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Maneuverable Reentry Vehicle Trajectory Footprints: Calculation and Properties

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Maneuverable Reentry Vehicle Trajectory Footprints: Calculation and Properties

ABSTRACT

We have obtained an analytical solution of Maneuverable Reentry Vehicle (MARV) trajectory equations that provides useful estimates of a MARV's potential impact footprint on the ground. We concentrate on quick footprint estimation, as required for battle management of a terminal strategic defense system, but we also present other applications such as evaluating tradeoffs between MARV trajectories, design capabilities, and resulting footprints. We estimate the footprint by analytically solving three types of maximal maneuvering trajectories in which the MARV combines maximal turns, lifts or dives through specified angles, finally attempting to extend level flight as long as possible.

We display several quantitative results, for a range of MARV capabilities, including variation of footprint area with trajectory incidence angle, altitude, and lift capability. We find that footprint area decreases rapidly as altitude decreases due to the exponentially increasing atmospheric drag. Further, the eccentricity of the footprints depends sensitively on the altitude, incidence angle and velocity, ranging from 0.1 to 0.8. Footprint area increases as lift capability of the MARV increases, roughly as the $3/2$ power of lift for a trajectory with incidence angle of 20 degrees.

1. Summary

We have obtained an analytical solution of MANeuverable Reentry Vehicle (MARV) trajectory equations that provides useful estimates of a MARV's potential impact footprint on the ground. Although we concentrate on quick footprint estimation, there are as well other applications such as evaluating tradeoffs between MARV trajectories, design capabilities, and resulting footprints. Further, straightforward extension of these results would support quick detailed trajectory solutions.

The results given here provide a method for quickly (in terms of computer time) estimating the footprint of an incoming MARV, a task required for battle management of a terminal strategic defense system. For that purpose, a simple elliptical model of the footprint can be obtained for each incoming MARV by deriving initial conditions from radar tracking data and assuming an appropriate lift capability. The data required are the position, velocity, and an estimate of the drag coefficient. Given these values, or conservative estimates of them, we only need three points to obtain the elliptical model: the maximum and minimum downrange extent, and the maximum crossrange extent. The first two are straightforward evaluations of the solutions here for 0 turn angle. The third point requires finding the maximum crossrange extent over a range of turn angle. Once these points are obtained, the elliptical parameters, semimajor and semiminor axes, are simply half the distance between the x-axis extrema and the value of the y-axis extreme. Placing the ellipse about the aimpoint and performing any coordinate transformations necessary completes the footprint estimate. This calculation is generally quick enough to perform as rapidly as the position (and possibly lift and drag coefficients) are updated.

We can use the footprints to examine quantitatively the trade-offs between different MARV trajectories and advantages of maneuvering. For example, we could calculate at what point the cost of flatter trajectories, with smaller angles of incidence, balances the advantage of larger footprints, resulting in optimal trajectories for various cases of interest.

The results can also be used immediately to determine what effect changes in the capabilities of the MARV would have on the performance in terms of impact footprint, again yielding cost/benefit evaluation of proposed MARV capabilities, such as increased lift or reduced drag.

Finally, the trajectory could be divided into several incremental steps, alternating as necessary between the three solutions given here. The lifting and diving solutions we have obtained are quite general and do not require the final specific angular boundary conditions. Thus we could generate any trajectory by piecing together short lifting, diving and turning segments, saving computer time at the possible expense of approximation error.

2. Results

By 'footprint' we mean the following: Given that the MARV is incident at altitude z_0 , with velocity $\vec{v}$, what is the boundary of the region on the ground, at $z = 0$, that the MARV can reach if it maneuvers maximally? For example, in Figure 1, a MARV is incident with ballistic aimpoint B and velocity $v_x = v \cos \gamma_0$, $v_y = 0$, $v_z = -v \sin \gamma_0$, where γ denotes the angle between the MARV velocity and the horizontal plane. By diving, the MARV could extend its footprint back to $x_{\min}$, by lifting extend out to $x_{\max}$, and by turning extend out to either $y_{\min}$ or $y_{\max}$. The footprint is bounded because the MARV quickly loses speed via drag in the atmosphere, and the drag is increased during maneuvers, which are aerodynamically achieved. While trajectory extension is possible with skipping trajectories or exo-atmospheric maneuvers, here we only examine the aerodynamically controlled phase of the trajectory, typically below about 100 Kft in altitude.

