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Intrabeam scattering formulas for fast numerical evaluation

Sergei Nagaitsev
FNAL, P.O. Box 500, Batavia, IL 60510, USA

Abstract
Expressions for small-angle multiple intrabeam scattering (IBS) emittance growth rates are normally expressed through integrals, which require a numeric evaluation at various locations of the accelerator lattice. In this paper, I demonstrate that the IBS growth rates can be presented in closed-form expressions with the help of the so-called *symmetric elliptic integral.* This integral can be evaluated numerically by a very efficient recursive method by employing the duplication theorem. Several examples of IBS rates for a smooth-lattice approximation, equal transverse temperatures and plasma temperature relaxation are given.

1. Introduction

This paper presents the results, previously obtained by Bjorken and Mtingwa [1], as closed-form analytic expressions. In fact, all of the rates, presented here, are strictly identical to the ones in Ref. [1]. Also, the notations are essentially the same as in Ref. [1]. Suppose that the bunched-beam distribution function, *f*, is described by the following expression:

$$f = \exp(-S), \tag{1}$$

where

$$S = \frac{1}{2} A_{\alpha\beta} \frac{p_\alpha}{p} \frac{p_\beta}{p} + \frac{1}{2} B_{\alpha\beta} x_\alpha \frac{p_\beta}{p} + \frac{1}{2} C_{\alpha\beta} x_\alpha x_\beta , \tag{2}$$

x_α (=*x, y* or *s*) is the particle coordinate, p_α is its momentum and $p=\beta\gamma Mc$. It is normalized to a total 6-dimensional rms invariant phase-space volume, Γ, as

$$\Gamma = \int f \, \mathrm{d}^3x \mathrm{d}^3 p \, . \tag{3}$$

There have been attempts in the past [1, 2] to express the IBS rates through Legendre's incomplete elliptic integrals. Let me now introduce a *symmetric* elliptic integral of the second kind, $R_D(x,y,z)$, following Carlson's definition [3]:

$$R_D(x,y,z) = \frac{3}{2} \int_0^\infty \frac{\mathrm{d}t}{\sqrt{(t+x)(t+y)(t+z)^3}} \, . \tag{4}$$

The following are some useful properties of this integral:

$$R_D(x,x,x) = x^{-3/2} , \tag{5}$$

$$R_D(x,y,z) + R_D(y,z,x) + R_D(z,x,y) = \frac{3}{\sqrt{xyz}} , \text{ and} \tag{6}$$

$$R_D(hx,hy,hz) = h^{-3/2} R_D(x,y,z) \text{ (for } h > 0). \tag{7}$$

There exists the so-called *duplication* theorem [4] for this integral, which allows for a very efficient numerical evaluation using a recursive method [5]. Only rational operations and square roots are required. Such a numerical method gives, in my opinion, the main advantage for expressing the IBS rates through this integral.

The IBS formulas, I am proposing in this paper, require evaluating the symmetric elliptic integral, with its variables cycled, three times at each point of the accelerator lattice. Actually, Eq. (6) allows to reduce the number of integrals to two.

Some of the IBS rates for special cases are expressed by the following combination of elliptic integrals:

$$\Psi(x,y,z) = -2xR_D(y,z,x) + yR_D(z,x,y) + zR_D(x,y,z)\,. \tag{8}$$

For example, the Eq. (3.6) in B-M [1] paper can be written as follows:

$$\frac{1}{\tau} \equiv \frac{1}{\Gamma}\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \\ = \frac{2\pi^2 N\alpha^2 M L_C}{3\gamma\Gamma}\int_0^L \frac{\mathrm{d}s}{L\sqrt{\lambda_1\lambda_2\lambda_3}}\left[\lambda_1\Psi\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2},\frac{1}{\lambda_3}\right) + \text{two cyclic permutations}\right] \tag{9}$$

where λ_1, λ_2, and λ_3 are the eigenvalues of the matrix $\boldsymbol{A}$ in Eq. (2), and L is the ring circumference.

