Reduction of Field Equations and Saturated Bounds for Vortex-Line Theories*

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Abstract

The Euler-Lagrange equations following from the two-dimensional Higgs-type Lagrangian and from the chirally nonlinear Lagrangian are shown to be reducible to one second order differential equation. This simplification is possible for an arbitrary configuration of static parallel vortex-lines but only for a specific potential. In both cases the result is connected with a saturated bound for the energy and also with conditions arising from the scale variation.

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I. Introduction

When one looks for vortex-line solutions to gauged scalar field theories usually the first step is to simplify the coupled set of nonlinear differential equations by using some kind of an ansatz. With the standard radial ansatz^{1,2} the set of four partial differential equations can be reduced to two ordinary differential equations.

In this paper we study this reduction or simplification problem by separating the gauge-dependent phase part and the amplitude part for arbitrary configurations of parallel, static vortex-lines. The equations decouple except for boundary conditions: The scalar field amplitude must have an n 'th order zero at the position of a vortex-line with circulation n . This reduction works only for a specific potential term. In those cases we can also show that a bound for the energy is saturated.³

The organization of the paper is as follows: In Section II we show how the reduction works for the Higgs-type Lagrangian when the coupling constant $\beta = 1$. Some of these results have been obtained earlier by Harden and Arp⁴ and by Bogomolny.³ In Section III we extend the method to the nonlinear Lagrangian of chiral $SU(2) \otimes SU(2)$. This is also related to a saturated bound. Finally in Section IV we point to a connection between these saturated bounds and virial type conditions coming from the scale variation.

II. The Higgs-type Lagrangian

A. Field equations

Let us consider the Higgs-type Lagrangian with a general potential.

We are looking for stationary solutions of vortex-line type^{1,2} and therefore our object is to extremize the energy

$$E = \int d^2x \left\{ \frac{1}{4} F_{ij}^2 + \frac{1}{2} (D_i \Phi_a)^2 + V(\Phi^2) \right\}, \quad (1)$$

where

$$\begin{aligned} F_{ij} &= \partial_i A_j - \partial_j A_i, \\ D_i \Phi_a &= \partial_i \Phi_a + e \epsilon_{ab} \Phi_b A_i, \end{aligned} \quad (2)$$

$i, j, a = 1, 2$ and $\epsilon_{12} = +1$. In the gauge $\partial_i A_i = 0$ the Euler-Lagrange equations are

$$\square A_i + e \epsilon_{ab} \Phi_a \partial_i \Phi_b - e^2 \Phi^2 A_i = 0, \quad (3)$$

$$\square \Phi_a + 2e \epsilon_{ab} \partial_i \Phi_b A_i - e^2 \Phi_a A_i^2 - 2\Phi_a V'(\Phi^2) = 0. \quad (4)$$

To simplify these equations we parametrize the two fields Φ_a as follows⁵

$$\begin{aligned} \Phi_1 &= \phi \cos \chi, \\ \Phi_2 &= \phi \sin \chi. \end{aligned} \quad (5)$$

Then the field equations can be written in the form

$$\square A_i + e \phi^2 (\partial_i \chi - e A_i) = 0, \quad (6)$$

$$\square \phi - \phi (\partial_i \chi - e A_i)^2 - 2\phi V'(\phi^2) = 0, \quad (7)$$

$$2 \partial_i \phi (\partial_i \chi - e A_i) + \phi \square \chi = 0. \quad (8)$$

Before we go on with these equations let us discuss the properties of χ . The only gauge invariant requirement for χ is that it should change by $2\pi n$ as we go around a vortex-line with winding number n ¹. A solution to this is to take

$$\chi(\vec{x}) = \sum_a n_a \theta(\vec{R}_a, \vec{x}), \quad (9)$$

where n_α are integers and $\Theta(\vec{R}, \vec{x})$ is the angle in the positive direction between a reference direction from the point $\vec{R}$ and the vector $\vec{x} - \vec{R}$. A change in the reference direction would give an overall constant change, which would not appear in the field equations (6)-(8). [These reference lines originating from each vortex correspond to the Dirac strings of three-dimensional theories.] Because of the discontinuity at $\vec{R}_\alpha$ we have

$$\epsilon_{ij} \partial_i \partial_j \chi(\vec{x}) = 2\pi \sum_\alpha n_\alpha \delta^2(\vec{R}_\alpha - \vec{x}). \quad (10)$$

