CONTINUING CRACK TIP DEFORMATION AND FRACTURE
FOR PLANE-STRAIN CRACK GROWTH IN ELASTIC-PLASTIC SOLIDS

by J.R. Rice and E.P. Sorensen[†]

Division of Engineering, Brown University, Providence, R.I. 02912

October 1977

ABSTRACT

Analysis of the deformation field consistent with a Prandtl stress distribution travelling with an advancing plane-strain crack reveals the functional form of the near-tip crack profile in an elastic-plastic solid. The crack opening is shown to have the form $\delta \sim r \ln(\text{const.}/r)$ at distance r from the tip. This observation coupled with data generated from finite element investigations of growing cracks in small scale yielding permits the construction of a relation characterizing the deformation at an extending crack tip. A crack growth criterion consisting of the attainment of a critical opening at a small characteristic material distance from the tip is adopted. Predictions of the stability of a growing crack for both small scale yielding specimens and those subject to general yielding are discussed.

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Department of Energy, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

[†]Present affiliation: General Motors Research Laboratory, Warren, Mich. 48090

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

1. INTRODUCTION

The distributions of stress and strain accompanying stationary cracks in various materials and various modes of loading are available from analytic and numeric solutions, e.g. Rice (1976). These investigations together with extensive experimental studies, e.g., Begley and Landes (1972,1976) and Green and Knott (1975), have led to the characterization of the onset of crack growth in terms of a critical value of the J integral, J_{Ic} for plane strain, or equivalently a critical value of the crack opening displacement, δ_{Ic} , assuming that certain minimum specimen size limits are exceeded. These parameters are useful in quantifying the onset of fracture; however, the question of whether such fracture is initially stable or catastrophic remains unresolved and there is evidence for stable crack growth in many ductile metals, e.g., Clarke et al. (1976), Griffis and Yoder (1976) and Paris et al. (1977).

The analytical investigation of extending cracks is difficult mathematically, although some progress has been made in the Mode III or anti-plane strain case, McClintock and Irwin (1965), and the form of the crack tip singular strain field for such a problem in the tensile or Mode I case has been studied by Rice (1968a; 1975). Results indicate that for crack extension in an elastic-ideally plastic material, the strain at the crack tip is governed by a $\ln r$ singularity as opposed to the stronger $1/r$ singularity in the stationary case. This reduced strain concentration originates in the irreversible straining in the plastic zone surrounding the crack tip so that complete refocusing of the strain at the tip of the extended crack tip is prevented. In this sense, elastic strains are destabilizing, since their reversibility promotes additional straining at an advancing crack tip, Drucker and Rice (1970). The reduced strain concentrations accompanying

extending cracks partly explain the phenomenon of stable crack growth.

Numerical investigations of stable crack growth include the plane-strain studies of Kfourri and Miller (1976) and Sorensen (1977) and the plane stress studies of de Koning (1977) and Andersson (1973). These latter two investigators conclude that a possible parameter for the characterization of continuing fracture is the crack tip opening angle, the angle between the separating crack surfaces behind the extending crack tip. The difficulty with this parameter is that as shown by Rice (1975), an extending crack in an elastic-plastic material exhibits a displacement profile proportional to $r \ln r$ (where r is measured from the crack tip) and this implies a vertical tangent at the crack tip with a correspondingly ambiguous definition of the crack tip opening angle. However, this objection is not present at a small but finite distance away from the crack tip where perhaps a meaningful definition of this parameter may be made.

Kfourri and Miller (1976) propose a Griffith-like critical separation energy as a criterion for continuing fracture. This quantity is defined as the work per unit depth of specimen expended in extending a crack by a distance Δl divided by that distance. Although intuitively appealing, such a separation energy is zero for infinitesimal crack advance when the stress level at the advancing crack tip is finite, Rice (1965). A more serious shortcoming of this quantity is its failure to account for strain accumulations at the crack tip as the critical separation energy is traction controlled. In the limiting case of an elastic-perfectly plastic material at limit load, similar values of the critical separation energy may be obtained for crack advance under the same stress state but vastly different levels of near tip straining. For strain dominated fracture mechanisms

such as void growth and localization of shear it seems inappropriate to employ a fracture parameter which does not reflect the strain state at the crack tip.

The recent experiments of Griffis and Yoder (1976) and Clarke et al. (1976) indicate that under monotonic loading the (nominal) J integral continues to rise monotonically with crack advance. Green and Knott (1975) report similar increases of the (nominal) crack opening displacement with crack extension. These observations have prompted attempts to characterize stable crack growth using J and δ as macroscopically measurable quantities, e.g., Paris et al. (1977). The following discussion lends support to this kind of characterization but also suggests limitations to the universality of the stable growth representations thus obtained.

2. A RELATION FOR CHARACTERIZING CONTINUING FRACTURE

Sorensen (1977) presents finite element solutions for crack advance under well-contained, small scale yielding, plane strain conditions. These solutions simulate crack advance through the finite element grid by the successive relaxation of nodal constraints; the external load is held constant during crack advance and is incremented at constant crack length between crack growth steps of one element each. Typically during the growth process the maximum radius of the plastic zone was 8 to 12 times the size of the near tip elements (although the plastic zone extends directly in front of the crack only over 2 to 3 elements). The material was modelled as an elastic-plastic Mises material satisfying the Prandtl-Reuss flow rule, and strain hardening was modelled by a power law hardening formulation such as that used by Tracey (1976), namely

$$\frac{\bar{\sigma}}{\sigma_0} = \left[\frac{\bar{\sigma}}{\sigma_0} + \frac{3E\bar{e}^P}{2(1+\nu)\sigma_0} \right]^N$$

where $\bar{\sigma}$ is the equivalent tensile stress, σ_0 is the yield strength in tension, $\bar{e}^P$ is the equivalent plastic strain, E is the elastic tensile modulus, ν is Poisson's ratio, and N is the hardening exponent. Numerical results for hardening and non-hardening ($N=0$) analyses indicate that the stress distribution ahead of an advancing crack tip is essentially invariant when plotted against distance normalized by $(K_I/\sigma_0)^2$. Further, this stress distribution is similar to those developed by Tracey (1976) for monotonically loaded stationary cracks. Figure 1 presents a plot of the tensile stress ahead of a crack tip before crack growth and following several increments of crack advance; curves taken from Tracey (1976) are included on the plot. The essentially unchanging stress field (except that given stress

levels extend over wider zones of material as K_I is increased) argues for a deformation dependent fracture criterion, at least when the fracture mechanisms are "strain controlled", to control possible stable crack growth following the initiation of crack advance.

