Conf-9208142--16

LBL-32286
UC-414

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

Accelerator & Fusion Research Division

Presented at the Fourteenth International Free Electron Laser Conference, Kobe, Japan, August 23–28, 1992, and to be published in the Proceedings

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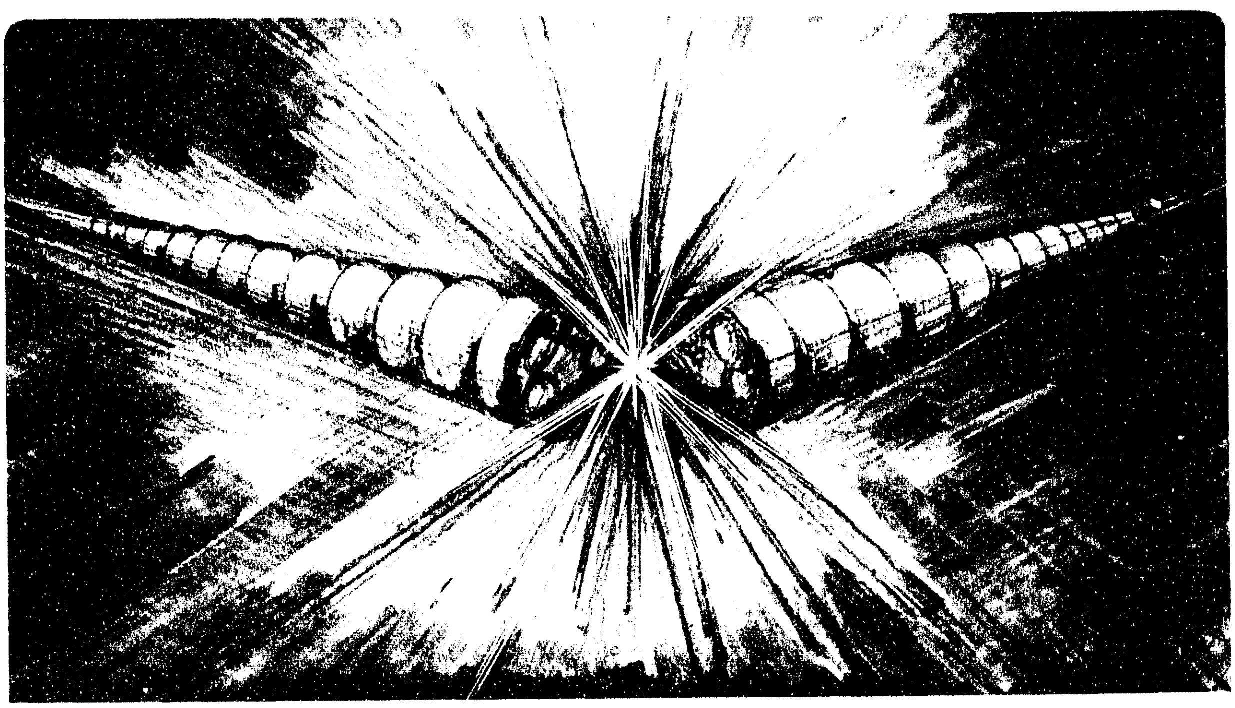
August 1992



Prepared for the U.S. Department of Energy under Contract Number DE-AC03-76SF00098

LBL--32286

DE93 004768

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Calculation of 3-D Free Electron Laser Gain:Comparison with Simulation and Generalization to Elliptical Cross Section*

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Abstract

In the previous paper [1], we have derived a dispersion relation for the free electron laser (FEL) gain in the exponential regime taking account the diffraction and electron's betatron oscillation. Here, we compare the growth rates obtained by solving the dispersion relation with those obtained by simulation calculation for the waterbag and the Gaussian models for the electron's transverse phase space distribution. The agreement is found to be good except for the limiting case where the Rayleigh length is much longer than the gain length (1-D limit). We also generalize the analysis to the case where the electron beam cross section is elliptical as is usually the case in storage rings, and derive the first-order dispersion relation.

I. Introduction

In the previous paper [1], we have presented a three-dimensional (3-D) FEL theory based upon the Maxwell-Vlasov equations including effects of the energy spread, emittance, and betatron oscillations of the electron beam. In this theory, the orthogonal expansion of the electron distribution function converts the combined Maxwell-Vlasov equations into a matrix equation, from which a dispersion relation for the FEL gains is derived. The series expansion converges very quickly, unless the Rayleigh range is much longer that the gain length of the one-dimensional (1-D) theory (in which case the 3-D effects are unimportant). An accurate FEL gain for the fundamental mode can be obtained

* This work was supported by the Director, Office of Energy Research, Office of Basic Energy Sciences, Materials Sciences Division, the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

by taking only the lowest-order expansion term except for the 1-D limit. In this approximation, the matrix the dispersion relation is reduced to a single scalar equation. The gain for the fundamental mode can be obtained for any initial electron transverse phase space distribution including the hollow beam, the waterbag, and the Gaussian models. We have compared the growth rates obtained by solving the scalar dispersion relation with those obtained by Yu, Krinsky, and Gluckstern's variational approach [2] for the waterbag model of the electron transverse phase space distribution. Good agreement was found.