It is useful to define an auxiliary function $G(\vec{x})$ by

$$\partial_i \chi(\vec{x}) = - \epsilon_{ij} \partial_j G(\vec{x}), \quad (11)$$

then

$$G(\vec{x}) = \sum_\alpha n_\alpha \ln |\vec{x} - \vec{R}_\alpha| + \text{const.} \quad (12)$$

Since we now have $\square \chi = 0$ everywhere the equation (8) suggests the following coordinate independent ansatz

$$A_i = \frac{1}{e} \partial_i \chi + \epsilon_{ij} \partial_j f(\phi), \quad (13)$$

where $f(\phi)$ is some function that will be determined later. With this choice the gauge condition $\partial_i A_i = 0$ and the equation (8) are automatically satisfied.

We next use the freedom in f to reduce (6) and (7) to a single equation.

Substituting (13) to (6) gives first

$$\square \epsilon_{ij} \partial_j \left[f(\phi) - \frac{1}{e} G(\vec{x}) \right] - e^2 \phi^2 \epsilon_{ij} \partial_j f(\phi) = 0. \quad (14)$$

Since we assume A_i to be smooth the singularities in $f(\phi)$ and $G(\vec{x})$ must cancel in the above expression, i.e.

$$f(\phi) \sim \frac{n_\alpha}{e} \ln |\vec{x} - \vec{R}_\alpha|, \text{ as } |\vec{x} - \vec{R}_\alpha| \rightarrow 0. \quad (15)$$

We can then commute ∂_j and $\square$ to get

$$\square f(\phi) - e^2 \int_{\mathbb{R}} t^2 f'(t) dt = \frac{1}{e} \epsilon_{ij} \partial_i \partial_j \chi. \quad (16)$$

Then since $\epsilon_{ij} \partial_i \partial_j \chi / f'(\phi) = 0$ everywhere⁶ we can write equations (16) and (7) as

$$\square \phi + (\partial_i \phi)^2 \frac{f''}{f'} - e^2 \frac{1}{f'} \int_{\mathbb{R}} t^2 f'(t) dt = 0, \quad (17)$$

$$\square \phi - (\partial_i \phi)^2 e^2 \phi (f')^2 - 2 \phi V'(\phi^2) = 0. \quad (18)$$

We now determine f by comparing coefficients of the $(\partial_i \phi)^2$ terms; the only solution with $|f'| \rightarrow \infty$ as $\phi \rightarrow 0$ is

$$f = \pm \frac{1}{e} \ln |\phi| \quad (19)$$

The condition (15) can now be written as

$$|\phi| \sim |\vec{x} - \vec{R}_\alpha|^{\pm n_\alpha}, \text{ as } |\vec{x} - \vec{R}_\alpha| \rightarrow 0. \quad (15')$$

This means that whichever sign we choose in (19) we can have vortices of one sign only. As can be seen from (13) this overall sign can be taken care of by redefining $A_i \rightarrow -A_i$ if necessary. [The only possibility of having vortices of different signs is to have two dimensional regions with $|\phi| = 1$ at the boundary. Then $f = 0$ at the boundary and we could try to match continuously the regions with different signs.]