2. IBS rates for uncoupled betatron oscillations

Uncoupled betatron oscillations can be described by the following expressions:

$$x = \sqrt{2J_x\beta_x}\cos(\varphi_x) + D_x\delta, \tag{10}$$

$$\frac{p_x}{p} = x' = D_x'\delta + \frac{\beta_x'}{2\beta_x}(x - D_x\delta) - \sqrt{\frac{2J_x}{\beta_x}}\sin(\varphi_x), \tag{11}$$

$$y = \sqrt{2J_y\beta_y}\cos(\varphi_y), \quad \frac{p_y}{p} = y' = \frac{\beta_y'}{2\beta_y}y - \sqrt{\frac{2J_y}{\beta_y}}\sin(\varphi_y), \tag{12}$$

where

$$J_x = \frac{1}{2\beta_x}(x - D_x\delta)^2 + \frac{\beta_x}{2}\left(x' - \frac{\beta_x'}{2\beta_x}x - \Phi\delta\right)^2 \text{ and} \tag{13}$$

$$J_y = \frac{y^2}{2\beta_y} + \frac{\beta_y}{2}\left(y' - \frac{\beta_y'}{2\beta_y}y\right)^2 \tag{14}$$

are the action variables, φ_x, φ_y are the betatron phase variables of unperturbed oscillations and δ is the relative laboratory frame momentum spread. The function Φ is expressed as follows:

$$\Phi = D_x' - \frac{\beta_x' D_x}{2\beta_x}\,. \tag{15}$$

Similarly, the synchrotron action variable for a parabolic potential well is described by

$$J_s = \frac{s^2}{2k_s} + \frac{k_s}{2}\delta^2\,. \tag{16}$$

Suppose now that the distribution function, f, in Eq. (1) can be written as

$$f = \exp\left[-\frac{J_x}{\varepsilon_x} - \frac{J_y}{\varepsilon_y} - \frac{J_s}{\varepsilon_s}\right], \tag{17}$$

where ε_x, ε_y are the rms non-normalized transverse emittances, and $\varepsilon_s = \sigma_p\sigma_s$ with σ_s and σ_p being the rms bunch length and the rms relative lab-frame momentum spread.

Before presenting the IBS rate formulas, I will first define several parameters. Let

$$a_x = \frac{\beta_x}{\varepsilon_x} \equiv \frac{1}{\theta_\mathrm{x}^{\ 2}}, \quad a_y = \frac{\beta_y}{\varepsilon_y} \equiv \frac{1}{\theta_\mathrm{y}^{\ 2}}, \tag{18}$$

$$\sigma_x = \sqrt{D_x^{\ 2}\sigma_p^{\ 2} + \varepsilon_x\beta_x}, \ \sigma_y = \sqrt{\varepsilon_y\beta_y}, \tag{19}$$

$$a_s = a_x\left(\frac{D_x^{\ 2}}{\beta_x^{\ 2}} + \Phi^2\right) + \frac{1}{\sigma_p^{\ 2}}, \text{ and} \tag{20}$$

$$a_1 = \frac{1}{2}\left(a_x + \gamma^2 a_s\right), \ a_2 = \frac{1}{2}\left(a_x - \gamma^2 a_s\right). \tag{21}$$

The three eigenvalues of the matrix $\boldsymbol{A}$ (Eq. 2) can be now expressed as follows

$$\lambda_1 = a_y, \tag{22}$$

$$\lambda_2 = a_1 + \sqrt{a_2^{\ 2} + \gamma^2 a_x^{\ 2}\Phi^2}, \tag{23}$$

$$\lambda_3 = a_1 - \sqrt{a_2^{\ 2} + \gamma^2 a_x^{\ 2}\Phi^2}. \tag{24}$$

The three integrals are calculated at each location of the ring lattice as follows:

$$R_1 = \frac{1}{\lambda_1} R_D\left(\frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \frac{1}{\lambda_1}\right), \tag{25}$$

$$R_2 = \frac{1}{\lambda_2} R_D\left(\frac{1}{\lambda_3}, \frac{1}{\lambda_1}, \frac{1}{\lambda_2}\right), \tag{26}$$

$$R_3 = \frac{1}{\lambda_3} R_D\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}\right). \tag{27}$$

Here, Eq. (7) can be used to avoid extreme values in the arguments of $R_D(x,y,z)$ and Eq. (6) to reduce the number of integrals to two.