We estimate the footprint by analytically solving three types of maximal maneuvering trajectories in which the MARV combines maximal turns, lifts or dives through specified angles, finally attempting to extend level flight as long as possible. The three trajectories are denoted Trajectory A, B and C, respectively. We briefly describe them here to clarify the results discussed below. A detailed derivation is given in Section 3.

In Trajectory A, the MARV exerts maximal lift to decrease γ from γ_0 to 0, i.e. level flight. Then the MARV continues to exert only enough lift to maintain level flight, but exerts maximal turning force to turn its velocity vector through an angle α_T in the horizontal plane. Finally, the MARV continues to exert only enough lift to fly level until its speed is reduced by drag to a minimum value, v_f , at which time it falls ballistically.

In Trajectory B the MARV exerts maximal turning force aligned so as to turn its velocity vector in the x,y plane through an angle α_T , while maintaining constant $\gamma = \gamma_0$. After turning through α_T , the MARV lifts maximally, with zero turning force, reducing γ from γ_0 to 0, level flight. The MARV then continues to exert just enough lift to hold level flight until it runs out of speed and falls ballistically.

Trajectory C is identical to A, except that the MARV dives maximally, instead of lifting, increasing γ from γ_0 to 180° , then turns through α_T and continues level flight until its speed reaches v_f .

Trajectories A and B approximate maximal front and side reach, Trajectory C provides maximal backward reach. Figures 2 and 3 illustrate the trajectories with top and side views.

We characterize the dynamic capabilities of the MARV by coefficients of lift, turn, and drag, denoted β_γ , β_α , β_d , respectively, and defined as the ratio of the weight of the MARV to the force exerted in lifting, turning, or drag. During maneuvers we set β_γ or β_α equal to the maximal capability of the MARV and constant, and we increase the non-maneuvering drag an amount proportional to the maneuvering force. During level flight the lift is set equal to the weight of the vehicle.

2.1. Computed Footprints and Elliptical Fits

Figure 4 shows a typical computed footprint. The curve labeled Trajectory A connects the end points of A-type trajectories in which the turn angle α_T ranges from zero to a maximal value, about 130° , in steps of 5° . Recall that we set initially $v_y = 0$. For $\alpha_T = 0$, Trajectory A levels off (in this case at about 70 Kft) and cruises until drag reduces its speed to $v_f = 2$ Kft/s at about 1200 Kft downrange from the ballistic aimpoint, shown by a boxed cross. The next point on the curve marked Trajectory A arises from a path that levels off, then turns through $\alpha_T = 5^\circ$, then flies level and straight until $v = v_f$. Each subsequent point on the plot represents a path with α_T increased by 5° . As α_T increases, more of the speed available after leveling off is used in turning so that the straight, level portion of the path shortens, until at $\alpha_T = 130^\circ$, $v = v_f$ at the completion of the turn, where the trajectory ends. Similarly, the curve marked B connects the end points of B-type paths with successively larger values of α_T . Note that on the downrange axis, A and B curves coincide, as they should, when

$\alpha_T = 0$. The curve labeled C is so small for this case that on this scale it appears to be a point (in fact it has 10 points). Curve C connects the points reached by diving trajectories with successive values of α_T . The C-type paths give the maximal uprange extent of the footprint in most cases.

We estimate the footprint as the envelope containing the curves A, B, and C. In principle, we could 'fill in' the footprint boundary between the A, B, and C paths by breaking the trajectories into more than 3 cases and piecing together solutions, but in practice an elliptical model is a reasonable fit to the footprints. Figures 5-9 show footprints calculated for the indicated initial conditions, with γ_0 assuming values of 5, 10, 15, 20, 25, 30, 45, 60, and 85 degrees. All the plots are adjusted so that the ballistic aimpoint is at the origin, and in all cases the initial altitude is 100 Kft.