In this paper, we compare the growth rates obtained by solving the dispersion relation with those obtained by simulations using the code TDA [3] for the waterbag and the Gaussian models for the electron transverse phase space distribution. This will provide a further verification of the present FEL theory. We also generalize the analysis to the case where the electron beam cross section is elliptical and the betatron focusing is asymmetrical in the x- and y-directions. Such a case is interesting for application to storage ring FEL systems. Further details can be found in ref. [4].

II. Comparison with Simulations

The growth rate of the fundamental guided mode can be expressed in a scaled form using four dimensionless scaling parameters. One form of such a scaling relation convenient when the total beam current is constant is

$$\frac{Re(q)}{k_w D} = F(2k_1\varepsilon_x, \frac{\sigma_\gamma}{D}, \frac{k_\beta}{k_w D}, \frac{k-k_1}{k_1 D}), \tag{1}$$

where $Re(q)$ is the growth rate in the exponential growth regime. The growth rate $Re(q)$ is related to the power gain length L_G as $Re(q)L_G = 1/2$. The dispersion relation for the Gaussian beam model, for instance, can be written in the above scaling form as

$$\begin{aligned} 1 &= \frac{i}{4\sqrt{2\pi}} \frac{\frac{k_\beta}{k_w D}}{2k_1\varepsilon_x} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{e^{-\frac{x^2}{2}} x^3 e^{-\frac{t^2}{2}} \, dx dt}{(\frac{q}{k_w D} + 2i\frac{\sigma_\gamma}{D}t - \frac{i}{4} 2k_1\varepsilon_x \frac{k_\beta}{k_w D} x^2)^2} \\ &\times \int_0^{\infty} \frac{e^{-x^2} x dx}{\frac{q}{k_w D} + i\frac{k-k_1}{k_1 D} + i\frac{x^2}{2k_1\varepsilon_x}\frac{k_\beta}{k_w D}}, \end{aligned} \tag{2}$$

where we apply a rule that the integral signs in the multiple integrals are paired with the differential signs from inside to outside, unless otherwise specified. Here, k_w is the wiggler wave number, $k_1 = 2k_w\gamma_r^2/(1+K^2)$ is the resonant radiation wave number corresponding to the resonant energy γ_r of the reference electron in units of its rest mass mc^2, c is the speed of light, K is the rms wiggler parameter, ε_x is the rms transverse emittance of the electron beam, σ_γ is the rms relative energy spread, k_β is the betatron wave number, and $k = \omega/c$ is the wave number of the radiation field. The quantity D is the scaling parameter defined by

$$D = \sqrt{\frac{8}{\gamma_r}\frac{K^2}{1+K^2}\frac{I_0}{I_A}}\ [\mathrm{JJ}], \tag{3}$$

where I_0 is the total beam current, $I_A = ec/r_e \approx 17.05\mathrm{kA}$ is the Alfvén current, e is the electron charge, and r_e is the electron classical radius. For a planar wiggler, the Bessel factor [JJ] is given by

$$[\mathrm{JJ}] = \mathrm{J}_0\left(\frac{k}{k_1}\frac{K^2}{2(1+K^2)}\right) - \mathrm{J}_1\left(\frac{k}{k_1}\frac{K^2}{2(1+K^2)}\right), \tag{4}$$

where $\mathrm{J}_m(x)$ is the Bessel function. For a helical wiggler, [JJ]=1. The parameter D was originally introduced by Yu, Krinsky, and Gluckstern [2]. However, the value of D defined here is smaller than that defined by Yu, Krinsky, and Gluckstern by a factor of $\sqrt{2}$.[†] The scaling parameter D is related to the Pierce parameter [5] ρ as

$$\frac{D}{\rho} = \frac{2\sqrt{2}}{3^{1/4}}\left(\frac{L_R}{L_G^{(1-D)}}\right)^{1/2} \approx 2.15\left(\frac{L_R}{L_G^{(1-D)}}\right)^{1/2}. \tag{5}$$

Here, $L_G^{(1-D)}$ is the power gain length of the one-dimensional theory given by

$$L_G^{(1-D)} = \frac{1}{2\sqrt{3}\rho k_w}. \tag{6}$$

Also, L_R is the Rayleigh range given by

$$L_R = \frac{2\Sigma_\perp}{\lambda_1} = \frac{k_1\Sigma_\perp}{\pi}, \tag{7}$$

where λ_1 is the radiation wave length, and $\Sigma_\perp$ is the transverse beam area (defined by $I_0/\Sigma_\perp$ = peak current density on axis).

