"Creep of beams using a state-variable approach"

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ABSTRACT

The technical theory of bending of beams is formulated in terms of Hart's state-variable equations. Then the bending of rectangular beams under constant and variable moments is solved by a standard routine. The stationary creep of a beam in pure bending is obtained in closed form; this analysis serves as a check of some of the transient creep computations. All the results compare qualitatively with those using classical creep theories and in addition they show the strong influence upon creep deformation of the state-variable called "hardness", which is absent in the other theories.

INTRODUCTION

The phenomenon of creep in metals has been known for a long time [1] - generally the creep rate rises very markedly with increased temperature and stress. With the development of nuclear reactor and aircraft components working at increasingly higher temperature and pressure, it has become essential to incorporate accurately the effect of creep in analysis and in design [2].

One of the basic elementary structural members is the beam. Although abundant creep data based on tensile tests are readily available, experimental investigations of creep in beams are relatively few in number. MacCallough [3], and Tapsell and Johnson [4] tested lead beams in pure bending while Davis [5] experimented with steel beams. Tests on cantilever aluminum beams of constant strength were reported by Strum, Dumont and Howell [6], and a comparative study between tension and bending on aluminum at room temperature was initiated by Marin and Zwissler [7]. Theoretical calculations were carried out by Rabotnov [8] using a strain-hardening law, Lubahn [9] and Penny [10] using a time-hardening law directly, and Kachanov [11] using a time-hardening law with a variational principle.

It is well known that seemingly identical specimens often give rise to wide scatter in creep data. This undoubtedly is due to the fact that they have undergone different histories of loading. The classical creep laws are incapable of accounting for the effect of past history upon subsequent deformation and hence they fail to predict experimentally observed phenomena such as recovery. To remedy such drawbacks, two alternative approaches have been introduced, namely, the integral representation and the state-variable representation. The former approach has been presented by Rabotnov [8], Pipkin and Rogers [12], and Krempl [13], while the latter has been reported by Onat and Fardshisheh [14], Lagneborg [15], Bodner and Partom [16], Miller [17] and Hart [18]. As reviewed by Onat and Fardshisheh

[14], the state-variable representation is superior to the integral representation from both conceptual and computational points of view. Of all the state-variable theories (some of which are mathematically equivalent), Hart's law has received the most recent attention. It is explicitly expressed in terms of well-defined parameters, has been extensively tested in tensile experiments [19,20], and was recently chosen for analysis of spheres and cylinders by Kumar and Mukherjee [21].

In this paper, first a general technical theory of bending of beams is formulated in terms of Hart's state-variable equations. It is felt that incorporation of a state variable, called "hardness", will account for the past history of loading in the material and thus produce more accurate and consistent predictions. Note that "hardness" is a material parameter which can be determined experimentally [20]. The principal objective here is to develop, using the general theory, solutions to classical problems in bending of beams which can easily be tested experimentally, and can be compared to the results of analysis using other theories such as time-hardening and strain-hardening. Such comparisons may provide another basis for acceptance of this new model.

Following the theoretical development, we construct numerical solutions to problems of bending of rectangular beams under constant and variable moment distributions. Stationary creep in beams, a phrase similarly used in the classical literature, is studied and the results compared with long-time behavior of beams analyzed earlier.

Our study indicates that results based on Hart's creep law and the concept of "hardness" are consistent with previous experience; past history of loading, i.e. hardness, has a strong influence on creep behavior.

FORMULATION OF A TECHNICAL BEAM THEORY

We take as a basic problem that of a beam undergoing creep, that is, we seek to determine its deflection as a function of time due to a suddenly imposed load which may subsequently vary with time. For engineering design purposes, it is equally essential to study other features of its mechanical behavior such as the stress redistribution at various points of the beam.

