from Figure 4 of Lubow and Shu.⁷¹ The collision of the returning stream with the incident stream creates two shocks which act to deflect the stream. Some sort of equilibrium disk is established as shown in part (c).

Fig. 102 Similar to Figures 47 and 49 except that the Her X-1 track is followed more closely in the lower frame to improve the accuracy of the Mean Pulsed Fraction profile in the middle frame.

Fig. 103 The pulsed radiation pattern which would emerge from a hypothetical rectangular cloud of certain dimensions due to the reprocessing of x rays into visible radiation. The optical radiation then transmits relatively well in most directions but is seen to pulse strongly only in certain directions because of differing transit times.

Fig. 104 The various constraints on the temperature, T , and the electron density, N , of the stream-disk cloud as derived in the text. The triangles mark the points where the optical pulsation level was calculated.

Fig. 105 The HZ Her data folded as in Figures 47-50, 85 and 102 but with a period equal to $P_0 \cdot 20.5/21.5 = 1.621087$ days -- the higher beat frequency between the 1.700165 days orbital period (P_0) and the 34.85-day period. The boxes in the lower frame are distributed non-randomly and the mean pulsed fraction profile of the middle frame is not flat. Thus a significant amount of correlation of the data exists with the higher beat frequency.

Fig. 106 Like Figure 105 except that the data has been folded with the period $P = P_0 \cdot 20.5/19.5 = 1.7831$ days -- the lower beat frequency between the 1.7-day orbital frequency and the 34.85-day frequency. The boxes in the lower frame are far more randomly grouped than in Figure 105 and the mean pulsed fraction profile is not as significant. Comparison between Figures 105 and 106 indicates

that the HZ Her - Her X-1 system favors *backward-marching* structure with respect to the 1.7-day phase.

Fig. 107 The HZ Her data folded at $P = P_0 \cdot 20.75/21.75 = 1.660161$ days --halfway between Figure 105 and Figures 47-50, 85 and 102.

The lower frame shows the boxes spaced very randomly and the middle frame shows a four-fold structure which is the result of the two-fold structure of Figure 105 being shifted by the different folding period.

Fig. 108 The HZ Her data folded with the period, $P = P_0 \cdot 20.25/19.75$ -- half-way between Figure 106 and Figures 47-50, 85 and 102. The structure shown is completely random.

Fig. 109 The HZ Her data folded at the period, $P = P_0 \cdot 20.50/22.5 = 1.5490392$ days -- the higher beat frequency between the 1.7-day orbital period and $\frac{1}{4}$ of the 34.85 period. The results are significant.

Fig. 110 The HZ Her data folded at the period, $P = P_0 \cdot 20.5/18.3 = 1.8839666$ d -- the lower beat frequency between the 1.7-day period and $\frac{1}{4}$ of the 34.85-day period. The results are not as significant as those of Figure 109.

Fig. 111 The constraints on the altitude, θ_0 , and the $\frac{1}{2}$ -width, $\delta\theta/2$, of the Her X-1 radiation pattern, where θ_0 is measured from the orbital plane. The axes are labelled in units of $|90-i|$, where i is the inclination of the orbit. The constraints were derived from considerations of the durations of the x ray "on" and "off" intervals.

Fig. 112 A schematic diagram of the sequencing of the events of the

34.85-day cycle. Outside of the inner circle, the numbers with the + superscripts refer to the cycle of the 35-day phase at which the upper lip of the x ray radiation pattern is pointing directly at the companion at the particular binary as labelled on the outermost regions. When this occurs at favorable orbital phases, optical pulsation features I^+ and II^+ are generated. The x ray "on" is also marked because it is also a consequence of the upper lip (as defined) of the x ray radiation beam. The inside circle is used to diagram the phenomena associated with the lower lip of the x ray beam, namely Features I^- and II^- .

Fig. 113 Similar to Figure 112 except that the constraints on θ_0 and $\delta\theta/2$ are plotted for the Slaved Orienting Disk model, such as that of Roberts.³⁵ The axes are also plotted in units of $|90-i|$.