We have compared the growth rates obtained by solving the dispersion relation with those obtained by simulations using the code TDA [3]. The nominal FEL parameters

[†]The difference of the definitions of D by a constant factor dose not affect any physical results such as the gain length. We have used the definition of D given by Eq. (3) in our previous paper [1], and have done the computations for the present paper based on this definition. Therefore, to avoid confusion, we also use this definition in the present paper. However, the best way of defining D might be such that the ratio of D to ρ becomes $D/\rho = (L_R/L_G^{(1-D)})^{1/2}$ instead of that in Eq. (5). In this way, D becomes a natural generalization of the quantity ρ introduced in the one-dimensional theory.

used in the simulation are as follows: the radiation wavelength $\lambda_1 = 7.5\mu m$, the average electron energy $\gamma_r = 100$, $K = 2$, $\lambda_w = 3cm$, the electron beam current $I_0 = 53.28A$, the betatron wavelength $\lambda_\beta = 2\pi/k_\beta = 2.12132m$, and the scaling parameter $D = 0.014142$. Here, we have chosen the FEL parameters such that the scaled betatron wave number $k_\beta/(k_w D) = 1$, a value large enough to show clearly the effects of Landau damping due to the betatron focusing and the emittance. The detuning parameters used for the simulations are identical to those for the analytical results which yield the maximum growth rates. In Fig. 1(a), we plot the scaled growth rate $Re(q)/(k_w D)$ as a function of $2k_1\varepsilon_x$ for the zero energy spread for the Gaussian and the waterbag beam distributions. The agreement is excellent. The benchmark for the non-zero energy spread $\sigma_\gamma/D = 0.2$ is shown in Fig. 1(b) for the Gaussian beam distribution. The agreement is also excellent.

We have examined the accuracy of the approximate dispersion relation (2) obtained by truncating the exact dispersion relation at the lowest-order of the azimuthal and the radial expansions [4]. We found that the truncated dispersion relation provides a good approximate growth rate, unless Rayleigh range is much longer than the gain length of the one-dimensional theory (typically, $\frac{L_R}{L_G^{(1-D)}} \gtrsim 30$), so that 3-D effects such as the diffraction effect become negligible.

III. Generalization to Elliptical Cross Section

The method can be generalized to the case where the electron beam cross section is elliptical and the betatron focusing is asymmetric in the x- and y-directions, that may be more realistic in a storage ring FEL system with a planar wiggler. The procedure closely follows the formulation for the round beam case [4].

We consider the electron beam moving in the z-direction through a planar wiggler with a parabolic pole face with longitudinal wave number k_w. For a small transverse displacement from the wiggler axis, the vector potential of a planar wiggler with a parabolic pole face, $\boldsymbol{A}_w$, can be approximated by

$$\boldsymbol{A}_w = A_w[\boldsymbol{i}_x \frac{k_{wy}}{k_w}(1 + \frac{1}{2}k_{wx}^2 x^2 + \frac{1}{2}k_{wy}^2 y^2)\sin k_w z - \boldsymbol{i}_y \frac{k_{wx}}{k_w} k_{wx} k_{wy} xy \sin k_w z], \qquad (8)$$

where

$$k_{wx}^2 + k_{wy}^2 = k_w^2. \qquad (9)$$

Here, k_{wx} and k_{wy} are the wave numbers of the wiggler field in the x- and y-directions, respectively. We choose z, the distance from the wiggler entrance, as the independent variable. The transverse trajectory of the electron consists of the betatron motion and the wiggler motion. The betatron oscillations are governed by the equations of motion for a simple harmonic oscillator with the betatron wave number $k_{\beta x}$ and $k_{\beta y}$ on x- and y-planes, respectively (in the absence of external focusing, $k_{\beta x} = Kk_{wx}/\gamma$, and $k_{\beta y} = Kk_{wy}/\gamma$). Here, γ is the electron energy, and $K = eB/(\sqrt{2}mc^2k_w)$ is the rms value of wiggler parameter where B is the peak magnetic field on axis. The transverse variables to be used in the Vlasov equation are the betatron oscillation vector $\boldsymbol{x}_\beta$ and its canonical momentum conjugate $\boldsymbol{p}_\beta$. The longitudinal variables are $\tau = t - t/v_r$, the relative position of an electron from the resonant electron in time units (v_r is the longitudinal velocity of the reference electron), and the electron energy γ.