The total 6-dimensional emittance growth rate in Eq. (9) can be written as

$$\begin{aligned} \frac{1}{\tau} &\equiv \frac{1}{\Gamma}\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \frac{1}{\varepsilon_x}\frac{\mathrm{d}\varepsilon_x}{\mathrm{d}t} + \frac{1}{\varepsilon_y}\frac{\mathrm{d}\varepsilon_y}{\mathrm{d}t} + \frac{1}{\sigma_p^{\ 2}}\frac{\mathrm{d}\sigma_p^{\ 2}}{\mathrm{d}t} = \\ &= \frac{N r_p^{\ 2} c L_C}{12\pi\beta^3\gamma^5\sigma_s}\int_0^L \frac{\mathrm{d}s}{L\sigma_x\sigma_y}\left[\lambda_1\Psi\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}\right) + \text{two cyclic permutations}\right], \end{aligned} \tag{28}$$

where

$$\Gamma = 8\pi^3\beta^3\gamma^3 M^3 c^3 \varepsilon_x\varepsilon_y\sigma_s\sigma_p. \tag{29}$$

The partial emittance growth rates can be now written as follows

$$\frac{\mathrm{d}\sigma_p^{\ 2}}{\mathrm{d}t} = \frac{N r_p^{\ 2} c L_C}{12\pi\beta^3\gamma^5\sigma_s}\int_0^L \frac{\mathrm{d}s}{L\sigma_x\sigma_y} S_p, \tag{30}$$

$$\frac{\mathrm{d}\varepsilon_y}{\mathrm{d}t}=\frac{Nr_p^{\ 2}cL_C}{12\pi\beta^3\gamma^5\sigma_s}\int_0^L\frac{\beta_y\mathrm{d}s}{L\sigma_x\sigma_y}\Psi\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2},\frac{1}{\lambda_3}\right), \tag{31}$$

$$\frac{\mathrm{d}\varepsilon_x}{\mathrm{d}t}=\frac{Nr_p^{\ 2}cL_C}{12\pi\beta^3\gamma^5\sigma_s}\int_0^L\frac{\beta_x\mathrm{d}s}{L\sigma_x\sigma_y}\left[S_x+\left(\frac{D_x^{\ 2}}{\beta_x^{\ 2}}+\Phi^2\right)S_p+\Phi S_{xp}\right], \tag{32}$$

where functions S_p, S_x, and S_{xp} are expressed through the functions defined in Eqs. (25-27) as

$$S_p=\frac{\gamma^2}{2}\left[2R_1-R_2\left(1-\frac{3a_2}{\sqrt{a_2^{\ 2}+\gamma^2a_x^{\ 2}\Phi^2}}\right)-R_3\left(1+\frac{3a_2}{\sqrt{a_2^{\ 2}+\gamma^2a_x^{\ 2}\Phi^2}}\right)\right], \tag{33}$$

$$S_x=\frac{1}{2}\left[2R_1-R_2\left(1+\frac{3a_2}{\sqrt{a_2^{\ 2}+\gamma^2a_x^{\ 2}\Phi^2}}\right)-R_3\left(1-\frac{3a_2}{\sqrt{a_2^{\ 2}+\gamma^2a_x^{\ 2}\Phi^2}}\right)\right], \tag{34}$$

$$S_{xp}=\frac{3\gamma^2a_x\Phi}{\sqrt{a_2^{\ 2}+\gamma^2a_x^{\ 2}\Phi^2}}\left[R_3-R_2\right]. \tag{35}$$

Thus, by computing the three integrals R_1, R_2, and R_3 one fully defines the IBS rates at a given lattice location.

3. Smooth approximation

In a smooth-lattice approximation case I will assume $\Phi = 0$, thus

$$\lambda_1=a_y, \tag{36}$$

$$\lambda_2=a_x, \tag{37}$$

$$\lambda_3=\gamma^2a_s. \tag{38}$$

The functions S_p, S_x, and S_{xp} can be written as

$$S_p=\Psi\left(\frac{1}{\lambda_3},\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right), \tag{39}$$

$$S_x=\Psi\left(\frac{1}{\lambda_2},\frac{1}{\lambda_3},\frac{1}{\lambda_1}\right), \tag{40}$$

$$S_{xp}=0. \tag{41}$$

These formulas can be simplified even further if one neglects the dispersion function, D_x. With this assumption, one obtains the following partial growth rates:

$$\frac{\mathrm{d}\sigma_p^{\ 2}}{\mathrm{d}t}=\frac{Nr_p^{\ 2}cL_C}{12\pi\beta^3\gamma^3\sigma_s}\int_0^L\frac{\mathrm{d}s}{L\sigma_x\sigma_y}\Psi\left(\frac{\sigma_p^{\ 2}}{\gamma^2},\theta_y^{\ 2},\theta_x^{\ 2}\right), \tag{42}$$