Figure 5 shows the cases of 15° and 20°. The dashed line shows an elliptical model obtained by letting the semimajor axis, a , be 1/2 the distance between the minimum and maximum downrange extent, and the semiminor axis, b , is set equal to 1/2 the maximal crossrange extent. Since all the curves are symmetric about the x-axis, only three points are required to obtain the elliptical model. For small γ_0 (less than about 15° in this case) the diving trajectory runs out of speed ($v \leq v_f$) and fails to level off. In this case the C curve is a single point significantly uprange of the most uprange portions of the A and B curves. We expect that uprange impact points would comprise a narrow spike uprange from the A and B curves to the C curve. The simple elliptical model slightly overestimates the uprange area of the footprint in these cases. As an example of one simple way of handling these cases, the semi-dotted line shows a footprint obtained by computing an ellipse based only on the A and B curves, then attaching a small triangle at the left with vertices at the C path and the maximal uprange extent of the A and B curves.

In Figures 5-7 γ_0 ranges from 5° to 30° and the footprint size expands as γ_0 decreases. This results from less speed being lost in leveling off and leveling off at a higher altitude, consequently allowing a longer level flight.

Figures 8 and 9 show footprints for steep reentry trajectories with γ_0 equal 45, 60, and 85 degrees. Since the MARV loses altitude quickly, and must lift or dive through a large angle to level off, these footprints are much smaller. Further, the lifting A-type curve shrinks and the diving C-type

curve grows, until at 85° they are nearly equal. The B curve grows and becomes a very circular spiral. At 90° we expect by symmetry a circular footprint with equal lift and dive components.

2.2. Elliptical Footprint Properties

The footprint size, shape, and orientation depend on MARV capabilities and trajectory parameters such as γ_0 , z_0 , and the β parameters. By varying these parameters, we can obtain the footprint of an arbitrary MARV at any point in its trajectory. Figures 10-14 show some of the major features of MARV footprints.

Figure 10 shows how footprint area and eccentricity depend on trajectory angle γ_0 , for a MARV incident at $z_0 = 100\text{Kft}$, with various values of γ_0 . The ellipse eccentricity, $(= \sqrt{1 - b^2/a^2})$, where a and b are the semimajor and semiminor axes of the ellipse, decreases as γ_0 approaches either 0 or 90° . At 90° the MARV is incident straight downward, and its footprint should be symmetric about the z -axis, or circular, with eccentricity 0. We plot only one point with $\cos \gamma_0$ close to 0, but it agrees with our expectation. As discussed above, at small γ_0 the footprint has a sharp backward spike and the simple ellipse slightly overestimates the area by overestimating the semimajor axis. If the area is reduced as discussed, the resulting body of the footprint, neglecting the backward spike, yields the eccentricities shown by triangles, continuing the trend to eccentricity 0. The footprint area, scaled logarithmically, increases smoothly and rapidly with $\cos \gamma_0$, the log increasing by 4.9 or a factor of 135 over the areas plotted.

The footprint area decreases rapidly as altitude decreases due to the exponentially increasing atmospheric drag. Figure 11 plots the $\log(\text{footprint area})$ versus initial altitude, for a range of γ_0 values. Between 30 and 100Kft, areas are plotted for 7 values of γ_0 . The areas shown here are for a MARV incident with γ_0 and initial $v_0 = 22\text{Kft/s}$ at $z_0 = 100\text{Kft}$. As z_0 is decreased in 10 Kft increments, the initial velocity v_0 is decreased by the speed lost to drag along the heretofore straight trajectory. Therefore these plots should be interpreted as showing the footprint area given that a MARV incident with γ_0 begins its maneuver at successively lower altitudes. In general, the areas trend downward toward 0 and the spread narrows. The greater the incidence angle, the smaller the footprint, and the flatter the trend with altitude. The altitude trend is flatter because less velocity is lost

while descending with a steeper trajectory. Note that at about 30Kft the order of some of points is changing, that is the 10° trajectory footprint area is less than all the others, the 45° footprint is smaller than the 60° , etc. This results from the flatter trajectories having traveled more distance and lost more speed to reach the indicated altitudes than the steeper trajectories.