The linearized Vlasov equation for the perturbed part of the distribution function, $f_1(\boldsymbol{x}_\beta, \boldsymbol{p}_\beta, \tau, \gamma; z)$, is written as

$$\frac{\partial f_1}{\partial z} + p_{\beta x}\frac{\partial f_1}{\partial x_\beta} - k_{\beta x}^2 x_\beta \frac{\partial f_1}{\partial p_{\beta x}} + p_{\beta y}\frac{\partial f_1}{\partial y_\beta} - k_{\beta y}^2 y_\beta \frac{\partial f_1}{\partial p_{\beta y}} + \frac{d\tau}{dz}\frac{\partial f_1}{\partial \tau} + \frac{d\gamma}{dz}\frac{\partial f_0}{\partial \gamma} = 0, \qquad (10)$$

where f_0 is the unperturbed electron distribution. In the following, we assume that the focusing in the wiggler is matched to the electron beam so that f_0 is a function of $x_\beta^2+(p_{\beta x}/k_{\beta x})^2$, $y_\beta^2+(p_{\beta y}/k_{\beta y})^2$, and γ only (i.e., f_0 is uniform in the longitudinal direction). Furthermore, the distribution in γ is usually sharply peaked around an average value. It is then a good approximation to assume that f_0 can be factorized as follows:

$$f_0 = f_{0\perp}(x_\beta^2 + (p_{\beta x}/k_{\beta x})^2, y_\beta^2 + (p_{\beta y}/k_{\beta y})^2) \cdot f_{0\parallel}(\gamma), \qquad (11)$$

where f_0 is normalized so that its integral over six dimensional phase space is equal to the total number of electrons, N. The equation of motion of τ is given by

$$\frac{d\tau}{dz} = \frac{1}{c}[-2\frac{k_w}{k_1}\frac{\gamma - \gamma_r}{\gamma_r} + \frac{1}{2}(p_{\beta x}^2 + k_{\beta x}^2 x_\beta^2 + p_{\beta y}^2 + k_{\beta y}^2 y_\beta^2)], \qquad (12)$$

where γ_r is the resonant energy of the reference electron with zero transverse oscillation amplitude. The energy change by the radiation field is given by

$$\frac{d\gamma}{dz} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\frac{1}{2\pi i}\int_{q_0-i\infty}^{q_0+i\infty}[\int_{-\infty}^{\infty} P_{\omega q}(k_x, k_y)\rho_{\omega q}(\boldsymbol{k}_\perp)e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\beta}d^2\boldsymbol{k}_\perp]e^{qz}dq\}e^{-i\omega\tau}d\omega, \qquad (13)$$

where $\rho_{\omega q}(\boldsymbol{k}_\perp)$ is the Laplace-Fourier transform of the charge density, which is related to $f_1(\boldsymbol{x}_\beta, \boldsymbol{p}_\beta, \tau, \gamma; z)$ by

$$\rho_{\omega q}(\boldsymbol{k}_\perp) = \int_{-\infty}^{\infty}\{\int_0^{\infty}[\int_{-\infty}^{\infty}(\int_1^{\infty}\int_{-\infty}^{\infty} f_1(\boldsymbol{x}_\beta, \boldsymbol{p}_\beta, \tau, \gamma; z)d^2\boldsymbol{p}_\beta d\gamma)e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\beta}d^2\boldsymbol{x}_\beta]e^{-qz}dz\}e^{i\omega\tau}d\tau. \qquad (14)$$

In the limit of small amplitude of the wiggler motion, $P_{\omega q}$ is approximately given by

$$P_{\omega q}(k_x, k_y) = -\sum_{p=-\infty}^{\infty}\frac{r_e}{2\pi c}(\frac{K}{\gamma_r})^2\frac{[\mathrm{JJ}]_p^2/2}{\sqrt{1-(|\boldsymbol{k}_\perp|/k)^2}[q - i(pk_w + \sqrt{k^2 - |\boldsymbol{k}_\perp|^2} - \frac{w}{v_r})]}, \qquad (15)$$

where

$$[\mathrm{JJ}]_p = \mathrm{J}_{\frac{p-1}{2}}(\frac{k}{k_1}\frac{K^2}{2(1+K^2)}) - \mathrm{J}_{\frac{p+1}{2}}(\frac{k}{k_1}\frac{K^2}{2(1+K^2)}). \tag{16}$$

Higher-order harmonics terms ($|p| > 1$) arise because of the longitudinal velocity modulation due to the planar wiggler field. The function $P_{\omega q}$ can be further approximated by retaining only the fundamental harmonic term of the forward radiation, $p = 1$. In that case, we simply denote $[\mathrm{JJ}]_1$ as $[\mathrm{JJ}]$ defined by Eq. (4) in what follows.