The formulation is developed for a beam whose cross-section is symmetric about one principal axis of bending as in fig. 1. Let the length of the beam be l , the width of the profile be $b(y)$, and the depth be $h_1 + h_2$ where the X-axis is chosen so that

$$\int_{-h_2}^{h_1} by dy = 0.$$

It follows that the moment of inertia about the x-axis is

$$I = \int_{-h_2}^{h_1} by^2 dy. \quad (1)$$

The assumptions adopted in the technical beam theory to be developed are:

- (i) strain is small,
- (ii) plane sections remain plane,
- (iii) deflection slope is small,
- (iv) loading is steady or slowly varying, and
- (v) maximum stress induced by the initial load is less than the yield stress.

We also assume that the total longitudinal strain is the sum of an elastic strain e and a non-elastic strain e^n ,

$$e^t = e + e^n. \quad (2)$$

From assumption (ii), it follows that

Fig. 1

$$\dot{e}^t = Ky \quad (3)$$

where K is the curvature of the beam defined with assumption (iii) by the relation

$$K = - d^2W/dz^2 \quad (4)$$

where W is the deflection. Note that assumption (ii) is purely geometrical and bears no relation to the properties of the material [11]. In a slender beam, i.e. one whose length is several times a typical cross-sectional dimension, the shear stresses due to lateral loading are considerably less than the normal stresses due to bending and the pressure between the longitudinal filaments is thus very small. Consequently, the cross sections suffer little distortion and it is reasonable to assume plane sections remain plane.

We denote the intensity of the distributed lateral load at a typical cross section, z , of the beam by $q(z)$, the shearing force $V(z)$ and the bending moment $M(z)$ (fig. 1). Equilibrium requires

$$dV/dz = -q, \quad dM/dz = V \quad \text{and} \quad (5)$$

$$M = \int_{-h_2}^{h_1} \sigma y dy \quad (6)$$

As creep is a time-dependent process, the rate versions of the equations must be used,

$$\dot{e}^t = \dot{e} + \dot{e}^n \quad (7)$$

Two separate creep strain rate components have been experimentally identified [22], i.e.,

$$\dot{e}^n = \dot{e}^a + \dot{p} \quad (8)$$

where $\dot{e}^a$ is the recoverable creep strain rate and $\dot{p}$ is the irrecoverable creep strain rate. If the loading is steady or slowly varying as assumed, $\dot{e}^a$ will readily approach a saturation value $\dot{e}^{as}$ and consequently its rate $\ddot{e}^a \approx 0$. Henceforth

$$\dot{e}^t = \dot{e} + \dot{p} \quad (9)$$

The other rate equations follow from equations (3) and (6),

$$\dot{e}^t = \dot{K}_y$$

$$\dot{M} = \int_{-h_2}^{h_1} \dot{\sigma} b y dy \quad (10)$$

The constitutive relations include Hooke's law and Hart's kinetic equations of creep in terms of a state variable σ^* , called "hardness"[18].

$$e = \sigma/E \quad (11)$$

$$\dot{p} = f(\sigma, \sigma^*, T) \quad (\dot{}) = d()/dt \quad (12)$$

$$\dot{\sigma}^* = g(\sigma, \sigma^*, T)$$

where f and g are experimentally determined functions, σ is the normal stress, $\dot{p}$, the irrecoverable creep strain rate, T , the temperature and E , the Young's modulus. Equations (12) are sometimes referred to as evolutionary equations [14]. It should be emphasized that the beam theory formulated here is, as in ordinary strength of materials, based on uniaxial stress-strain relations. Higher order theories will be the subject of subsequent reports.

We group the equations as follows:

$$\begin{aligned} \dot{p}(t) &= f(\sigma, \sigma^*, T) \\ \dot{\sigma}^*(t) &= g(\sigma, \sigma^*, T) \\ \dot{K}(t) &= \dot{M}(t)/(EI) + \left(\int_{-h_2}^{h_1} \dot{p} b y dy \right) / I \\ \dot{\sigma}(t) &= E \dot{K}_y - E \dot{p} \end{aligned} \quad (13)$$

Under a set of loading rate conditions, $\dot{M}(t)$ is given by the rate version of (5). The corresponding initial conditions are:

$$\begin{aligned} \sigma^*(0) &= \sigma_0^* \\ K(0) &= M(0)/(EI) \\ \sigma(0) &= E K_y \\ p(0) &= e^n(0) = 0 \end{aligned} \quad (14)$$

Note that these are the results obtained from the elastic beam (assumption (v)), and σ_0^* , the initial hardness, is determined experimentally [20].