The numerical investigations of Sorensen (1977) provide crack surface displacements throughout the prescribed loading history. Figure 2 presents crack surface profiles prior to crack advance and following the final step of crack growth. For the stationary crack the value of $\delta_t/(J/\sigma_o)$ in the non-hardening case (where δ_t is the crack tip opening displacement, extrapolated to $r = 0$ as shown in the figure, and J is equal to $(1-\nu^2)K_I^2/E$) is 0.66. For the hardening cases we use the HRR (Hutchinson, 1968; Rice and Rosengren, 1968) theoretical result that $\delta \sim r^{N/(1+N)}$ very near the crack tip, and write

$$\delta = \delta_1 (r/r_1)^{N/(1+N)}$$

where δ_1 is the opening displacement at the first node behind the crack tip and r_1 is the position of that node. This extrapolation is shown by the dashed lines on the figure. Then, following the procedure of Tracey (1976), we define δ_t as the opening δ at the place where a 45° line drawn back from the crack tip would intersect the deformed crack surface. Thus we set $r = \delta_t/2$ in the above equation and solve for δ_t to obtain $\delta_t/(J/\sigma_o) = 0.47$ and 0.27 for the two hardening cases $N = 0.1$ and 0.2 , respectively. For comparison, Tracey (1976) presents an approximate relation for δ_t , based on his finite element solutions with singularity elements and the 45° line intersection procedure,

$$\delta_t \approx 0.49 \frac{K_I^2}{E \sigma_o} (1+N) \left[\frac{2(1+\nu)\sigma_o(1+N)}{\sqrt{3} E N} \right]^N$$

and Parks (1975) argues that the factor 0.49 should be approximately 20% higher, at least for the non-hardening case, due to the artificial path dependence of J that seems (through comparison with a corresponding "deformation theory" solution) to be directly traceable to Tracey's non-hardening singularity element. Values of $\delta_t/(J/\sigma_0)$ obtained from the above expression when Poisson's ratio ν is taken equal to .3 and $\sigma_0/E = 1/1000$, corresponding to the value used in Sorensen's (1977) analyses, are 0.65, 0.47 and 0.30 for $N = 0.0, 0.1$ and 0.2 and using Parks' correction to the above equation. The analyses of Tracey (1976) employ a small strain formulation and singular elements, whereas McMeeking (1977) obtains results for crack tip openings using a finite strain formulation. The latter investigator reports values for the non-dimensionalized crack opening displacement between 0.55 and 0.67 in the non-hardening case, between 0.41 and 0.44 for $N = 0.1$ and between 0.27 and 0.30 for $N = 0.2$. Thus we conclude that the above estimates of δ_t based on Sorensen's (1977) numerical results, namely $\delta_t/(J/\sigma_0) = 0.66, 0.47$, and 0.27 for $N = 0.0, 0.1$, and 0.2 , respectively, are in reasonably close agreement with the earlier, and presumably more accurate, solutions of Tracey, Parks and McMeeking for monotonic loading of a stationary crack.

Figure 2 also shows the crack profiles after the final steps of crack advance. These seem to be compatible with the logarithmic dependence of displacement on distance r from the tip noted by Rice (1968a, 1975) for crack extension in an elastic-perfectly plastic material. This logarithmic displacement dependence is now derived and its consequences explored.

2.1 Theoretical analysis of incremental deformations at a growing crack tip

The Prandtl slip line construction, fig. 3, is thought to provide the limiting plane-strain stress distribution, $\sigma_{ij}(\theta)$, as $r \rightarrow 0$ for contained

plastic yielding at the tip of a sharp crack in an elastic-perfectly plastic Mises material, Rice (1968b). The theoretical basis is discussed further by Rice and Tracey (1973) who note that stress fields given by such slip line constructions become asymptotically valid solutions of the 3-D Mises equations, even in elastically compressible materials, provided that $D_{33}^P/D_{\max}^P \rightarrow 0$ as $r \rightarrow 0$. Here D_{ij}^P is the plastic rate of deformation tensor and $D_{\max}^P$ represents its maximum value in the plane of straining; the limit of zero is anticipated since the crack tip is the site of unbounded in-plane strain intensification while the plastic strain in the x_3 direction is bounded by the plane strain constraint and must be the negative of the (bounded) elastic strain. The stress field given by the Prandtl construction arises as the non-hardening limit of the HRR crack tip singular fields for power-law strain hardening materials, Hutchinson (1968) and Rice and Rosengren (1968), and it also agrees closely with numerical stress predictions by finite element methods for stationary cracks in elastic-plastic materials, Levy et al. (1971), Rice and Tracey (1973). Further, recent finite element results by Sorensen (1977) for growing plane strain cracks in elastic-plastic materials likewise validate the Prandtl field as the limiting crack tip stress state in the ideally plastic case.

Accordingly, for the present analysis of deformation fields at the tip of a growing crack, we assume that the Prandtl stress state $\sigma_{ij}(\theta)$ prevails in some small zone near the crack tip and examine the consequences of having this zone move through the material with the advancing crack. As such, our work carries out in detail an integration of the incremental deformation equations as considered in general terms by Rice (1975). Note that for the Prandtl field the deviatoric part, s_{ij} , of the stress tensor is such that $s_{33} = 0$ and $\sigma_{33} = \sigma$ where $\sigma = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$ is the mean

normal stress.

Within the usual assumptions of the "small strain" approximation in stress analysis, the Prandtl-Reuss flow rule takes the form

$$D_{ij} = D_{ij}^e + D_{ij}^p = \left[\frac{1+\nu}{E} \dot{s}_{ij} + \frac{1-2\nu}{E} \delta_{ij} \dot{\sigma} \right] + \dot{\lambda} s_{ij} \quad (1)$$

where $D_{ij} = \frac{1}{2} (\partial v_i / \partial x_j + \partial v_j / \partial x_i)$, v_i is the velocity field, and the distinction between the deformed and undeformed geometry is neglected; E is the elastic modulus, ν the Poisson ratio, and $\dot{\lambda}$ a non-negative scalar.