If we substitute Eqs. (12) and (13) into Eq. (10) and take its Fourier-Laplace transform, the linearized Vlasov equation becomes

$$\begin{aligned}[q - i\omega\frac{d\tau}{dz}]f_{\omega q} + p_{\beta x}\frac{\partial f_{\omega q}}{\partial x_\beta} &- k_{\beta x}^2 x_\beta \frac{\partial f_{\omega q}}{\partial p_{\beta x}} + p_{\beta y}\frac{\partial f_{\omega q}}{\partial y_\beta} - k_{\beta y}^2 y_\beta \frac{\partial f_{\omega q}}{\partial p_{\beta y}} = \\ &-f_{0\perp}\frac{df_{0\|}}{d\gamma}\int P_{\omega q}(k_x, k_y)\rho_{\omega q}(\boldsymbol{k}_\perp)e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\beta}d^2\boldsymbol{k}_\perp. \end{aligned} \tag{17}$$

Let us introduce the transverse polar-coordinates as

$$x_\beta = r_x\cos\phi_x, \quad y_\beta = r_y\cos\phi_y, \tag{18}$$

$$\frac{p_{\beta x}}{k_{\beta x}} = r_x\sin\phi_x, \quad \frac{p_{\beta y}}{k_{\beta y}} = r_y\sin\phi_y. \tag{19}$$

The matching condition, Eq. (11), can be written in terms of r_x and r_y as

$$f_0 = f_{0\perp}(r_x, r_y)\cdot f_{0\|}(\gamma). \tag{20}$$

Now, due to the periodic boundary condition for $f_{\omega q}$ in the azimuthal angles ϕ_x and ϕ_y, $f_{\omega q}$ can be Fourier decomposed with respect to ϕ_x and ϕ_y into an infinite series of azimuthal modes:

$$f_{\omega q}(\boldsymbol{x}_\beta, \boldsymbol{p}_\beta, \gamma) = \sum_{m,n=-\infty}^{\infty} F_{\omega q}^{(m,n)}(r_x, r_y, \gamma)e^{im\phi_x}e^{in\phi_y}, \tag{21}$$

where m and n are integers. If we insert Eqs. (14) and (21) into Eq. (17), divide the resulting Vlasov equation by $[q - i\omega\frac{d\tau}{dz} - i(mk_{\beta x} + nk_{\beta y})]$, and integrate the both sides of the equation over γ, we obtain an integral equation for

$$R_{\omega q}^{(m,n)}(r_x, r_y) = \int_1^\infty F_{\omega q}^{(m,n)}(r_x, r_y, \gamma)d\gamma. \tag{22}$$

The result is

$$\begin{aligned} R_{\omega q}^{(m,n)}(r_x, r_y) &= -f_{0\perp}(r_x, r_y)\int_1^\infty \frac{\frac{df_{0\|}(\gamma)}{d\gamma}d\gamma}{q - i\omega\frac{d\tau}{dz}(r,\gamma) - i(mk_{\beta x} + nk_{\beta y})} \\ &\times \sum_{m',n'}\int_0^\infty\int_0^\infty K_{\omega q}^{(m,n,m',n')}(r_x, r_y|r'_x, r'_y)R_{\omega q}^{(m',n')}(r'_x, r'_y)r'_x dr'_x r'_y dr'_y, \end{aligned} \tag{23}$$

where the kernel $K_{\omega q}^{(m,n,m',n')}$ is given by

$$K_{\omega q}^{(m,n,m',n')}(r_x, r_y|r'_x, r'_y) = i^{|m|+|n|-(|m'|+|n'|)} 2\pi k_{\beta x} \cdot 2\pi k_{\beta y} \times \int_{-\infty}^{\infty} P_{\omega q}(k_x, k_y) \left[\mathrm{J}_{|m|}(k_x r_x) \mathrm{J}_{|n|}(k_y r_y) \right] \cdot \left[\mathrm{J}_{|m'|}(k_x r'_x) \mathrm{J}_{|n'|}(k_y r'_y) \right] d^2 \boldsymbol{k}_\perp. \quad (24)$$

The integral equation (23) can be solved by expanding the radial function $R_{\omega q}^{(m,n)}$ using a complete set of orthogonal functions $f_k^{(|m|,|n|)}(r_x, r_y)$ as [6]

$$R_{\omega q}^{(m,n)}(r_x, r_y) = W_\perp(r_x, r_y) \sum_{k=0}^{\infty} a_k^{(m,n)} f_k^{(|m|,|n|)}(r_x, r_y) r_x^{|m|} r_y^{|n|}. \quad (25)$$

Here, the weight function $W_\perp(r_x, r_y)$ is defined by

$$W_\perp(r_x, r_y) = C f_{0\perp}(r_x, r_y), \quad (26)$$

where C is a normalization constant to be chosen. The functions $f_k^{(|m|,|n|)}(r_x, r_y)$ are determined so as to satisfy the following orthogonality relationship

$$\int_0^\infty \int_0^\infty W_\perp(r_x, r_y) f_k^{(|m|,|n|)}(r_x, r_y) f_l^{(|m|,|n|)}(r_x, r_y) r_x^{2|m|+1} r_y^{2|n|+1} dr_x dr_y = \delta_{kl}. \quad (27)$$