Figure 12 shows the eccentricities of the elliptical footprints for the seven values of γ_0 used in Figure 11. Recall that low eccentricity corresponds to very circular footprints, high eccentricity to very elongated footprints. Note that, as before, each line shows the eccentricity of a footprint given that a MARV waits until altitude z to begin to maneuver. Consequently, at a given altitude, trajectories with greater γ_0 have a greater velocity. Low inclination (10° to 30°) trajectories have low eccentricities (about 0.2) at 100 Kft trend up to about 0.4 at 60 Kft, then trend downward at lower altitudes. Higher inclinations have higher eccentricities at 100 Kft and are more or less flat until about 40 kft, where the eccentricities turn downward. Eccentricity for any given inclination angle varies about 0.2 over the range of altitudes plotted.

Figure 13 shows how the MARV footprint evolves along a particular MARV trajectory. The figure shows footprints along a single trajectory, that begins at 100 Kft with $\gamma_0 = 20^\circ$, and executes a spiraling maneuver. The maneuver is such that the MARV maintains $\gamma = \gamma_0$, but turns to its left constantly. The trajectory is a maximal one, in the sense that if it were continued, it would reach the footprint envelope at the point labeled T. The footprints are plotted after the MARV has turned through 0, 10, 45, and 90° , at $z_0 = 100, 79.5, 54.5$, and 40.7 Kft respectively. Note first that each subsequent footprint is a subset of the previous footprints, as expected since the footprint is designed to be the maximal extent possible at each altitude. Note further that each successive footprint is tangent at the point T, as expected for a maximal trajectory. The current ballistic aimpoint is shown for each stage of the trajectory by the crosses, showing how the (geometrically projected) aimpoint evolves during the path. The footprint area decreases by a factor of 225 between 100 and 40 Kft.

Footprint area increases as lift capability of the MARV increases, roughly as the $3/2$ power of lift for the case of $\gamma_0 = 20$. Figure 14 is a log-log plot of footprint area and lift/drag, where lift/drag = $\beta_{d0} / \beta_\gamma$. In our numerical work here, we have set the lift capability and turn capability equal for sim-

plicity. We have plotted data from $\beta_{d0} / \beta_\gamma = 1$ to 8 using a value of $\beta_{d0} = 1000$ lb-force/sqft, so β_γ ranged from 1000 to 125 lb-force/sqft. All of the points assume $z_0 = 100$ Kft and $\gamma_0 = 20^\circ$. The straight line shown is a simple linear regression of the data that yielded $\log(\text{Area}) = 12 + 1.5 \log(\text{Lift/Drag})$, which is a fair approximation, even though the relation is clearly nonlinear.

3. Analytical Derivation

We obtain analytical estimates of MARV footprints by computing impact points for three specific 'maximally maneuvering' trajectories denoted here A, B, and C. In trajectory A (C), the MARV maximally lifts (dives) to level off, then maximally turns through an angle α_T while flying level, and then continues to fly level until its speed is reduced to a limiting final speed v_f . (Trajectory C is a special case of trajectory A, the case in which the MARV 'lifts' negatively, i.e. dives.) In trajectory B, the MARV first maximally turns through angle α_T while maintaining constant angle between its velocity and the horizontal, then maximally lifts to level off, then cruises at constant altitude until its speed is reduced to v_f .

We use a (flat, stationary earth) rectangular (xyz) coordinate system, with z pointing up and x pointing downrange so the x-y plane is horizontal, the z axis is vertical and directed upwards. The origin is at ground level, directly under the MARV initial position. The angle γ is the angle between the MARV velocity and the x-y plane, and α is the angle in the x-y plane between the x axis and the projection of the velocity on the x-y plane. We denote initial values by $x_0, y_0, z_0, v_0, t_0, \gamma_0, \alpha_0$, and set initial conditions

$$x_0 = y_0 = t_0 = \alpha_0 = 0.$$

We align the x-y plane so that initially $v_x = v_0 \cos \gamma_0$ and $v_y = 0$.