Thus the problem of finding the time-history of lateral deflection, $W(t)$, in a beam under prescribed loading history (an initial "elastic" load followed by time-varying loads) consists of solving the initial value problem with the rate (evolutionary) equations (13), throughout the cross-section of the beam, subject to the constraint $\dot{e}^t = \dot{K}_y$ (plane sections remain plane) and under prescribed initial conditions (14). Deflection is obtained by integrating (4).

Forward integration in time is achieved with any standard routine. For example Euler's method would lead to an increment of stress determined by the relation

$$\sigma|_{t_2} = \sigma|_{t_1} + \dot{\sigma}|_{t_1} (t_2 - t_1) \quad (15)$$

and can provide numerically accurate results as long as Δt is sufficiently small, say $\Delta t < 0.1\sigma/\dot{\sigma}$, which has been adequate for the problems discussed here. More refined numerical techniques such as Runge-Kutta or predictor-corrector can be used with consequent increases in accuracy. The integration is continued until $\dot{\sigma} = \epsilon$ where ϵ is a sufficiently small number.

RECTANGULAR BEAM

As an example of explicit calculations of deflection and stress, consider a rectangular beam of length ℓ , width b and depth $2h$. For the sake of definiteness, a well-established form of Hart's law is used[20],

$$\dot{p} = (\sigma^*/D)^m [\log(\sigma^*/\sigma)]^{-(1/\lambda)}$$

$$\dot{\sigma}^* = \Lambda_p \dot{\sigma}^\delta / (\sigma^*)^{\beta-1} \quad (\dot{}) = d()/dt \quad (16)$$

where m , λ , δ , β and D are experimentally determined quantities. D is the only strongly temperature-dependent parameter.

Let the initial applied moment on a cross-section be M_0 and introduce the following constant quantities in terms of M_0 . Note that σ_0 is defined in terms of M_0 .

$$\sigma_0 = 3M_0/(2bh^2), \quad e_0 = \sigma_0/E, \quad K_0 = \sigma_0/(Eh),$$

$$t_0 = (\sigma_0/E)(\sigma^*/D)^{-m} (\log \sigma^*/\sigma_0)^{1/\lambda} \quad \text{and} \quad D_0 = D/\sigma_0 \quad (17)$$

The significance of t_0 will be discussed in the Appendix. The following dimensionless quantities are defined:

$$\bar{\sigma} = \sigma/\sigma_0, \quad \bar{\sigma}^* = \sigma^*/\sigma_0, \quad \bar{p} = p/e_0, \quad \bar{K} = K/K_0,$$

$$\bar{\Lambda} = \Lambda_p \sigma_0^{\delta-\beta} (\sigma_0/E), \quad \bar{M} = M/M_0, \quad \bar{y} = y/h, \quad \bar{z} = z/\ell,$$

$$\bar{t} = t/t_0, \quad (\dot{}) = d()/d\bar{t} \quad (18)$$

Henceforth, for simplicity, overbars will no longer be used to denote dimensionless quantities, except where emphasis is necessary.

Two classical cases are examined in detail.

A. Pure Bending ($M' = 0$, constant moment or $M' > 0$, monotonically increasing moment).

Let the elastic deflection at the mid-point of the beam ($z = 1/2$) (inset, fig. 2) be

$$W_0 = K_0 \ell^2/8 \quad \text{and define}$$

$$W_{z=1/2} = W/W_0 \quad (19)$$

At $t = 0$, (14) i.e. initial conditions in terms of (17), (18) and (19) are

$$\begin{aligned}
\sigma^*(0,y) &= \sigma_0^*, & K(0) &= 1 \\
\sigma(0,y) &= y, & p(0,y) &= 0, \text{ and} \\
W_{z=1/2}(0) &= 1.
\end{aligned}
\tag{20}$$