Now that $f(\phi)$ has been determined the form of the potential $V(\phi^2)$ is fixed by comparing the last terms in the equations (17) and (18). We find

$$V(\phi^2) = \frac{e^2}{8} (\phi^2 - \mathbb{R}^2)^2. \quad (20)$$

For this potential $\beta = M_S^2/M_V^2 = 1$, or in terms of the usual parameter of superconductivity², $\kappa = 1/\sqrt{2}$. The arbitrariness in the parameter F gives no essential generalization as it can be scaled away along with e .⁷ We then get the differential equation in the normal form

$$\square \ln \phi^2 - \phi^2 + 1 = 2 \epsilon_{ij} \partial_i \partial_j \chi, \quad (21)$$

with the boundary condition $\lim_{|\vec{x}| \rightarrow \infty} \phi^2 = 1$. Unfortunately no closed form solutions have been found to this differential equation.

B. The Saturated Bound

Closely related to the above reduction scheme is the saturated bound result obtained by Bogomolny.³ He notes that for the potential (20) the various terms in the energy (1) can also be grouped as follows

$$E = \int d^2x \left\{ \frac{1}{4} \left[F_{ij} - \epsilon_{ij} e \frac{1}{2} (F^2 - \Phi^2) \right]^2 + \frac{1}{4} \left[\epsilon_{ij} D_j \Phi_a + \epsilon_{ab} D_i \Phi_b \right]^2 + W \right\}, \quad (22)$$

where

$$\begin{aligned} W &= \frac{1}{2} \epsilon_{ij} \partial_i A_j e (F^2 - \Phi^2) + \frac{1}{2} \epsilon_{ij} \epsilon_{ab} D_i \Phi_a D_j \Phi_b \\ &= \frac{1}{2} \epsilon_{ij} \partial_i \left[e A_j (F^2 - \Phi^2) + \epsilon_{ab} \Phi_a \partial_j \Phi_b \right]. \end{aligned} \quad (23)$$

Since W can be expressed as a curl of a vector we can calculate that integral and get the topological charge

$$\int d^2x W = \pi \sum_{\alpha} n_{\alpha} F^2. \quad (24)$$

The energy is therefore minimized when the first two terms in (22) are identically zero. This gives us the field equations³

$$F_{ij} + \frac{e}{2} \epsilon_{ij} (\Phi^2 - F^2) = 0, \quad (25)$$

$$\epsilon_{ij} D_j \Phi_a + \epsilon_{ab} D_i \Phi_b = 0. \quad (26)$$

Now from (26) we can solve for A_i and obtain (13) with (19), then substitution to (25) gives (21). For the standard radial ansatz (25) and (26) reduce to a coupled set of first order scale field equations³ first found by Harden and Arp⁴. [From these elimination of ϕ leads to the form studied recently by de Vega and Schaposnik⁸.]

III. The Chiral $SU(2) \otimes SU(2)$ Theory

The previous analysis can also be carried out for a theory based on a nonlinear realization of the chiral $SU(2) \otimes SU(2)$. For the moment we will use σ - model coordinates. To obtain stationary vortex-line solutions our object would be to extremize the energy

$$E = \int d^2x \left[\frac{1}{4} F_{ij}^2 + \frac{1}{2} (D_i \pi_a)^2 + \frac{1}{8} \frac{(\partial_i \pi^2)^2}{f^2 - \pi^2} + V(\pi^2) \right]. \quad (27)$$

In analog with (5) we now let

$$\begin{aligned} \pi_1 &= f s \cos \chi, \\ \pi_2 &= f s \sin \chi, \\ \pi_3 &= 0, \end{aligned} \quad (28)$$

and then the Euler-Lagrange equations are

$$\square A_i + e f^2 s^2 (\partial_i \chi - e A_i) = 0, \quad (29)$$

$$\square s + (\partial_i s)^2 \frac{s}{1-s^2} - s(1-s^2) (\partial_i \chi - e A_i)^2 - 2s(1-s^2) V'(f^2 s^2) = 0, \quad (30)$$

$$2 \partial_i s (\partial_i \chi - e A_i) + s \square \chi = 0. \quad (31)$$

Since the nonlinearities do not change the gauge coupling part the previous comments on χ still apply and we can again use the coordinate independent ansatz

$$A_i = \frac{1}{e} \partial_i \chi + \epsilon_{ij} \partial_j g(s). \quad (32)$$