We observe that

$$\dot{\sigma}_{ij} = \sigma'_{ij}(\theta) \dot{\theta} = \sigma'_{ij}(\theta) \sin \theta \dot{l}/r \quad (2)$$

where $\dot{\theta}$ is computed at a fixed material point and the prime denotes a derivative with respect to θ . Further, within the centered fan zone where the focusing slip lines allow strain singularities to develop $\sigma'(\theta) = -2\tau_0$, where $\tau_0 = \sigma_0/\sqrt{3}$ is the yield strength in shear. Also, letting $e_i(\theta)$ and $h_i(\theta) [= e'_i(\theta)]$ be unit vectors in the r and θ directions respectively, one has

$$e_i s'_{ij} e_j = (e_i s_{ij} e_j)' - 2h_i s_{ij} e_j = \frac{d}{d\theta} (s_{rr}) - 2s_{r\theta} = -2\tau_0 \quad (3)$$

since $s_{rr} = 0$ in the fan and $s_{r\theta} = \tau_0$.

Thus when we multiply both sides of (1) by $e_i e_j$ to form the radial component, we obtain

$$e_i D_{ij} e_j \equiv D_{rr} \equiv \frac{\partial v_r}{\partial r} = -2(2-\nu) \frac{\tau_0}{E} \frac{\sin \theta}{r} \dot{l}, \quad (4)$$

where v_r is the radial velocity.

Hence

$$v_r = 2(2-\nu) \frac{\tau_0}{E} \sin \theta \dot{l} \ln \frac{\bar{R}}{r} + f'(\theta) \quad (5)$$

where $\bar{R}$ is introduced as some effective cut-off radius to the logarithmic term and the function $f'(\theta)$ describes the velocity distribution at $r = \bar{R}$.

By noting that $s_{11} + s_{22} = 0$ in the fan one has also from (1) that

$$D_{11} + D_{22} \equiv \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} = -4(1-2\nu) \frac{\tau_o}{E} \frac{\sin \theta}{r} \dot{\epsilon} \quad (6)$$

From this we can determine v_θ as

$$v_\theta = -2(2-\nu) \frac{\tau_o}{E} \left(\frac{1}{\sqrt{2}} - \cos \theta \right) \dot{\epsilon} \left[-\frac{3\nu}{2-\nu} + \ln \frac{\bar{R}}{r} \right] - f(\theta) + g(r) \quad (7)$$

where $g(r)$ is an arbitrary function of integration.

Since $\sigma'_{ij}(\theta) = 0$ in the "constant stress" zones of the Prandtl field, where both slip lines are straight, the elastic strain rates vanish in those zones and (1) reduces to $D_{ij} = \dot{\epsilon} s_{ij}$. This implies that the normal components of D_{ij} relative to the slip lines vanish and hence the velocity component in the direction of a given slip line is uniform along the length of that line. Thus the function $g(r)$, giving v_θ on $\theta = \pi/4$, must be finite at the crack tip for, otherwise, an unbounded velocity would be propagated along the slip line at the front border of the centered fan zone below the crack tip.

We note that a velocity field of the type described by the terms with $f(\theta)$ and $g(r)$ in the expressions for v_r , v_θ is of the same form as that which describes opening of a stationary crack, Rice (1968b), and thus, as remarked generally by Rice (1975), the total velocity field is a combination of these with logarithmically singular terms proportional to $\dot{\epsilon}$.

Now consider a point on the upper crack surface in the constant stress zone behind the crack tip, fig. 3. If the total opening between upper and lower crack surfaces is δ , $v_2 = \dot{\delta}/2$ and

$$\dot{\delta}/2 = (v_\alpha - v_\beta)/\sqrt{2} \quad (8)$$

Here v_α , v_β are velocities along the slip lines as shown, so $v_\beta = v_\theta$ of the fan with $\theta = 3\pi/4$ and v_α is some bounded velocity in the other slip line direction. Thus the opening rate at a small distance r behind the crack tip is

$$\dot{\delta} = 4(2-\nu) \frac{\tau_o}{E} \dot{\ell} \ln \frac{\bar{R}}{r} + \dot{A} \quad (9)$$

where $\dot{A}$ embodies all terms except the logarithmically infinite term at the tip. In general $\dot{A}$ will be linear in $\dot{\ell}$ and in rates of parameters describing the external loading. If, for example, the intensity of the external loading is correlated in terms of the J integral value in the deformation field far from the crack tip, we can write

$$\dot{\delta} = \alpha \dot{J}/\sigma_o + \beta \frac{\sigma_o}{E} \dot{\ell} \ln \frac{R}{r} \quad (10)$$

where $\beta = 4(2-\nu)/\sqrt{3} = 3.93$ for $\nu = 0.3$, where the additional terms in $\dot{\ell}$ within $\dot{A}$ have been absorbed into the definition of R (which replaces $\bar{R}$), and where α is some dimensionless coefficient. The analysis gives no indication of the value of R but, by dimensional considerations, we expect that it should scale with the size of the crack tip plastic zone.

We observe from (10) significant differences in the kinematics of opening of stationary vs. growing cracks (see also Rice, 1968a, 1975). Load increase ($\dot{J} > 0$) at fixed crack length ($\dot{\ell} = 0$) gives a discrete opening displacement immediately at the tip; its full details can be resolved only by a finite deformation analysis of the tip opening as by Rice and Johnson (1970) and McMeeking (1977). But for $\dot{\ell} > 0$ during load alterations we may integrate (10) to obtain

$$\delta = \frac{\alpha}{\sigma_o} \frac{dJ}{d\ell} r + \beta \frac{\sigma_o}{E} r \ln \frac{eR}{r} \quad (11)$$

where e is the natural logarithm base and, for simplicity, we assume that

r is small enough so that α , R , and dJ/dl can be considered as being sensibly constant during crack length changes of size comparable to r .

This shows that the crack opening displacement is zero at the crack tip although the slope $d\delta/dr$ of the opening is logarithmically infinite at the crack tip. The result suggests, therefore, that a reasonably unambiguous definition of a "crack tip opening angle" can be made only when the dJ/dl term is large enough to effectively dominate the logarithmic term except for very small values of r . Indeed, as Rice (1975) showed, the concept of a crack tip opening angle is unambiguous only for crack growth in rigid-plastic solids; elastic effects lead to the logarithmic singularity of crack surface slope.

We now examine (10) and (11) as a basis for analysis of Sorensen's (1977) finite element results for growing cracks.