Using $f_k^{(|m|,|n|)}(r_x, r_y)$, we expand the Bessel functions as

$$\mathrm{J}_{|m|}(k_x r_x) \mathrm{J}_{|n|}(k_y r_y) = \sum_{k=0}^{\infty} C_{|m|,|n|,k}(k_x, k_y) \cdot f_k^{(|m|,|n|)}(r_x, r_y) \cdot r_x^{|m|} r_y^{|n|}, \quad (28)$$

where

$$C_{|m|,|n|,k}(k_x, k_y) = \int_0^\infty \int_0^\infty \mathrm{J}_{|m|}(k_x r_x) \mathrm{J}_{|n|}(k_y r_y) W_\perp(r_x, r_y) f_k^{(|m|,|n|)}(r_x, r_y) r_x^{|m|+1} r_y^{|n|+1} dr_x dr_y. \quad (29)$$

Inserting Eqs. (25) and (28) into Eq. (23), multiplying by $f_k^{(|m|,|n|)}(r_x, r_y) r_x^{|m|+1} r_y^{|n|+1}$ and integrating over r_x and r_y, we have a matrix equation for the coefficients $a_k^{(m,n)}$:

$$a_k^{(m,n)} + \sum_{m',n',l,j} \beta_{k,l}^{m,n} M_{m',n',j}^{m,n,l} a_j^{(m',n')} = 0, \quad (30)$$

where the matrix elements are given by

$$\beta_{k,l}^{m,n} = \int_1^\infty \int_0^\infty \int_0^\infty \frac{W_\perp(r_x, r_y) f_k^{(|m|,|n|)}(r_x, r_y) f_l^{(|m|,|n|)}(r_x, r_y) r_x^{2|m|+1} r_y^{2|n|+1}}{q - i\omega \frac{d\tau}{dz}(r_x, r_y, \gamma) - i(m k_{\beta x} + n k_{\beta y})} \frac{df_{0\|}}{d\gamma} dr_x dr_y d\gamma, \quad (31)$$

and

$$M_{m',n',j}^{m,n,l} = i^{|m|+|n|-(|m'|+|n'|)} \frac{(2\pi)^2 k_{\beta x} k_{\beta y}}{C} \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_{\omega q}(k_x, k_y) C_{|m|,|n|,l}(k_x, k_y) C_{|m'|,|n'|,j}(k_x, k_y) dk_x dk_y, \quad (32)$$

respectively. The matrix equation can be symbolically written as

$$(\boldsymbol{I}+\boldsymbol{\beta}\cdot\boldsymbol{M})\boldsymbol{a}=0, \tag{33}$$

where $\boldsymbol{a}$ is the vector of the coefficient $a_k^{(m,n)}$, $\boldsymbol{I}$ is the unit matrix, and the matrix elements of $\boldsymbol{\beta}$ and $\boldsymbol{M}$ are given by Eqs. (31) and (32), respectively. The nontrivial solution of Eq. (33) requires that

$$det(\boldsymbol{I}+\boldsymbol{\beta}\cdot\boldsymbol{M})=0. \tag{34}$$

This dispersion relation gives eigenvalues q as a function of ω or vice versa.

If we retain only the lowest-order term $m=n=k=0$ in the azimuthal and radial expansions as we have done for the round beam case [1,4], the dispersion relation (34) can be written in a general form as

$$\begin{aligned}1 \;=\;& i\frac{k}{k_1}\frac{r_e}{c}\left(\frac{K}{\gamma_r}\right)^2\frac{k_w}{\gamma_r}\frac{[\mathrm{JJ}]^2}{2\pi}\int_0^\infty\int_0^\infty\int_1^\infty\frac{f_{0\|}(\gamma)d\gamma f_{0\perp}(r_x,r_y)(2\pi)^2k_{\beta x}k_{\beta y}r_xdr_xr_ydr_y}{[\;q+2i\frac{k}{k_1}k_w\frac{\gamma-\gamma_r}{\gamma_r}-i\frac{1}{2}k(k_{\beta x}^2r_x^2+k_{\beta y}^2r_y^2)\;]^2}\\ &\times\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{dk_xdk_y}{q+ik_w\frac{k-k_1}{k_1}+i\frac{k_x^2+k_y^2}{2k}}\\ &\times\left[\int_0^\infty\int_0^\infty f_{0\perp}(r_x,r_y)\mathrm{J}_0(k_xr_x)\mathrm{J}_0(k_yr_y)(2\pi)^2k_{\beta x}k_{\beta y}r_xdr_xr_ydr_y\right]^2,\end{aligned} \tag{35}$$

where $f_{0\perp}(r_x,r_y)$ is normalized such that

$$\int_0^\infty\int_0^\infty f_{0\perp}(r_x,r_y)(2\pi)^2k_{\beta x}k_{\beta y}r_xdr_xr_ydr_y=1. \tag{36}$$