$$\frac{\mathrm{d}\varepsilon_y}{\mathrm{d}t}=\frac{Nr_p^{\ 2}cL_C}{12\pi\beta^3\gamma^5\sigma_s}\int_0^L\frac{\beta_y\mathrm{d}s}{L\sigma_x\sigma_y}\Psi\left(\theta_y^{\ 2},\theta_x^{\ 2},\frac{\sigma_p^{\ 2}}{\gamma^2}\right), \tag{43}$$

$$\frac{\mathrm{d}\varepsilon_x}{\mathrm{d}t} = \frac{N r_p^{\,2} c L_C}{12\pi\beta^3\gamma^5\sigma_s}\int_0^L \frac{\beta_x \mathrm{d}s}{L\sigma_x\sigma_y}\Psi\left(\theta_x^{\,2}, \frac{\sigma_p^{\,2}}{\gamma^2}, \theta_x^{\,2}\right). \tag{44}$$

4. Round beam approximation

Further useful approximation can be obtained by assuming equal transverse beam sizes ($\sigma_x = \sigma_y = \sigma$) and temperatures ($\theta_x^2 = \theta_y^2 = \theta^2$). One can then introduce a single variable, z, such that

$$z = \frac{\sigma_p^{\,2}}{\gamma^2\theta^2} \tag{45}$$

is the ratio of the longitudinal to transverse temperatures in the beam rest frame. The partial IBS growth rates are expressed as follows:

$$\frac{\mathrm{d}\sigma_p^{\,2}}{\mathrm{d}t} = \frac{N r_p^{\,2} c L_C}{8\beta^3\gamma^3\sigma_s\sigma^2\theta} F(z), \tag{46}$$

$$\frac{\mathrm{d}\varepsilon_{x,y}}{\mathrm{d}t} = -\frac{N r_p^{\,2} c L_C}{16\beta^3\gamma^5\sigma_s\varepsilon_{x,y}\theta} F(z), \tag{47}$$

where

$$F(z) = \frac{2}{3\pi}\Psi(z,1,1). \tag{48}$$

The function $F(z)$ can be expressed analytically through hyperbolic functions, instead I propose an approximate function, $G(z)$, which deviates from function $F(z)$ by no more than 20% in the range of z values from 0 to 10.

$$\frac{2}{3\pi}\Psi(z,1,1) \approx G(z) = \left(1 - z^{\frac{1}{4}}\right)\frac{\ln(z+1)}{z}. \tag{49}$$

Both of these functions are shown in Figure 1.