Fig. 114 A schematic of the HZ Her system as seen by looking down at the orbital plane with the stream and disk of Lupow and Shu⁷¹ drawn to scale. The x ray and optical paths which create the various observed Features of the HZ Her optical pulsation are shown. The system is assumed to be rotating in a right-handed fashion.

Fig. 1

XBL 7511-8927

Fig. 2

XBL 7511-8925

XBL 7511-9055

Fig. 3

Fig. 4

XBL 7511-8926

Fig. 5

XBL 7511-8939

Fig. 6

NRL 752-151

Fig. 7

Fig. 8

g. 9

Fig. 10

Fig. 11

Fig. 13

Fig. 14

Fig. 16

Fig. 18

Fig. 15

Fig. 17

Fig. 22

Fig. 21

Fig. 23

Fig. 24

Fig. 25

XBL 7511-8897

Fig. 31

Fig. 32

Fig. 33

Fig. 34

Fig. 35

Fig. 36

Fig. 37

Fig. 38

Fig. 39

Fig. 40

Fig. 41

Fig. 42

Fig. 43

Fig. 44

Fig. 45

Fig. 46

Fig. 51

Fig. 52

Fig. 53

Fig. 56. PULSED REACTION OF BINDER PASTE FOR EACH OF 21 CYCLES

Fig. 57. PULSED REACTION OF BINDER PASTE FOR EACH OF 21 CYCLES

Fig. 54. PULSED REACTION OF BINDER PASTE FOR EACH OF 21 CYCLES

Fig. 55. PULSED REACTION OF BINDER PASTE FOR EACH OF 21 CYCLES

Fig. 58 PULSED FRACTION (x1000) vs. PHASE ANGLE (deg) for star 10-71-15-211. Mean On 1.7-181.

Fig. 59 PULSED FRACTION (x1000) vs. PHASE ANGLE (deg) for star 10-71-15-211. Mean On 1.7-181.

Fig. 60 PULSED FRACTION (x1000) vs. PHASE ANGLE (deg) for star 10-71-15-211. Mean On 1.7-181.

Fig. 61 PULSED FRACTION (x1000) vs. PHASE ANGLE (deg) for star 10-71-15-211. Mean On 1.7-181.

Fig. 62

XBI 7511 A907

Fig. 65

Fig. 64

Fig. 63

Fig. 68

Fig. 67

Fig. 66

Fig. 69

Fig. 70

Fig. 71

Fig. 72

Fig. 73

Fig. 74

Fig. 76

Fig. 77

$R=1.65$

$R=1.65$

Fig. 78

Fig. 79

Fig. 80

Fig. 81

Fig. 82

Fig. 83

XBL 7511 8910

Fig. 84

ARE COVERED

Fig. 85

XBL 7511-8911

Fig. 86

Fig. 87

Fig. 88 1.7-DAY ORBITAL PHASE (0=ECLIPSE)

Fig. 89 1.7-DAY ORBITAL PHASE (0=ECLIPSE)

Fig. 90

XBL 7511-8958

R=2.0

• Full

• 0.1RES

• NOSE

CLIP

$\theta=90^\circ$

- 1.65
- 1.65 OZRES
- 1.70 OZRES

$\theta=90^\circ$

- 1.65
- 1.65 OZRES
- 1.65 OZRES

Fig. 94. 1.7-DAY DRIFTING PHASE (OZELL-1)

Fig. 95

Fig. 96

Fig. 97 1.7-DAY ORBITAL PHASE 10=ECLIPSE

Fig. 98

Fig. 99

XRI. 7511-8918

Fig. 100

(a)

(b)

(c)