2.2 Comparison with finite element predictions of crack surface displacement

Now, for a stationary crack subject to monotonic load, (10) integrates to the well-known relation $\delta_t = \alpha J/\sigma_0$ for the crack tip opening displacement and α is a material dependent non-dimensional number. Sorensen (1977), in a finite element investigation of slow crack growth, prescribes various steps of crack advance followed by increments of external load at constant crack length and finds that the incremental relation

$$d\delta = \alpha dJ/\sigma_0 \quad (12)$$

between nominal crack opening displacement (that at the first node behind the advanced crack tip) and J is applicable for increments of load at constant crack length, with essentially the same α as for the stationary crack solution irrespective of the amount of crack growth. Values of α are within 7% of those corresponding to the stationary crack (see previous discussion) for

for hardening and non-hardening cases. Thus increments in the J value associated with the outer elastic field seem, at least for the small scale yielding case examined, to be related in an almost unique manner to increments in the crack opening very near the tip, although the non-proportional stressing and unloading that accompany crack growth invalidate path-independence of the line integral for J on paths within the crack tip plastic zone. The generality of this result, e.g., at large scale yielding, needs to be examined further.

Taking (12) to relate increments of crack opening displacement to increments of external load at constant crack length, the result in eq. (10) on the preceding page suggests correlation of the finite element results for increments of crack surface displacement accumulated during increments of crack growth with

$$d\delta = \alpha \frac{dJ}{\sigma_0} + \beta \frac{\sigma_0}{E} \ln \frac{R}{r} d\ell, \quad (13)$$

although the second term on the right-hand side of (13) is only rigorously valid as $r \rightarrow 0$ and for perfectly plastic materials. As such, equation (13) is at best approximate for hardening materials and for finite r , but it exhibits the appropriate behavior as $r \rightarrow 0$ in the non-hardening case and for external loading at constant crack length in the small scale yielding case.

Integration of (13) at a fixed position $x (= x_1$ coordinate in fig. 3) as the crack grows from ℓ_1 to ℓ_2 gives

$$\begin{aligned} & \delta(x, \ell_2) - \delta(x, \ell_1) \\ &= \frac{\alpha}{\sigma_0} \left[J(\ell_2) - J(\ell_1) \right] + \beta \frac{\sigma_0}{E} \left[(\ell_2 - x) \ln \left(\frac{eR}{\ell_2 - x} \right) - (\ell_1 - x) \ln \left(\frac{eR}{\ell_1 - x} \right) \right], \end{aligned}$$

where $l_2 > l_1 \geq x$ and to simplify the integration R is treated as constant during the growth. By taking $l_1 = x$, so that $\delta(x, l_1) = 0$, and treating dJ/dl as constant in the growth from l_1 to l_2 , this last equation gives the opening δ at a small distance Δl behind the tip as in (11):

$$\delta = \frac{\alpha}{\sigma_0} \Delta l \frac{dJ}{dl} + \beta \frac{\sigma_0}{E} \Delta l \ln \frac{eR}{\Delta l},$$

or upon rearranging into a form which we will later take as the basis for a fracture criterion,

$$\frac{E}{\sigma_0^2} \frac{dJ}{dl} = \frac{E}{\alpha \sigma_0} \frac{\delta}{\Delta l} - \frac{\beta}{\alpha} \ln \frac{eR}{\Delta l}. \quad (14)$$

The theoretical value of $\beta = 3.93$ based on the Prandtl field analysis has been noted earlier. A value of β may be obtained from the finite element solutions of Sorensen (1977) whose small scale yielding analyses prescribe increments of crack growth at constant external load so that the left side of (14) is zero. The plastic zone size and, indeed, all geometrical sizes in a small scale yielding solution for the monotonic loading of a stationary crack are proportional to EJ/σ_0^2 . We expect the parameter R to scale with the size of the plastic zone and we know from Sorensen's (1977) solutions that the maximum extent of the plastic zone, at a given J , is essentially the same for a stationary and a growing crack. Accordingly, we assume in the interpretation of numerical results that $R = \lambda EJ/\sigma_0^2$ where λ is a pure number to be determined. Hence, from (14) we expect the numerical result for crack growth at constant J to satisfy

$$\frac{\delta}{\Delta l} \frac{E}{\sigma_0} = \beta \ln \frac{\lambda e EJ}{\sigma_0^2 \Delta l}$$

when Δl is a sufficiently small fraction of the plastic zone dimension. Figure 4 presents a plot of the incremental displacement δ of the node immediately behind the crack tip versus $\ln(eEJ/\sigma_0^2 \Delta l)$ for both hardening and non-hardening analyses of one element growth steps at constant J . The close correlation with a straight line shown on the plot, not only for the non-hardening material but also for the hardening cases, lends support to our considerations, but the numerical results are best fit by $\beta = 9.5$ (vs. a theoretical value of 3.9) and $\lambda = 0.04$. The λ value corresponds to an R value which is approximately $1/4$ of the maximum radius of the plastic zone. The discrepancy from the theoretical value of β may arise because of constraints of the finite element grid over the one-element crack growth step, or because the manner of unloading the element to simulate growth cannot precisely simulate continuous crack extension. On the other hand, it is well to remember that fracture can be discontinuous on the microscale and this may result in different effective β values very near the tip over the size scale of the fracture process zone. For the second node behind the crack tip the integration of (13) at constant J is used as a theoretical basis, and, as shown in figure 5, the correlation is not as definite; the fits corresponding to the line shown are $\beta = 4.8$ and $\lambda = 0.16$. This latter value of β is nearer the theoretically predicted value of 3.9 and the value of λ corresponds to a value of R that is approximately equal to the maximum radius of the plastic zone. However, two element sizes may be too far from the crack tip, given the extent of the plastic zone (typically 8 to 12 elements), for the asymptotic formulation for δ to apply. To fully resolve the near tip openings numerically it seems necessary to perform an analysis for a substantially

finer mesh than the already rather fine mesh used by Sorensen (1977), so that the correlation of δ with Δl may be definitively investigated. Such an analysis is costly and appears feasible only if performed on the latest generation of high speed computers, with a grid size such that the steps by which the crack advances are smaller than, say, 1/50 of the maximum extent of the plastic zone. (We arrive at this estimate by noting that the finite element results can be expected to be valid only up to 2 to 3 elements from the crack tip, yet this distance must be very small compared to overall plastic zone dimensions for the asymptotic results of (10) to be valid).