For $t \geq 0$, (13) i.e. the rate equations in terms of (17), (18) and (19) are

$$\begin{aligned}
p'(t,y) &= (\sigma^*/\sigma_0^*)^m (\log(\sigma^*/\sigma)/\log \sigma_0^*)^{-(1/\lambda)} \\
\sigma'^*(t,y) &= \lambda p' \sigma^\delta / (\sigma^*)^{\beta-1} \\
K'(t) &= 3 \int_0^1 p'(t,\xi) \xi d\xi + M'(t) \\
\sigma'(t,y) &= yK' - p'
\end{aligned}
\tag{21}$$

If $\sigma'^* \approx 0$, the first equation of (21) can be replaced by

$$p'(t,y) = (1 - \log \sigma / \log \sigma_0^*)^{-(1/\lambda)} \tag{22}$$

$$\text{Note that } W_{z=1/2}(t) = K(t). \tag{23}$$

(i) $M' = 0$. Fig. 2 and 3 show the time history of stress on a typical cross-section of a rectangular beam in pure bending with two different values of initial hardness σ_0^* . As intuitively expected, the stress redistribution (i.e. change of stress in time) took place more rapidly in the softer material (i.e. smaller σ_0^* , fig. 3) than in the harder material (fig. 2). It is also confirmed by comparing their corresponding values of t_0 , which is a measure of the creep time span (see Appendix). The same conclusion can be drawn by examining figs. 4 and 5 which show the effects of hardness on stress redistribution at two different temperatures. It is interesting to note that, as seen in figs. 2 and 3, there was a small region (in the neighborhood of $y = 0.65$) where the stress almost remained constant. This was called the "skeletal point" by Marriot and Leckie [23].

For large t or more precisely when the stress rate σ' approached

TRANSIENT CREEP IN PURE BENDING

Fig. 2

Fig. 3

Fig. 4

Fig. 5

EFFECT OF HARDNESS ON STRESS REDISTRIBUTION

zero, the stress distribution also approached an asymptotic value as shown in figs. 2 and 3.

The time variation of the maximum deflections at two different temperatures of a beam in pure bending are given in figs. 6 and 7.

Again the influence of initial hardness σ_0^* was obvious, namely a softer material crept at a faster rate.

(ii) $M' > 0$. When the applied moment was not fixed but increasing at a constant rate, the deflection accelerated very rapidly. The effects at two temperatures were fully presented in figs. 8 and 9. The graphs show features similar to those reported in [10].

B. Cantilever beam with lateral force F at the free end ($z = 1$) sufficient to induce moment M_0 at the fixed end ($z = 0$).

Let the elastic deflection at the free end ($z = 1$) be W_0 and define

$$\bar{W} = W/W_0 \quad (24)$$

In this case K is no longer a function of t only but also of z . At $t = 0$, (14), i.e. initial conditions in terms of (24),

$$\begin{aligned} \sigma^*(0, y, z) &= \sigma_0^*, \quad K(0, z) = -1 + z, \\ \sigma(0, y, z) &= (-1+z)y, \quad p(0, y, z) = 0 \text{ and} \\ w(0, z) &= 3(z^2/2 - z^3/6) \end{aligned} \quad (25)$$

For $t \geq 0$, (13), i.e. the rate equations are

$$p'(t, y, z) = (\sigma^*/\sigma_0^*)^m (\log(\sigma^*/\sigma_0^*)/\log \sigma_0^*)^{-(1/\lambda)} \approx (1 - \log \sigma / \log \sigma_0^*)^{-(1/\lambda)} \quad \text{if } \sigma^* \text{ is small}$$

$$\dot{\sigma}^*(t, y, z) = A p' \sigma_0^{\beta-1} / (\sigma^*)^{\beta-1}$$

$$K'(t, z) = 3 \int_0^1 p'(t, \xi, z) \xi d\xi$$

$$\sigma'(t, y, z) = y K' - p' \quad (26)$$

$K, W_z = 1/2$

Fig. 6

TIME VARIATION OF DEFLECTION

Fig. 8

EFFECT OF $\bar{M}'$ ON DEFLECTION

$\sigma_0 = 13.79 \text{ MPa (2000 psi)}$ $T = 250^\circ\text{C}$

Fig. 9

$K, W_z = 1/2$

EFFECT OF $\bar{M}'$ ON DEFLECTION

$\sigma_0 = 51.71 \text{ MPa (7500 psi)}$ $T = 22^\circ\text{C}$

Finally from (2.8)

$$d^2W(t,z)/dz^2 = -K(t,z) / 3 \quad (27)$$

with boundary conditions $W(t,0) = dW(t,0)/dz = 0$.