Fig. 101

XBL 7511-9054

Fig. 102

XBL 7511-8919

XBL 7511-8956

Fig. 103

Fig. 1C4

XBI. 7511-8957

Fig. 109

PERIOD=1549039 d

Fig. 110

PERIOD=1883967 d

FIG. 111

XBL 7511-8955

Fig. 112

XBL 7511-8954

Fig. 113

XBL 7511-8953

Fig. 114

XBL 7511-8941

CAPTIONS FOR FIGURES IN THE APPENDIX

Figures	Caption
1A	The r th element of the Fourier transform of constant data. The vectors execute r left-handed (clockwise) " N/r -gons" in the lower half of the complex plane before returning exactly to the origin.
2A	A few steps in a Fourier random walk for $N/r = 8$. Such walks are the result of fluctuations in the data.
3A	The results of the n th bin of the Fourier transform of a cosine with the same frequency as the n th bin -- the vectors step off in the same direction.
4A	The results of the Fourier transform of a signal which has a higher frequency than the bin frequency by $+1/2T$ Hz. The response is advanced in complex phase by 90° and attenuated in amplitude by $2/\pi$.
5A	The results of the Fourier transform of a signal with a lower frequency than the bin frequency by $-1/2T$ Hz. The response lags by 90° in complex phase and is still attenuated by $2/\pi$ in amplitude.
6A	The plot of the comb function which is used to multiply continuous data so that it may be sampled discretely.
7A	The plot of the response of the absolute amplitude of the Fourier transform to a signal due to the convolving of the comb function with a dt integration period.
8A	The entire process of converting an infinite, continuous Fourier transform of a signal into a finite, discrete transform.

The top frame shows the signal, S and its Fourier transform -- the delta function. The next frame shows the comb function, C , and the resulting convolution of the delta function of the signal which gives rise to aliasing. The third frame shows the integration time window, B , which gets convolved with the signal times the comb function. The Fourier transform of B multiplies the second right frame to diminish the response for the frequencies outside of the sampling range. The last frame shows the run time function, D , which multiplies the convolution of B with the product of C and S . The Fourier transform of D is convolved with the product of the transform of B with the convolution of the transforms of C with S . The lowermost right frame shows the profile of the signal near f_0 caused by the convolution with the transform of D .

9A A signal vector, $\bar{s}$, adds to a noise vector, $\bar{n}$, to create a resultant vector, $\bar{r}$, in the complex plane for a particular Fourier transform bin.

10A The two-dimensional gaussian distribution of noise vectors tacked onto the end of the signal vector to illustrate the probability of obtaining any resultant vector.

11A The noise vector length squared is related to the r and s vector lengths by the law of cosines.

12A The three paths associated with P_+ , P_0 and P_- on the complex plane in the case where $z = 0.2$ and the signal frequency, f^0 , is equal to the folding frequency, f_0 , used in generating the P_0

13A The three paths associated with P_+ , P_0 and P_- in the case where the signal frequency, f^0 , exceeds f_0 by about $0.1/T$.

- 14A The data of the HZ Her runs with strong signals plotted on the A-B plane. The data were reduced with $z = 0.2$.
- 15A The frequency behavior vs time used to generate the path associated with P_+ from signal folding.
- 16A The paths on the complex plane associated with P_+ , P_0 and P_- and the measurement of $\dot{f}$ when the true $\dot{f}$ is equal to zero.
- 17A The response of the quantity, $(A-B)/z$ to the quantity, $T^2 \dot{f}$ for the three cases, $z = 1$, $z = 2$ and $z = 1$ with f_0 off center by $0.2/T$.
- 18A The response of the quantity, $(2-(A+B))/z^2$ to $T^2 \dot{f}$ for the same three cases.
- 19A Like Figure 14A, Figure 19A gives the data of the HZ Her runs with strong signals plotted on the A-B plane. The data were reduced with $z = 1$.

FIGURE 1A
rth element of Transform of constant

FIGURE 2A
a random walk for $N/r=8$

FIGURE 3A

FIGURE 4A

FIGURE 5A

FIGURE 8A

FIGURE 9A

FIGURE 10A

$$n^2 = r^2 + s^2 - 2rs \cos \theta$$

FIGURE 11A

FIGURE 17A

XBL 7511-8951

XBL 7511-8950

XBL 7511-8949