3. INVESTIGATION OF A POSSIBLE CRACK GROWTH CRITERION

Equation (14) may be used to phrase a fracture criterion which permits investigation of the stability of a growing crack in small scale yielding conditions. Take the term $E\delta/\alpha\sigma_0\Delta l$ to be a material dependent constant, in the sense that to achieve a growth step Δl in a material, where the size Δl is considered characteristic of the material, it is assumed that a critical value of the crack opening displacement δ at distance Δl must be achieved. This criterion requires that the near crack tip openings must, in some sense, remain similar in size for continuing crack growth. Note that with R proportional to J in (14), a J level exists at which the dJ/dl required for continued growth approaches zero. Denote J at these "steady state" conditions by J_{ss} . Then the term $E\delta/\alpha\sigma_0\Delta l$ in (14) can be identified in terms of J_{ss} and (14) becomes the differential equation

$$\frac{dJ}{dl} = \frac{\beta}{\alpha} \frac{\sigma_0^2}{E} \ln \left(\frac{J_{ss}}{J} \right) \quad (15)$$

describing the J history necessary for continuing stable crack growth.

This equation may be integrated from initial crack length l_0 to current

crack length l , where a value of J_{Ic} , marking the initiation of growth, is associated with l_0 . Of course, we assume $J_{Ic} < J_{ss}$ and the two parameters together describe the fracture resistance of the material. The result is

$$\frac{\sigma_o^2 (l-l_0)}{E J_{ss}} = \frac{\alpha}{\beta} \left[E_1 \left(\ln \frac{J_{ss}}{J} \right) - E_1 \left(\ln \frac{J_{ss}}{J_{Ic}} \right) \right], \quad (16)$$

where $E_1(x)$ is the exponential integral

$$E_1(x) = \int_x^{\infty} \frac{e^{-u}}{u} du.$$

The fact that $l-l_0$ becomes infinite when $J \rightarrow J_{ss}$ simply reflects the steady state condition which permits infinite crack growth by definition. The second term on the right hand side of (16) is a constant which does not affect the general features of the curves presented on figure 6. These curves are obtained by evaluating $E_1(\ln J_{ss}/J)$ for various values of J/J_{ss} and plotting the results for the two (α, β) pairs (.5, 9.5) and (.5, 3.9) corresponding to a representative value of α and the range of β obtained from numerical and theoretical considerations. The effect of the large β value is to reduce the amount of stable crack growth predicted. The curves on figure 6 are used as follows. For a particular material we know J_{Ic} and J_{ss} and we enter the curve at a level corresponding to J_{Ic}/J_{ss} as at the point so marked. The crack growth $l-l_0$ necessary for J to reach any given level is identified as shown on the curve, and the dimensionless amount of crack growth, $\sigma_o^2(l-l_0)/E J_{ss}$, is measured according to the scale marked on the horizontal axis.

For example, the two points shown correspond to $J_{Ic} = 0.6 J_{ss}$ and crack growth to a J level of $0.95 J_{ss}$. Taking $E/\sigma_o = 300$ and $\delta_{Ic} \approx J_{Ic}/2\sigma_o = 25\mu m$ as representative of medium to high strength brittle materials, the predicted $\ell - \ell_o$ based on the (α, β) pair (0.5, 3.9) is 6mm. For the given ratios of J_{Ic}/J_{ss} and J/J_{ss} , the predicted amount of growth increases proportionally with E/σ_o and with δ_{Ic} .

Equation (14) is now used to investigate initial values of $dJ/d\ell$ and steady state values J_{ss} in terms of parameters of the fracture criterion, assuming that $\delta_{critical} \approx \delta_{Ic} = \alpha J_{Ic}/\sigma_o$, or that the nominal critical opening required at the characteristic distance $\Delta\ell$ for crack advance is approximately equal to that required for the initiation of fracture. Substitution into (14) results in

$$\frac{E}{\sigma_o^2} \frac{dJ}{d\ell} = \frac{E J_{Ic}}{\sigma_o^2 \Delta\ell} - \frac{\beta}{\alpha} \ln \frac{eR}{\Delta\ell} \quad (17)$$

For small scale yielding, $R = \lambda EJ/\sigma_o^2$ so that at steady state conditions (i.e., for $dJ/d\ell = 0$)

$$\frac{E J_{ss}}{\sigma_o^2 \Delta\ell} = \frac{1}{\lambda} \exp \left[\frac{\alpha}{\beta} \frac{E J_{Ic}}{\sigma_o^2 \Delta\ell} - 1 \right] \quad (18)$$

Table I presents some numerical comparisons; the value of $(dJ/d\ell)_{initial}$ may be obtained directly from equation (14) with the substitutions

$\delta = \alpha J_{Ic}/\sigma_o$ and $R = \lambda E J_{Ic}/\sigma_o^2$. The conclusion drawn from Table I is that as the initial value of $dJ/d\ell$ increases, for the small scale yielding case the value of J required for steady state growth increases exponentially. The large effect on J_{ss} of a modest change in β underlines the sensitivity of the present model for continuing fracture to the value of this parameter and emphasizes again the importance of numerical

analysis with a considerably finer mesh so that parameters of the fracture characterization can be more reliably estimated. Note that the dimensionless J_{Ic} value in the table is equal to $R/\lambda \Delta l$ so that as $E J_{Ic}/\sigma_o^2 \Delta l$ decreases, the ratio $R/\Delta l$ decreases. For the present theory to be meaningful, the nominal plastic zone dimension R should presumably be large compared to the growth step Δl . Since Δl is generally expected to be on the order of a few times the crack opening displacement and R is of the order of the plastic zone size it is expected that R be several times Δl . As $R \rightarrow \Delta l$, results from equation (14) reflect the limits of the theory by predicting increasing values of $(dJ/dl)_{init}$ for decreasing values of J_{Ic} . This is why Table I has no entries for the case $\lambda = 0.04$ and $E J_{Ic}/\sigma_o^2 \Delta l = 20$.