For a Gaussian beam

$$f_{0\perp}(r_x,r_y)=\frac{1}{(2\pi)^2k_{\beta x}\sigma_x^2k_{\beta y}\sigma_y^2}e^{-\frac{r_x^2}{2\sigma_x^2}-\frac{r_y^2}{2\sigma_y^2}}, \tag{37}$$

$$f_{0\|}(\gamma)=\frac{N}{\hat{\tau}}\frac{1}{\sqrt{2\pi}\sigma_\gamma\gamma_r}e^{-\frac{(\gamma-\gamma_r)^2}{2\gamma_r^2\sigma_\gamma^2}}, \tag{38}$$

where $\hat{\tau}$ is the length of the electron beam in time units, the above dispersion relation can be written in a scaled form as

$$\begin{aligned}1 \;=\;& i\frac{\sqrt{\frac{k_{\beta x}}{k_wD}}\sqrt{\frac{k_{\beta y}}{k_wD}}}{4\pi\sqrt{2\pi}\sqrt{2k_1\epsilon_x}\sqrt{2k_1\epsilon_y}}\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\frac{e^{-\frac{s^2+u^2+v^2}{2}}dsdudvdv}{\{\frac{q}{k_wD}+2i\frac{\sigma_\gamma}{D}s-\frac{i}{4}[2k_1\epsilon_x\frac{k_{\beta x}}{k_wD}u^2+2k_1\epsilon_y\frac{k_{\beta y}}{k_wD}v^2]\}^2}\\ &\times\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{e^{-\alpha^2-\beta^2}d\alpha d\beta}{\frac{q}{k_wD}+i\frac{k-k_1}{k_1D}+i[\frac{\alpha^2}{2k_1\epsilon_x}\frac{k_{\beta x}}{k_wD}+\frac{\beta^2}{2k_1\epsilon_y}\frac{k_{\beta y}}{k_wD}]},\end{aligned} \tag{39}$$

where we have replaced k by k_1 except in the detuning term $(k - k_1)/(k_1 D)$ to a good approximation. The scaled growth rate $Re(q)/(k_w D)$ is a function of the six scaling parameters:

$$\frac{Re(q)}{k_w D} = F(2k_1\varepsilon_x, 2k_1\varepsilon_y, \frac{\sigma_\gamma}{D}, \frac{k_{\beta x}}{k_w D}, \frac{k_{\beta y}}{k_w D}, \frac{k - k_1}{k_1 D}). \tag{40}$$

The authors specially wish to thank J. S. Wurtele for kindly providing us with the code TDA.

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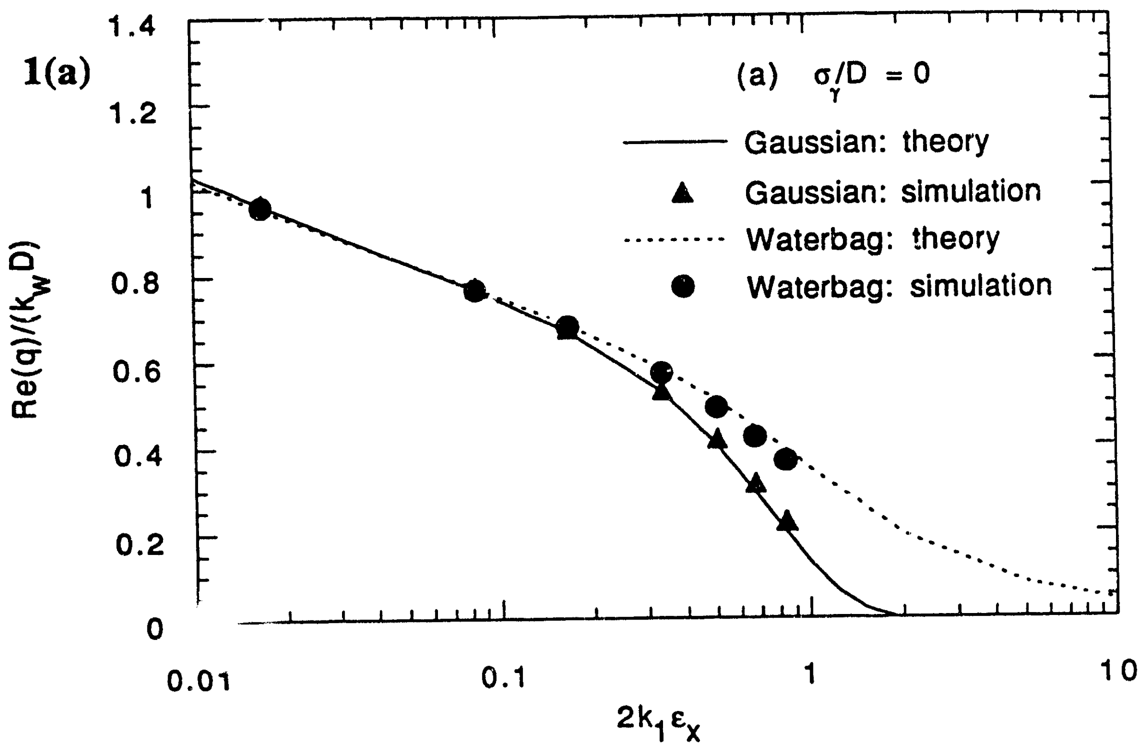
Figure Captions