The deflections at various distances from the fixed end of a cantilever at 250° C. and at room temperature are shown in figs. 10 and 11. Note that the general features of behavior are similar to those of the previous beams.

DEFLECTIONS OF A CANTILEVER BEAM

Fig. 10

$\sigma_0 = 6.89 \text{ MPa (1000 psi)}$ $T = 250^\circ\text{C}$

DEFLECTION OF A CANTILEVER BEAM

$W_{z=1}$

1.2

1.1

1.0

0

2

4

6

8

10

$$\sigma_o^* = 2.12$$

$$t_o = 1.04 \times 10^7 \text{ hr.}$$

$$\bar{t} = t/t_o$$

$$\sigma_o = 34.47 \text{ MPa (5000 psi)} \quad T = 22^\circ\text{C}$$

Fig. 11

STATIONARY STATE

In some engineering designs, an approximate solution to the creep problem at large time is sufficient. The complete transient analysis with initial elastic solutions, the corresponding rate equations and incremental time steps, as illustrated in the last section, is quite involved. Whenever possible, a stationary state analysis, which deals with constant stress distribution (hence constant strain rates) and predicts the asymptotic behavior of the system under consideration, should be carried out first. The stationary creep of a rectangular beam under pure bending is the subject of investigation in this section.

The stationary state with respect to Hart's law is hereby defined as the state at which

$$\sigma' = 0 \text{ and } \sigma^{*'} = 0 \text{ i.e.} \quad (28)$$

both stress and hardness remain unchanged.

For problems with high values of hardness, i.e. $\sigma_0^* \gg 1$, σ^{*} is very small and it is reasonable to assume

$$\sigma^* = \sigma_0^* = \sigma_{ss}^* \quad (29)$$

where ss denotes stationary state.

A rectangular beam under a constant moment M_0 with stationary state quantities σ_{ss} , p_{ss} , K_{ss} and W_{ss} is first analyzed. From (28) and the last equation of (21)

$$p_{ss} = yK_{ss} \quad (30)$$

Substitute (30) in the third equation of (21)

$$p_{ss} = 3y \int_0^1 p_{ss} \xi d\xi \quad (31)$$

or

$$p_{ss} = cy \quad (32)$$

where

$$c = 3 \int_0^1 p'_{ss} \xi d\xi \quad (32)$$

The second equation of (21) is of no interest as $\sigma^* = 0$

but the first one can be written as

$$p'_{ss} = p^*_{ss} (\log (\sigma^*_{ss} / \sigma_{ss}))^{-(1/\lambda)} \quad (33)$$

$$\text{with } p^*_{ss} = (\log \sigma^*_{ss})^{(1/\lambda)}$$

or

$$\sigma_{ss} = \sigma^*_{ss} \exp (-p'_{ss} / p^*_{ss}) \quad (34)$$

Substitute (32) in (34)

$$\sigma_{ss} = \sigma^*_{ss} \exp (-a \xi^{-\lambda}) \quad (35)$$

$$\text{where } a = (p^*_{ss} / c)^\lambda$$

Equilibrium (6) must be satisfied,

$$\int_0^1 \sigma_{ss} \xi d\xi = 1/3 \quad (36)$$

Substitute (35) in (36)

$$\sigma^*_{ss} \int_0^1 \exp (-a \xi^{-\lambda}) \xi d\xi = 1/3 \quad (37)$$

Let $a \xi^{-\lambda} = \eta$ and then (37) is transformed into

$$\int_a^\infty e^{-\eta} \eta^{-(2/\lambda+1)} d\eta = \lambda a^{-(2/\lambda)} / (3 \sigma^*_{ss})$$

$$\text{or } f(a) = \Gamma(-2/\lambda, a) - \lambda a^{-(2/\lambda)} / (3 \sigma^*_{ss}) = 0, \lambda \neq \text{integers} \quad (38)$$

where $\Gamma(-2/\lambda, a)$ is the incomplete Gamma Function. Hence a is the

root of $f(a)$; σ_{ss} , p'_{ss} are then determined through (35) and (32) respectively.