We use dynamical equations describing the motion of a point mass undergoing lift and drag (Martin, 1966):

$$\frac{g \rho v^2}{2 \beta_d} + \dot{v} = g \sin \gamma \quad (1)$$

$$\frac{g \rho v^2}{2 \beta_\gamma} + \dot{\gamma} v = g \cos \gamma \quad (2)$$

$$\frac{g \rho v^2}{2 \beta_\alpha} + v \dot{\alpha} \cos \gamma = 0 \quad (3)$$

where ρ is atmospheric pressure, g is the (assumed constant) gravitational acceleration, and v is the speed of the MARV. Here β_d , β_γ , and β_α , are coefficients describing drag, lift, and turn aerodynamic capabilities, defined as the ratios of MARV weight to drag force, lift force, or turning force respectively, namely W/L_d , W/L_γ , W/L_α . Note that the smaller the magnitude of the beta coefficient, the greater the lift, drag, etc. We adopt the convention that positive β_γ decreases γ , and negative β_γ increases γ , where γ is measured from the horizontal plane to the velocity. A dot over a quantity stands for the time derivative, d/dt of that quantity. We use the following kinematic relations,

$$\dot{x} = v \cos \gamma \cos \alpha \quad (4)$$

$$\dot{y} = v \cos \gamma \sin \alpha \quad (5)$$

$$\dot{z} = -v \sin \gamma \quad (6)$$

Here we solve eqs. (1-3) by assuming simple forms for β_d , β_α , and β_γ , and neglecting gravitational acceleration relative to lift and turn when appropriate, rendering analytically soluble equations and approximating a maximal extension of the RV footprint via maneuvering. We assume that at any time in the trajectory either

- 1 $L_\alpha = 0$ and $L_\gamma = \text{constant and maximal}$ or
- 2 $L_\gamma = 0$ and $L_\alpha = \text{constant and maximal}$ or
- 3 $L_\alpha = \text{constant and maximal}$, $L_\gamma = \text{minimum required to cruise with } \dot{\gamma} = 0$, $\gamma = 0$ or 180° or
- 4 $L_\alpha = 0$ and $L_\gamma = \text{minimal required to maintain level flight, with } \dot{\gamma} = 0$, $\gamma = 0$ or 180° .

Generally, we neglect the control commands, angle of attack, etc. necessary to achieve these values, except insofar as they affect point-mass motion. Therefore, we must increase the drag on the

MARV when lift or turn force is applied. Accordingly we model the drag coefficient as

$$\beta_d = \frac{\beta_{d0}}{(1 + c_1 \frac{\beta_{d0}}{|\beta_\gamma|} + c_2 \frac{\beta_{d0}}{|\beta_\alpha|})} \quad (7)$$

where c_1 and c_2 are positive coupling parameters and β_{d0} is the ballistic drag coefficient when $\beta_\alpha = \beta_\gamma = \infty$.

Below we derive expressions for the MARV position and velocity in each of the trajectories considered. For reference convenience, the resulting formulae are collected in the Appendix.

3.1. Trajectory A (and C)

Trajectory A has three ordered parts which we denote $\Delta\gamma$, $\Delta\alpha$, and cruise. In the derivation below, the results for Trajectory A or C are alternately obtained by taking the final $\gamma = \gamma_f$ to be 0 or 180° respectively, and letting $\beta_\gamma > 0$ for Trajectory A, and $\beta_\gamma < 0$ for Trajectory C.

Part 1, $\Delta\gamma$.