Examining the entries on the right in Table I, based on the theoretical β value from the Prandtl field analysis, it is seen that $(dJ/dl)_{init}$ is predicted to fall to zero (i.e., $J_{ss} = J_{Ic}$) when $E J_{Ic}/\sigma_o^2 \Delta l = 15$. This implies that when the dimensionless J_{Ic} value is 15 or less a material will fracture spontaneously in the K_{Ic} test when the criterion for onset of crack growth is met, according to the present analysis, without exhibiting stable crack growth. We note further that

$$\frac{E J_{Ic}}{\sigma_o^2 \Delta l} = \frac{E}{\alpha \sigma_o} \frac{\delta_{Ic}}{\Delta l} \approx 2 \frac{E}{\sigma_o} \frac{\delta_{Ic}}{\Delta l}.$$

Thus materials for which the characteristic growth step Δl is large compared to δ_{Ic} and/or for which the strength level σ_o is high are predicted to be more "brittle" in terms of the present analysis, in that

J_{ss} is less far removed from J_{Ic} and in that the amounts of stable crack growth to reach some level like $0.95 J_{ss}$ are reduced. Indeed, the effect of strength level on the amount of growth is especially pronounced in that we see from figure 6 that the growth $l-l_0$ associated with a given J_{Ic}/J_{ss} ratio is inversely proportional to σ_0^2 . Strain hardening effects are also evident in the above expression for $E J_{Ic}/\sigma_0^2 \Delta l$ in that α decreases with increasing hardening. Thus for a given δ_{Ic} , Δl and σ_0 , increases in the strain hardening exponent increase $E J_{Ic}/\sigma_0^2 \Delta l$ and therefore imply a higher J_{ss}/J_{Ic} . It is clear from the entries in the table that for sufficiently ductile materials, i.e. high values of $(E/\sigma_0)(\delta_{Ic}/\Delta l)$, J_{ss} is so elevated over J_{Ic} that, for all practical purposes, unstable fracture will be impossible under contained yielding and can occur only under fully plastic conditions with continuing increases in imposed deformation (or in the nominal J value) being required for continuing crack growth.

We note, however, that the attainment of J_{ss} is not a necessary requirement for unstable fracture. Rather, instability will occur when, under the given conditions of loading, the increment in the applied J value associated with a given virtual increment of crack extension equals or exceeds the increment of J required to continue to meet the fracture criterion. For example, if Q denotes the intensity of loading, the function $J_A(Q, l)$ denotes the applied J value, and the function $J(l-l_0)$ denotes the crack growth criterion, then

$$J_A(Q, l) = J(l-l_0)$$

must be satisfied throughout the growth process, and instability ($dQ/dl = 0$) occurs when

$$\partial J_A(Q, \ell) / \partial \ell = dJ(\ell - \ell_0) / d\ell .$$

The simultaneous meeting of these two conditions is often represented graphically in terms of a "resistance diagram" (e.g., Rice, 1968a). Thus the value of $dJ/d\ell$ is the proper measure of stability against rapid fracture. As shown by fig. 6, this stability measure decreases from the initial values given in Table 1 with increasing crack growth, at least under the small scale yielding conditions envisioned, and rapid fracture occurs when it has fallen to the critical value associated with the particular cracked configuration under consideration.

3.1 Possible implications for fracture under fully plastic conditions

For a specimen deformed well into the general yielding range the spatial extent of yielding is limited by the dimension of the remaining ligament. Assuming that the analysis leading to (10) and (17) remains approximately valid and that conditions are met under which the near tip deformation increments can be suitably characterized in terms of some outer-field J value, it would appear that the dimension R , reflective of the scale of plastic yielding, would then be proportional by dimensional considerations to the ligament size, the coefficient of proportionality depending on specimen type and hardening. The second term on the right hand side of (17) is then expected to remain nearly constant during crack advance (at least for $(\ell - \ell_0) \ll \text{ligament size}$). Thus equation (17) predicts $dJ/d\ell \approx \text{constant}$. Paris et al. (1977) term the quantity $(E/\sigma_0^2)(dJ/d\ell)$ the tearing modulus, denote it by T , and present experimental data for fully yielded configurations that indicate an effectively constant value of T for crack growth typically on the order of a few mm. The range of T reported by Paris et al. (1977) extends from $T = 221$ for a cast steel

at 149°C (=300°F) to $T = 0.15$ for a stainless steel at -269°C (= -452°F); the majority of T values reported are in the range 25 to 125. The experimental evidence of an approximately constant T value is seen to agree with the predictions of equation (17). However, equation (17) indicates that T may vary between fully plastic specimens of the same material. This is because the nominal plastic zone dimension R may depend on specimen geometry and loading so that although $E J_{Ic} / \sigma_o^2 \Delta l$ may remain constant, T values may differ. The amount of variation may be quite small for highly ductile materials (high $E J_{Ic} / \sigma_o^2 \Delta l$ values) and quite significant for materials of lower ductility. For example, for a material with $E J_{Ic} / \sigma_o^2 \Delta l = 100$, a threefold difference in R causes a change in T of $\approx 8 \ln 3$ or about 8% where $\frac{\beta}{\alpha}$ is taken as 8, and for a material with $E J_{Ic} / \sigma_o^2 \Delta l = 25$, such a difference in R implies a 35% variation in T . These values indicate that the specimen geometry and loading dependence of T in the fully plastic range is controlled by the ductility of the material; for highly ductile materials, T is essentially specimen independent.

This discussion is not necessarily at variance with the experimental results of Begley and Landes (1976) which indicate very different dJ/dl values for fully plastic compact tension and center cracked specimens of the same material. For the ductile rotor steel which they studied this discrepancy may be caused by different paths of crack advance: along 45° lines in the center cracked panel and straight ahead in the compact tension specimen. The occurrence of 45° crack growth would seem to mean that conditions in the center cracked specimens were well removed from those for existence of a unique near-tip HRR singular field, characterized by J . But for 6061-T651 aluminum specimens (J.A. Begley and J.D. Landes,

unpublished private communication) for which crack advance is straight ahead in both specimens, the significantly different T values (approximately 3 for compact tension, 23 for center cracked panel) might be explained by a low value of ductility (implying a low value of $E J_{Ic} / \sigma_0^2 \Delta l$) and different values of R . As discussed by Rice (1976), the region over which the characteristic HRR singularities dominate the strain field is expected, from consideration of the ideally plastic limit, to be considerably larger in a bend specimen than in a fully plastic tensile specimen (if, indeed, any such region exists in the latter case — unpublished numerical results by R.M. McMeeking and D.M. Parks do not suggest one in the fully plastic regime) and the parameter R should have a similar trend. Assuming the same to be true in contrasting compact tension specimens and center cracked panels, the specimen dependent dJ/dl values for aluminum may be consistent with equations (14) and (17) since R is expected to be much smaller for the center cracked case. These considerations are, of course, highly speculative but their more detailed examination seems approachable within the framework that we develop here assuming that suitably mesh-refined numerical calculations are carried out in future work, for fully plastic specimen geometries as well as for the contained yielding case remarked upon earlier.