FIG. 1(a). Comparison of the scaled growth rate $Re(q)/(k_w D)$ with the simulation results for the Gaussian and the waterbag beam distributions. Here, the scaled betatron wave number $k_\beta/(k_w D) = 1$ and the scaled energy spread $\sigma_\gamma/D = 0$. The solid and the broken curves show the solutions of the dispersion relations for the Gaussian and the waterbag beam distributions, respectively, while the triangles and the circles show the simulation results for the Gaussian and the waterbag beam distributions, respectively.

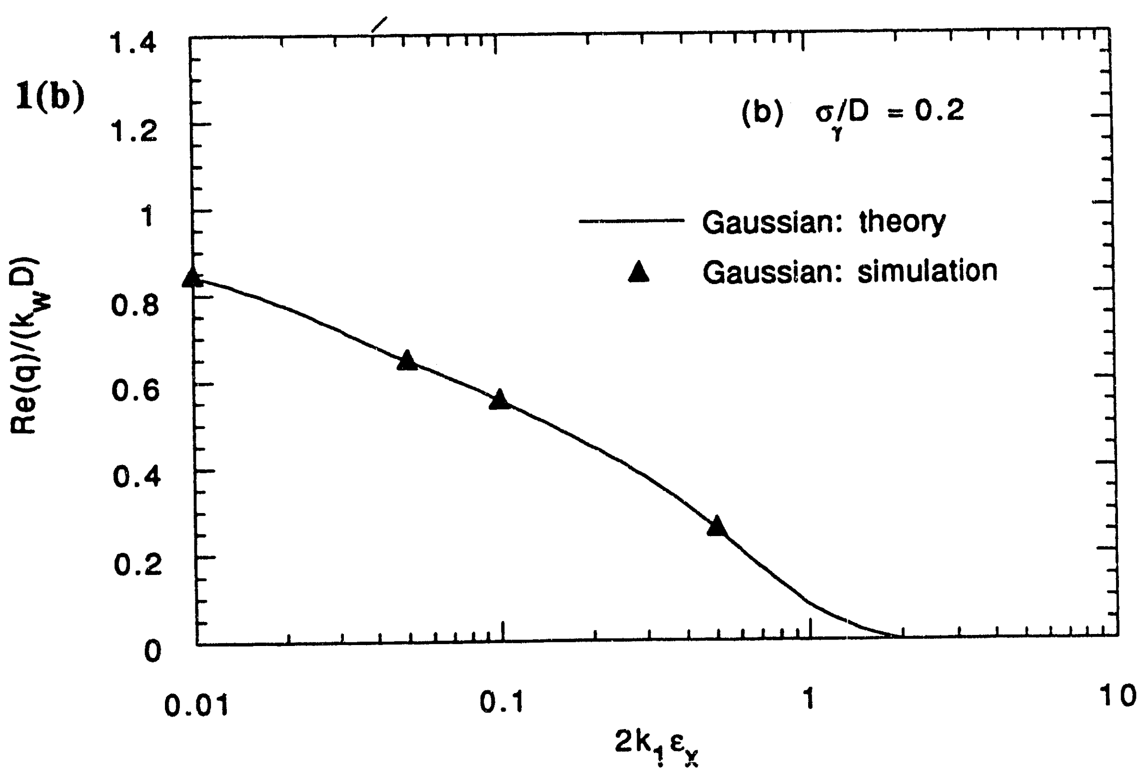
FIG. 1(b). Comparison of the scaled growth rate $Re(q)/(k_w D)$ with the simulation results for the Gaussian beam distribution. Here, $k_\beta/(k_w D) = 1$ and $\sigma_\gamma/D = 0.2$. The solid curve shows the solution of the dispersion relation for the Gaussian beam distribution, while the triangles show the simulation results for the Gaussian beam distribution.

Fig. 1(a)



(a) $\sigma_\gamma/D = 0$
Gaussian: theory
Gaussian: simulation
Waterbag: theory
Waterbag: simulation
Re(q)/(k_WD)
$2k_1\varepsilon_x$
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.01
0.1
1
10

Fig. 1(b)



(b) $\sigma_\gamma/D = 0.2$
Gaussian: theory
Gaussian: simulation
Re(q)/(k_WD)
$2k_1\varepsilon_x$
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.01
0.1
1
10

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