Finally K'_{ss} and W'_{ss} are obtained from (30) and (4).

The stationary state stress distributions as derived from the analysis of this section for various initial hardness σ^*_0 are given in fig. 12.

STATIONARY CREEP IN PURE BENDING

Fig. 12

It is worthwhile to observe that these values and their corresponding asymptotic values shown in figs. 2 and 3 were indistinguishable.

DISCUSSION

The mechanical equation of state proposed by Hart has been verified in uniaxial tension for a variety of metals at high temperature [18] - [20]. The extension of this new theory to predict component behavior in bending is, however, not obvious. The aim of this paper is twofold: to formulate a technical beam theory and then to analyze and discuss results obtained by solving practical problems in light of classical theoretical and experimental results.

From the analyses of the previous sections, it was observed that

(i) All the results indicated strong influence upon creep deformation (and also stress redistribution) of the state variable called "hardness" which is absent in classical creep theories.

(ii) The results from pure bending compared qualitatively with those from classical theories and experiments - a) $M' = 0$, figs. 2-5 with [8], [10] and [11]; figs. 6-7 with [7] and [9]. b) $M' > 0$, figs. 8-9 with [10]. No theoretical analysis of a cantilever was available for comparison but figs. 10-11 showed features similar to those reported in [6].

(iii) Figs. 2-3 confirmed the existence of a "skeletal point" at $y \approx 0.65$ where stress almost remained constant. This was first reported by Marriot and Leckie [23] and is the basis for the concept of reference stress, which has recently become one of the major research interests in design for creep.

(iv) The stationary state solutions given by fig. 12 were indistinguishable from the asymptotic values of the transient analysis shown in figs. 2-3. That served as a check and no additional numerical analysis was attempted in the latter case.

(v) The choice of time steps was crucial in the numerical analysis using Euler's method. Nonlinear steps (small in the beginning) based on stress redistribution have been used. A general guide-line was to choose

$\Delta t < 0.1 \sigma / \sigma'$. Working in terms of the dimensionless constitutive equations given by (A.1), the choice of Δt was independent of temperature and hardness.

(vi) In all the computations, the initial hardness changed during the course of creep by an insignificant amount and hence the assumption made in the stationary state analysis (29) as well as in the simplification of Hart's first equation (22) was justified.

In view of the analyses in the previous sections and the above observations, it is believed that the object of this paper has been achieved. However, more theoretical and experimental work will be needed to construct a rational basis for design.

The complete creep analysis as described in Sections II and III was quite involved, due to the fact that a direct constitutive relation between global parameters, namely, moment, curvature rate and the related state-variable was not available. This will be the subject of a subsequent report; the bending of plates will also be considered.

Finally a component's response to a complex loading history like cyclic loading is the ultimate test of any creep theory. Under that kind of environment, $\dot{\epsilon}^a$, the recoverable strain rate is no longer negligible. An explicit evolution equation is available [21] and analyses including $\dot{\epsilon}^a$ are now being pursued.

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In all calculations based on the previously defined problem, the interval $0 \leq y \leq 1$ was divided into 20 equal segments. The cantilever was divided into 20 segments lengthwise.