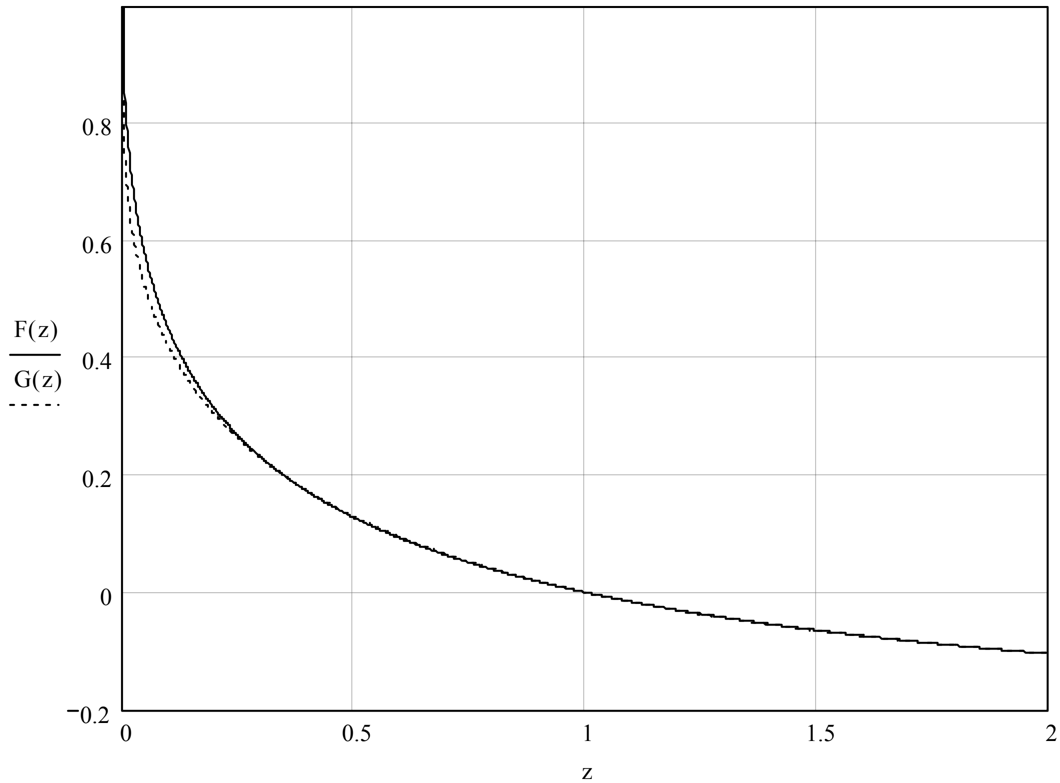

Figure 1: The exact and the approximate IBS functions for a round beam.

The round-beam approximation, presented here, is identical to that, obtained from the theory of Ichimaru and Rosenbluth [6] for a nonrelativistic plasma with initially unequal longitudinal and transverse temperatures. This theory has been recently confirmed by experiments [7] with a nonneutral electron plasma.

In a general case of unequal temperatures, if T_x, T_y, and T_z are the plasma electron temperatures, the plasma temperature relaxation rates can be written as

$$\frac{\mathrm{d}T_x}{\mathrm{d}t} = \frac{4\sqrt{\pi}L_C}{3} n r_e c \sqrt{\left(mc^2\right)^3}\,\Psi\left(T_x, T_y, T_z\right), \tag{50}$$

where ions have been treated as a stationary background and the external fields have been neglected. Rates for T_y and T_z are obtained by cycling the variables in Eq. (50).

5. Coasting beam IBS rates

The total 6-dimensional emittance growth rate for a coasting beam can be written as

$$\begin{aligned} \frac{1}{\tau} &\equiv \frac{1}{\Gamma}\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \frac{1}{\varepsilon_x}\frac{\mathrm{d}\varepsilon_x}{\mathrm{d}t} + \frac{1}{\varepsilon_y}\frac{\mathrm{d}\varepsilon_y}{\mathrm{d}t} + \frac{1}{2\sigma_p{}^2}\frac{\mathrm{d}\sigma_p{}^2}{\mathrm{d}t} = \\ &= \frac{N r_p{}^2 c L_C}{6\sqrt{\pi}\beta^3\gamma^5 L}\int_0^L \frac{\mathrm{d}s}{L\sigma_x\sigma_y}\left[\lambda_1 \Psi\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2},\frac{1}{\lambda_3}\right) + \text{two cyclic permutations}\right]. \end{aligned} \tag{51}$$

The partial emittance growth rates can be now written as follows

$$\frac{\mathrm{d}\sigma_p{}^2}{\mathrm{d}t} = \frac{N r_p{}^2 c L_C}{3\sqrt{\pi}\beta^3\gamma^5\sigma_s}\int_0^L \frac{\mathrm{d}s}{L\sigma_x\sigma_y} S_p \,, \tag{52}$$

$$\frac{\mathrm{d}\varepsilon_y}{\mathrm{d}t} = \frac{N r_p{}^2 c L_C}{6\sqrt{\pi}\beta^3\gamma^5\sigma_s}\int_0^L \frac{\beta_y \mathrm{d}s}{L\sigma_x\sigma_y} \Psi\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2},\frac{1}{\lambda_3}\right), \tag{53}$$

$$\frac{\mathrm{d}\varepsilon_x}{\mathrm{d}t} = \frac{N r_p{}^2 c L_C}{6\sqrt{\pi}\beta^3\gamma^5\sigma_s}\int_0^L \frac{\beta_x \mathrm{d}s}{L\sigma_x\sigma_y}\left[S_x + \left(\frac{D_x{}^2}{\beta_x{}^2} + \Phi^2\right) S_p + \Phi S_{xp}\right], \tag{54}$$

where functions S_p, S_x, and S_{xp} are defined in Eqs. (33- 35).

6. Conclusions

Starting from results reported in Ref. [1], I have expressed the IBS rates in a convenient form containing symmetric elliptic integrals. These integrals can be evaluated numerically by an efficient recursive procedure, requiring only rational and square-root operations. This allows to make accurate IBS rate calculations with commonly used software packages, such as MathCAD.

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