Finally, let the subscripts gy denote "general yield" and ssy denote "small scale yielding". Then the following equations may be written from (14):

$$\frac{E}{\sigma_0^2} \left(\frac{dJ}{dl} \right)_{gy} = \frac{E J_{Ic}}{\sigma_0^2 \Delta l} - \frac{\beta}{\alpha} \ln \left(e \frac{R_{gy}}{\Delta l} \right),$$

$$\frac{E}{\sigma_0^2} \left(\frac{dJ}{dl} \right)_{ssy} = \frac{E J_{Ic}}{\sigma_0^2 \Delta l} - \frac{\beta}{\alpha} \ln \left(e \frac{R_{ssy}}{\Delta l} \right).$$

Subtracting these equations,

$$\frac{E}{\sigma_0^2} \left[\left(\frac{dJ}{d\ell} \right)_{gy} - \left(\frac{dJ}{d\ell} \right)_{ssy} \right] = \frac{\beta}{\alpha} \ln \frac{R_{ssy}}{R_{gy}} \quad (19)$$

Equation (19) indicates that $(dJ/d\ell)_{gy} > (dJ/d\ell)_{ssy}$ so long as $R_{ssy} > R_{gy}$. This implies that for well contained yielding in large specimens (ssy) the initial $dJ/d\ell$ value is smaller than in much smaller fully plastic specimens (gy), say of the bend type, for which the ligament size is small compared to the plastic zone size in the contained yielding case. In turn, this suggests more stable crack growth behavior for the fully plastic specimens which seems to agree with experimental observations (Paris et al., 1977). Indeed, in these circumstances we expect rounded J versus $\ell - \ell_0$ curves as in figure 6 for contained yielding but essentially straight-line J versus $\ell - \ell_0$ curves, with a larger slope, in the fully plastic specimens.

Finally, we remark on the significance which Paris et al. (1977) attach to the dimensionless crack growth resistance parameter $T \equiv E(dJ/d\ell)/\sigma_0^2$ in fully plastic specimens. By modelling the specimen material as elastic-ideally plastic and considering virtual crack extensions under conditions for which the load-point displacement is held fixed, these authors calculate by methods similar to those of Rice (1975) the additional deformations imposed on the fully plastic uncracked ligament due to the incremental unloading of the elastic regions of the specimen. From this additional deformation they calculate the increment in the (nominal) applied J_A value, and hence calculate the smallest value of the material resistance $dJ/d\ell$ that is consistent with stability against rapid crack extension. The results take the form that for stability T must exceed some critical value which depends only on the geometric configuration (typically, on the ratios of

the ligament dimension to specimen length and width) but not on the specimen material. Paris et al. (1977) suggest from experiments on long edge-cracked bend specimens that a stability criterion of this type is consistent with observations of stable versus unstable crack growth under imposed displacement conditions.

4. CONCLUSIONS

A relation describing the deformation accompanying extending cracks in elastic-plastic solids is obtained from theoretical and numerical calculations. This relation is used as the basis for a fracture criterion for crack advance, and the model seems consistent with differences in experimentally observed features of crack growth under small scale yielding and fully plastic behavior. Namely, more stable growth is predicted for the fully plastic specimens and a nearly constant value of $dJ/d\ell$ results in this case with effects of specimen geometry dependent on material ductility. The quantitative aspects of the relation are less certain due to the imprecise nature of the nominal plastic zone dimension which is fundamental to the fracture model. To quantify more precisely the nominal plastic zone dimension, finite element investigations similar to those of Sorensen (1977) but with a higher degree of mesh refinement in the vicinity of the extending crack tip are desirable, particularly for fully plastic configurations. In addition, it is well to remember that the theoretical considerations of the paper are based on the simplest isotropic hardening (or non-hardening) "flow" theory of plasticity. This may not be fully adequate for characterizing deformations due to the strongly non-proportional stressing paths experienced near a growing crack tip.

Acknowledgement

This study was supported by the U.S. Energy Research and Development Administration under Contract EY-76-S-02-3084 with Brown University.

Table I

$(\lambda = 0.04 ; \beta = 10 ; \alpha = .5)$			$(\lambda = 0.16 ; \beta = 4 ; \alpha = .5)$		
$\frac{E J_{Ic}}{\sigma_0^2 \Delta \ell}$	$\frac{E J_{ss}}{\sigma_0^2 \Delta \ell}$	$\frac{E}{\sigma_0^2} \left(\frac{dJ}{d\ell} \right)_{init}$	$\frac{E J_{Ic}}{\sigma_0^2 \Delta \ell}$	$\frac{E J_{ss}}{\sigma_0^2 \Delta \ell}$	$\frac{E}{\sigma_0^2} \left(\frac{dJ}{d\ell} \right)_{init}$
20	--	--	15	15	0
40	68	11	40	341	17
60	184	22	60	4157	34
80	502	37	80	50644	52
100	1365	52	100	616974	70