The particular material chosen for analysis was 1100 Al whose creep properties were well-defined [20],

$$p' = (\sigma^*/\sigma_0^*)^{5.0} (\log(\sigma^*/\sigma) / \log \sigma_0^*)^{-9.09} \approx (1 - \log \sigma / \log \sigma_0^*)^{-9.09} \text{ if } \sigma^* \text{ is small}$$

$$\sigma^* = \Lambda p' \sigma^{7.82} / \sigma_0^{11.5} \quad ()' = d() / d\bar{t} \quad \bar{t} = t/t_0 \quad (A.1)$$

where $\Lambda = (A/\sigma_0)^{4.68} (\sigma_0/E)$, $A=133.8$ M Pa, σ_0 = maximum elastic stress, and

$$t_0 = (\sigma_0/E)(\sigma_0^*/D)^{-5.0} (\log(\sigma_0^*/\sigma_0))^{9.09} \quad (A.2)$$

The parameters E and D were temperature dependent - $E_{250^\circ\text{C}} = 60.56$ GPa, $E_{22^\circ\text{C}} = 68.86$ GPa, $D_{250^\circ\text{C}} = 0.74$ GPa - sec^{0.2} and $D_{22^\circ\text{C}} = 73.01$ GPa - sec^{0.2}.

Note that D was strongly temperature dependent, and did not appear in the dimensionless constitutive equations as given by (A.1) which were thus temperature insensitive. The effect of temperature and hardness upon creep was included in t_0 (A.2) which also gave an estimate of the creep time span - an invaluable piece of information in design. A wide range of initial hardness $\bar{\sigma}_0^*$ was chosen and the corresponding values of t_0 were listed in the following Table for reference. Note that high values of $\bar{\sigma}_0^*$ were selected at 250°C but low values at 22°C . It is because their alternate choices would lead to very large values of t_0 at 250°C and extremely small t_0 at 22°C - both unacceptable in engineering design.

TABLE

$\bar{\sigma}_o^* = \sigma_o^*/\sigma_o$	$t_o = (\sigma_o/E)(\sigma_o^*/D)^{-m} (\log (\sigma_o^*/\sigma_o))^{1/\lambda} / 3600 \text{ (in hr.)}$
3.54	8.37×10^{-2}
5.31	6.99×10^{-1}
10.62	8.23
21.23	4.26×10
	$T = 250^\circ\text{C}$
1.18	1.93×10
1.42	1.39×10^4
2.12	1.04×10^7
	$T = 22^\circ\text{C}$

$\sigma_o^* = 73.20 \text{ MPa } (1.0617 \times 10^4 \text{ psi})$ equivalent to 10% cold work [20].

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FIGURE CAPTIONS

- fig. 1 A beam with symmetry about one principal axis of bending and its free body diagram.
- fig. 2 Transient creep in pure bending ($\sigma_0^* = 21.23$)
- fig. 3 Transient creep in pure bending ($\sigma_0^* = 3.54$)
- fig. 4 Effect of hardness on stress redistribution
($\sigma_1^* = 5.31$, $\sigma_2^* = 2.66$, $\sigma_0 = 13.79$ MPa (2000psi), $T = 250^\circ\text{C}$)
- fig. 5 Effect of hardness on stress redistribution
($\sigma_1^* = 2.84$, $\sigma_2^* = 1.42$, $\sigma_0 = 51.71$ MPa (7500 psi), $T = 22^\circ\text{C}$)
- fig. 6 Time variation of maximum deflections in pure bending for three values of σ_0^* ($T = 250^\circ\text{C}$)
- fig. 7 Time variation of maximum deflections in pure bending for three values of σ_0^* ($T = 22^\circ\text{C}$)
- fig. 8 Effect of M' on maximum deflection ($\sigma_0 = 13.79$ MPa (2000 psi), $T = 250^\circ\text{C}$)
- fig. 9 Effect of M' on maximum deflection ($\sigma_0 = 51.71$ MPa (7500 psi), $T = 22^\circ\text{C}$)
- fig. 10 Deflections at various distances from the fixed end of a cantilever ($\sigma_0 = 6.89$ MPa (1000 psi), $T = 250^\circ\text{C}$)
- fig. 11 Maximum deflection of a cantilever ($\sigma_0 = 34.47$ MPa (5000 psi), $T = 22^\circ\text{C}$)
- fig. 12 Stationary state creep in pure bending with various hardness σ_0^* .