STABILITY OF BGK EQUILIBRIA

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K.R.Symon,* J.L.Schwarzmeier*

Dept. of Physics, University of Wisconsin, Madison, Wis., U.S.A.

and H.R.Lewis*

Los Alamos Scientific Laboratory, Los Alamos, N.M., U.S.A.

Abstract: An analysis of the stability of a BGK equilibrium solution of the Vlasov equation is presented, as an illustration of a general method of analysis of inhomogeneous equilibria which is particularly suited to numerical computations. Comparisons are made between the results of the analysis and numerical simulations.

The Problem. We consider a one-dimensional electron gas in a fixed ion background of density n_0 . We assume a stationary BGK wave solution $f_0(E)$, $\phi_0(x)$, where

$$E = 1/2mv^2 - e\phi_0(x), \quad \phi_0(x+a) = \phi_0(x). \quad (1)$$

The linearized Vlasov-Poisson equations for a small perturbation f , ϕ in density and electrostatic potential are

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \frac{e}{m} \frac{\partial \phi_0}{\partial x} \frac{\partial f}{\partial v} = - \frac{e}{m} \frac{\partial f_0}{\partial v} \frac{\partial \phi}{\partial x}, \quad \frac{\partial^2 \phi}{\partial x^2} = 4\pi e / f dv. \quad (2)$$

It is convenient to transform to a modified perturbed distribution function g as follows:

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$$f = P\phi + g, \quad P = -ef'_0 = -e \frac{\partial f_0}{\partial E} \quad (3)$$

Equation (3) is a special case of a general class of useful transformations in which P can be any suitably chosen operator. We substitute Eq.(3) into Eqs.(2) to obtain

$$\frac{\partial g}{\partial t} + \mathcal{L}g = ef'_0 \frac{\partial \phi}{\partial t}, \quad \Lambda \phi = 4\pi e f g dv, \quad (4)$$

where $\mathcal{L}$, Λ are the Liouville and Debye operators:

$$\mathcal{L} = v \frac{\partial}{\partial x} + \frac{e}{m} \frac{\partial \phi_0}{\partial x} \frac{\partial}{\partial v}, \quad (5)$$

$$\Lambda = \frac{\partial^2}{\partial x^2} - K_D^2(x), \quad K_D^2(x) = -4\pi e^2 \int f'_0(E) dv. \quad (6)$$

Note that K_D is the reciprocal Debye length. The algebraic merit of this transformation is that it eliminates $\partial \phi / \partial x$ from the Vlasov equation. The physical merit is that the Poisson equation now contains a term which may make the eigenfunctions of Λ resemble the x -dependence of the eigenfunctions of $\mathcal{L}$.

The Solution. We introduce eigenfunctions of the operators $\Lambda, \mathcal{L}$:

$$\Lambda \eta_n(x) = \lambda_n \eta_n(x), \quad (7)$$

$$\mathcal{L}[\delta(E-E') e^{ir\Omega(E')\tau(x,E)}] = ir\Omega(E') \delta(E-E') e^{ir\Omega(E')\tau(x,E)},$$

where $\Omega(E)$ is the frequency of the particle orbit with energy E , r is any integer, and $\tau(x,E)$ is the time required for a particle on the orbit of energy E to go from some reference point to the point X . There are ordinarily several orbits at each energy; integrals over E are then understood to include sums over each such orbit. All eigenfunctions can be chosen to have the Floquet form:

$$\eta_n(x) = e^{igx} h(x), \quad h(x+a) = h(x) \quad (9)$$

The entire problem now becomes diagonal in the index g . We assume a fixed choice of g ; in the example below, $g = \pi/a$. We expand the perturbations in these eigenfunctions:

$$g(x, v, t) = \sum_r \gamma_r(E, t) e^{ir\Omega(E)\tau(x,E)} \quad (10)$$

$$\phi(x, t) = \sum_n \alpha_n(t) \eta_n(x) \quad (11)$$

We Laplace transform the resulting equations for the coefficients γ_r, α_n , solve the Vlasov equation for the transformed coefficients $\tilde{\gamma}_r$, and substitute in the field equation to obtain:

$$\sum_{n'} D_{nn'}(\omega) \tilde{\alpha}_{n'} = I_n(\omega), \quad (12)$$

where ω is the Laplace transform frequency variable, $I_n(\omega)$ depends upon the initial conditions, and the dispersion matrix is

$$D_{nn'}(\omega) = \lambda_n \delta_{nn'} + 4\pi e^2 \sum_r \int dE \frac{\omega f'_0(E) K_{nr}^*(E) K_{n'r}(E)}{\omega - r \Omega(E)} \quad (13)$$

where

$$K_{nr}(E) = \int d\tau e^{-ir\Omega(E)\tau} \eta_n [x(\tau, E)] \quad (14)$$

The Dispersion Matrix. The matrix $D(\omega)$ has the interesting and useful property

$$[D_{nn'}(\omega)]^* = D_{n'n}(\omega^*) \quad (15)$$

A matrix with this property may be called a hermitian function. When Eq.(12) is solved for $\tilde{\alpha}_n$, and the Laplace transform inverted, the solution may be expressed as a superposition of van Kampen modes located along the cut in the real ω -axis associated with the matrix $D(\omega)$, plus a sum of discrete normal modes at complex frequencies ω which are roots of the dispersion relation

$$\det [D_{nn'}(\omega)] = 0. \quad (16)$$

The coefficients of the eigenfunctions $\eta_n(x)$ for any normal mode form an eigenvector corresponding to a null eigenvalue of the matrix $D(\omega)$ evaluated at the normal mode frequency.

Numerical Methods. The above analysis is particularly suited to the numerical determination of the frequencies, growth rates, and perturbed potentials and distribution functions for the normal modes. We discretize the energy spectrum of the orbits, choosing a finite number of energies suitable for describing the perturbation. Integrals over energy become sums. We also choose a small number of eigenfunctions $\eta_n(x)$. The transformation (3) results in an operator Λ whose eigenfunctions match rather well the low frequency normal modes. Thus a single eigenfunction $\eta_1(x)$ leads to fairly good results, and three lead to quite accurate results, as we shall see below.

Simulation results. The above analysis was applied to a particular case for which results from numerical simulations were available for the BGK equilibrium and for the unstable normal mode. The equilibrium potential is shown by the solid, approximately sinusoidal curve in the figure below. Also shown is the form of the perturbed potential associated with the unstable mode, and the corresponding eigenfunction (dashed curve) of the operator Λ for this case. If the perturbed potential is expanded in the first three eigenfunctions of Λ , the result is indistinguishable from the simulation result. In the table below, we show the theoretical and simulation re-

sults for the growth rate (in units of ω_p^{-1}) using a matrix $D(\omega)$ based upon one or three eigenfunctions of Λ , and for the coefficients of the three eigenfunctions in the expansion of the perturbed potential.

	Theory Simulation	
Growth rate (1X1)	3.03	$3.10 \pm .04$
Growth rate (3X3)	3.07	$3.10 \pm .04$
Coeff. of η_1	0.99894	0.99861
Coeff. of η_2	0.03765	0.04115
Coeff. of η_3	0.02662	0.03215

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