REFERENCES

ANDERSSON, H.	1973	"A Finite Element Representation of Stable Crack Growth," <u>Journal of the Mechanics and Physics of Solids</u> , Vol. 21, pp. 337-356.
BEGLEY, J.A. and LANDES, J.D.	1972	"The J Integral as a Fracture Criterion," in <u>Fracture Toughness</u> , ASTM Special Technical Publication 514, pp. 1-23.
BEGLEY, J.A. and LANDES, J.D.	1976	"Serendipity and the J Integral," <u>International Journal of Fracture Mechanics</u> , Vol. 12, pp. 764-766.
CLARKE, G.A., ANDREWS, W.R., PARIS, P.C. and SCHMIDT, D.W.	1976	"Single Specimen Tests for J_{IC} Determination," in <u>Mechanics of Crack Growth</u> , ASTM Special Technical Publication 590, pp. 27-42.
deKONING, A.U.	1976	"A Contribution to the Analysis of Quasi-Static Crack Growth in Sheet Metals", in <u>Fracture 1977</u> (Proc. 4th Int. Cong. Fracture, Waterloo, Canada, 1977); D.M.R. Taplin, ed., University of Waterloo Press, Vol. 3, pp. 25-31.
DRUCKER, D.C. and RICE, J.R.	1970	"Plastic Deformation in Brittle and Ductile Fracture," <u>Engineering Fracture Mechanics</u> , Vol. 1, pp. 577-602.
GREEN, G. and KNOTT, J.F.	1975	"On Effects of Thickness on Ductile Crack Growth in Mild Steel," <u>Journal of the Mechanics and Physics of Solids</u> , Vol. 23, pp. 167-183.
GRIFFIS, C.A. and YODER, G.R.	1976	"Initial Crack Extension in Two Intermediate Strength Aluminum Alloys," <u>Transactions ASME, Journal of Engineering Materials and Technology</u> , Vol. 98, pp. 152-158.
HUTCHINSON, J.W.	1968	"Singular Behavior at the End of a Tensile Crack in a Hardening Material," <u>Journal of the Mechanics and Physics of Solids</u> , Vol. 16, pp. 13-31.
KFOURI, A.P. and MILLER, K.J.	1976	"Crack Separation Energy Rates in Elastic-Plastic Fracture Mechanics," <u>Proceedings of the Institution of Mechanical Engineers</u> , Vol. 190, pp. 571-584.

- LEVY, N., MARCAL, P.V., 1971 "Small Scale Yielding Near a Crack in Plane
OSTERGREN, W.J. and Strain: A Finite Element Analysis," Inter-
RICE, J.R. national Journal of Fracture Mechanics,
Vol. 7, pp. 143-156.
- McCLINTOCK, F.A. and 1965 "Plasticity Aspects of Fracture Mechanics,"
IRWIN, G.R. in Fracture Toughness Testing and Its
Applications, ASTM Special Technical
Publication 381, pp. 84-113.
- McMEEKING, R.M. 1977 "Finite Deformation Analysis of Crack Tip
Opening in Elastic-Plastic Materials and
Implications for Fracture Initiation,"
Journal of the Mechanics and Physics of
Solids, (in press).
- PARIS, P.C., TADA, H., 1977 "Instability of the Tearing Mode of Elastic-
ZAHOOOR, A. and Plastic Crack Growth," to be presented at
ERNST, H. ASTM Symposium on Elastic-Plastic Fracture,
Atlanta, November 1977; submitted for
publication in ASTM STP. (Also, Washington
University Report NRC-77-1, St. Louis, Mo.)
- PARKS, D.M. 1975 "Some Problems in Elastic-Plastic Finite
Element Analysis of Cracks," Ph.D. Disser-
tation, Brown University, Chapter III.
(Available through University Microfilms,
Ann Arbor, Mich.)
- RICE, J.R. 1965 "An Examination of the Fracture Mechanics
Energy Balance from the Point of View of
Continuum Mechanics," in Proceedings of
the 1st International Congress on Fracture,
Sendai, Japan, edited by T. Yokobori et al.,
Japanese Society for Strength and Fracture,
Vol. 1, pp. 309-340.
- RICE, J.R. 1968a "Mathematical Analysis in the Mechanics of
Fracture," in Fracture: An Advanced
Treatise (ed. H. Liebowitz), Vol. 2,
Academic Press, pp. 191-311.
- RICE, J.R. 1968b "A Path Independent Integral and the
Approximate Analysis of Strain Concentra-
tion by Notches and Cracks," Journal of
Applied Mechanics, Vol. 35, pp. 379-386.
- RICE, J.R. 1975 "Elastic-Plastic Models for Stable Crack
Growth," in Mechanics and Mechanisms of
Crack Growth, (Proceedings of Conference
at Cambridge, England, April 1973), edited
by M.J. May, British Steel Corporation
Physical Metallurgy Centre Publication,
pp. 14-39.

- RICE, J.R. 1976 "Elastic-Plastic Fracture Mechanics," in The Mechanics of Fracture, edited by F. Erdogan, American Society of Mechanical Engineers, AMD Vol. 19, pp. 23-53.
- RICE, J.R. and JOHNSON, M.A. 1970 "The Role of Large Crack Tip Geometry Changes in Plane Strain Fracture," in Inelastic Behavior of Solids, M.F. Kanninen et al., eds., McGraw-Hill, pp. 641-672.
- RICE, J.R. and ROSENGREN, G.F. 1968 "Plane Strain Deformation Near a Crack Tip in a Power Law Hardening Material," Journal of the Mechanics and Physics of Solids, Vol. 16, pp. 1-12.
- RICE, J.R. and TRACEY, D.M. 1973 "Computational Fracture Mechanics," in Numerical and Computer Methods in Structural Mechanics, S.J. Fenves et al., eds., Academic Press, pp. 585-623.
- SORENSEN, E.P. 1977 "A Numerical Investigation of Plane Strain Stable Crack Growth Under Small Scale Yielding Conditions," to be presented at the ASTM Symposium on Elastic-Plastic Fracture, Atlanta, November 1977; submitted for publication in ASTM STP (Also Brown University Report EY-76-S-02-3084/55).
- TRACEY, D.M. 1976 "Finite Element Solutions for Crack-Tip Behavior in Small Scale Yielding," Journal of Engineering Materials and Technology, Transactions ASME, Vol. 98, pp. 146-151.

FIGURE CAPTIONS

- Figure 1. σ_{22} distributions ahead of the crack tip, from Sorensen (1977). Points marked "steady state" correspond to final advancing crack solution.
- Figure 2. Crack surface displacements, stationary and final advancing crack solutions for $N = 0.0, 0.1, \text{ and } 0.2$, based on numerical results by Sorensen (1977).
- Figure 3. Prandtl slip line field stress distribution.
- Figure 4. Correlation of crack opening displacement with crack advance at 1st node behind the crack tip in the analyses of Sorensen (1977).
- Figure 5. Correlation of crack opening displacement with crack advance at 2nd node behind the crack tip in the analyses of Sorensen (1977).
- Figure 6. Crack growth curves as predicted by equation (15). The curves are entered at a point corresponding to J_{Ic}/J_{ss} and the dimensionless growth $l-l_0$ corresponding to any J/J_{ss} is measured as shown according to the scale on the horizontal axis.

FIGURE I

FIGURE 2

FIGURE 3

FIGURE 4

FIGURE 5

FIGURE 6