The equations (29) and (30) now become

$$\square s + \frac{g''}{g'} (\partial_i s)^2 - e^2 f^2 \frac{1}{g'} \int_{\mp}^s t^2 g'(t) dt = 0, \quad (33)$$

$$\square s + \left(\frac{s}{1-s^2} - s(1-s^2)e^2(g')^2 \right) (\partial_i s)^2 - 2s(1-s^2)V'(f^2 s^2) = 0. \quad (34)$$

Comparing the coefficient of $(\partial_i s)^2$ gives us a differential equation for g ; its only solution satisfying $|g'(s)| \rightarrow \infty$, as $s \rightarrow 0$, is

$$g = \pm \frac{1}{e} \ln \left| s / [1 + (1-s^2)^{1/2}] \right|, \quad (35)$$

and then V can be found to be

$$V(\pi^2) = \frac{e^2 f^2}{2} \left[(f^2 - \pi^2)^{1/2} - (f^2 - \pi(\infty)^2)^{1/2} \right] \quad (36)$$

where $\pi^2(\infty) = \lim_{|\vec{x}| \rightarrow \infty} \pi^2(\vec{x})$. Our final form for the differential equation is then

$$\square \ln \left| s / [1 + (1-s^2)^{1/2}] \right| + e^2 f^2 [(1-s^2)^{1/2} - (1-s_\infty^2)^{1/2}] = \epsilon_{ij} \partial_i \partial_j \chi \quad (37)$$

or in terms of the Weinberg coordinates defined by $W = [1 - (1-s^2)^{1/2}] / s$, and after scaling by $\sqrt{2}ef$

$$\square \ln W^2 - \frac{W^2 - 1}{W^2 + 1} + \frac{W_\infty^2 - 1}{W_\infty^2 + 1} = 2 \epsilon_{ij} \partial_i \partial_j \chi \quad (38)$$

[compare this with (21)]. The boundary condition is now $\lim_{|\vec{x}| \rightarrow \infty} W(\vec{x}) = W_\infty$.

B. The Saturated Bound

For the chiral case we can also find a saturated bound for the energy. The terms in the energy (27) with (36) can now be grouped as follows

$$E = \int d^2x \left\{ \frac{1}{4} (F_{ij} + \epsilon_{ij} f e [(f^2 - \pi^2(\omega))^{\frac{1}{2}} - (f^2 - \pi^2)^{\frac{1}{2}}])^2 \right. \\ \left. + \frac{1}{4} [\epsilon_{ab} D_i \pi_b + \epsilon_{ij} D_j \pi_a - (\delta_{ab} \epsilon_{ij} + \epsilon_{ab} \delta_{ij}) \pi_b \partial_j \ln [f + (f^2 - \pi^2)^{\frac{1}{2}}]] \right. \\ \left. + W \right\}, \quad (39)$$

where

$$W = -\frac{1}{2} F_{ij} \epsilon_{ij} f e [(f^2 - \pi^2(\omega))^{\frac{1}{2}} - (f^2 - \pi^2)^{\frac{1}{2}}] + \frac{1}{2} \epsilon_{ab} \epsilon_{ij} D_i \pi_a D_j \pi_b \\ - \epsilon_{ab} \epsilon_{ij} D_i \pi_a \pi_b \partial_j \ln [f + (f^2 - \pi^2)^{\frac{1}{2}}] \\ = \epsilon_{ij} \partial_i \left\{ e A_j f [(f^2 - \pi^2)^{\frac{1}{2}} - (f^2 - \pi^2(\omega))^{\frac{1}{2}}] \right. \\ \left. + \frac{1}{2} \epsilon_{ab} \pi_a \partial_j \pi_b (1 + 2f [f - (f^2 - \pi^2)^{\frac{1}{2}}] / \pi^2) \right\}. \quad (40)$$

Remembering the singularly properties of χ we can now do the W integral as before and obtain

$$\int d^2x W = \pi \sum_{\alpha} n_{\alpha} \left\{ \pi^2(\omega) + 2f [f - (f^2 - \pi^2(\omega))^{\frac{1}{2}}] \right\}. \quad (41)$$

This is a bound for the energy and it is saturated if the first two non-negative terms in (39) vanish identically. This condition gives us the field equations

$$F_{ij} + \epsilon_{ij} e f [(f^2 - \pi(\omega)^2)^{\frac{1}{2}} - (f^2 - \pi^2)^{\frac{1}{2}}] = 0, \quad (42)$$

$$\epsilon_{ab} D_i \pi_b + \epsilon_{ij} D_j \pi_a - (\delta_{ab} \epsilon_{ij} + \epsilon_{ab} \delta_{ij}) \pi_b \partial_j \ln [f + (f^2 - \pi^2)^{\frac{1}{2}}] = 0. \quad (43)$$

Again we could solve for A_i from (43) which gives (32) with (35), then substitution to (42) gives (37).

At this point we would like to comment on the case $\pi^2(\infty) = 0$. A priori this seems to be possible as well but then by (41) energy would be zero. In fact the variation $\pi^a(\vec{x}) \rightarrow \xi \pi^a(\vec{x})$ suggests that no solutions exist confirmed by a study on (38). [If we should have $W(0) = 0$ and $W(\infty) = 0$ then at some point $\frac{1}{W} \square W \leq 0$ and $(\partial_i W)^2 = 0$, but this is not possible according to (38).] The same situation occurs also when there is no gauge coupling or potential⁹. The resolution of this problem is to consider instead a linear Lagrangian of four fields $\vec{\pi}, \sigma$ with the constraint $\vec{\pi}^2 + \sigma^2 = f^2$. Then the topological charge would be related to a kink in the radial function¹⁰ [sign change of σ] rather than just to the behavior of χ at infinity. This topological charge was not included in (41) because the possible singularity at $\pi^2 = f^2$ in the term $(\partial_i \pi^2)^2 / (f^2 - \pi^2)$ was ignored.

IV. Relation to Scale Variation

When one studies the existence of solutions to variational problems one can often use scale variations

$$\psi(\vec{x}) \rightarrow \psi^\lambda(\vec{x}) = \psi(\lambda \vec{x}), \quad (44)$$

$$A_i(\vec{x}) \rightarrow A_i^\lambda(\vec{x}) = \lambda A_i(\lambda \vec{x}), \quad (45)$$

to obtain some qualitative results⁹. When this variation is applied to (1) or (27) we get the following condition

$$\int d^2x \frac{1}{4} F_{ij}^2 = \int d^2x V(\psi^2). \quad (46)$$

We would now like to point out that for the saturated bound cases above this condition is satisfied pointwise, i.e., for the integrands rather than just for the integrals, see (25) and (42).

The same feature holds also for the three-dimensional case of Prasad and Sommerfield^{3,11}. The condition from scale variation is now

$$\int d^3x \frac{1}{4} (F_{ij}^a)^2 = \int d^3x \frac{1}{2} (D_i \Phi^a)^2, \quad (47)$$

on the other hand the bound saturating differential equations are^{3,12}

$$\frac{1}{2} \epsilon_{ijk} F_{jk}^a = D_i \Phi^a, \quad (48)$$

so that (46) is again satisfied pointwise. The scale variation might therefore be useful in guessing first integrals.

In conclusion we would like to stress again that the reduction method presented here applies to arbitrary configurations of static, parallel vortex-lines. It is likely that solutions exist only for certain configurations, unfortunately no simple closed form solutions have been found so far.

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5. Strictly speaking this is only possible for a simply connected region that excludes points where $\phi_1^2 + \phi_2^2 = 0$. The discontinuity at those points will be manifested by $\epsilon_{ij} \partial_i \partial_j \chi \neq 0$.
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