The MARV is incident with initial conditions specified above. At altitude z_0 , the MARV initiates lift and β_γ assumes a constant value. If $\beta_\gamma > 0$ the MARV lifts until γ decreases to 0. If $\beta_\gamma < 0$ the MARV dives and γ increases to 180 degrees. The magnitude of β_γ is a parameter chosen to characterize a particular MARV's maximal lift/dive capability. No turning force is applied. In this case, eqs. (1-7) become

$$\frac{g \rho v^2}{2 \beta_d} + \dot{v} = 0 \quad (8)$$

$$\frac{g \rho v^2}{2 \beta_\gamma} + \dot{\gamma} v = 0 \quad (9)$$

$$\dot{x} = v \cos \gamma \quad (10)$$

$$\dot{y} = 0 \quad (11)$$

$$\dot{z} = -v \sin \gamma \quad (12)$$

$$\beta_d = \frac{\beta_{d0}}{1 + c_1 \beta_{d0} / |\beta_\gamma|} \quad (13)$$

We neglect gravity in (8) and (9) because the lift force in this phase of the trajectory is typically much greater than gravity, particularly at altitudes less than about 100 Kft, i.e. $\frac{\rho v^2}{2 \beta_d} \gg 1$. We combine (9) and

(12) for

$$\frac{dz}{d\gamma} = \frac{\dot{z}}{\dot{\gamma}} = \frac{2 \beta_\gamma \sin \gamma}{g \rho}$$

or

$$\rho(z) dz = \frac{2 \beta_\gamma}{g} \sin \gamma d\gamma. \quad (14)$$

With an exponential atmosphere, $\rho = \rho_0 e^{-kz}$, $k = (\text{scale height } H)^{-1}$, eq. (14) has solution

$$\cos \gamma = \cos \gamma_0 + \frac{gH}{2 \beta_\gamma} [\rho(z) - \rho(z_0)] \quad (15)$$

relating z and γ .

Now we use (10) and (12) to express

$$\frac{dx}{dz} = \frac{\dot{x}}{\dot{z}} = \frac{-\cos \gamma}{\sqrt{1 - \cos^2 \gamma}}$$

With (15) and some manipulation this yields

$$\int_{z_0}^{z_1} dx = H \int_{\cos \gamma_0}^{\cos \gamma} \frac{ds (s + B_0)}{s (1 - B_0^2 - 2 B_0 s - s^2)^{1/2}}$$

where $s = \frac{g \rho_0 H}{2 \beta_\gamma} e^{-kz}$ and $B_0 = \cos \gamma_0 - s(z_0)$. This integral has solution (Gradshteyn and

Ryzhik, 1980, 2.261 and 2.266)

$$x_1 - x_0 = H(\gamma_0 - \gamma) \quad (16a)$$

$$- \frac{B_0 H}{\sqrt{1 - B_0^2}} \ln \left[\frac{\eta}{(\cos \gamma - B_0)} \frac{(1 - B_0 \cos \gamma + \sin \gamma \sqrt{1 - B_0^2})}{(1 - B_0 \cos \gamma_0 + \sin \gamma_0 \sqrt{1 - B_0^2})} \right]$$

for $B_0^2 < 1$ and

$$x_1 - x_0 = H(\gamma_0 - \gamma_1) - \frac{HB_0}{\sqrt{B_0^2 - 1}} \left[\arcsin \left[\frac{1 - B_0 \cos \gamma_1}{\cos \gamma_1 - B_0} \right] - \arcsin \left[\frac{1 - B_0 \cos \gamma_0}{\cos \gamma_0 - B_0} \right] \right] \quad (16b)$$

for $B_0^2 > 1$. Here $\eta = \frac{gH\rho(z_0)}{2\beta_Y}$.

Finally, we use (8), (9), and (13) to relate γ and v by

$$\frac{d\gamma}{dv} = \frac{\dot{\gamma}}{\dot{v}} = \frac{\beta_d}{\beta_Y v}$$

resulting in

$$v_1 = v_0 \exp \left[\frac{\beta_Y}{\beta_d} (\gamma - \gamma_0) \right] \quad (17)$$

Since $\dot{y} = 0$,

$$y_1 = y_0 = 0. \quad (18)$$

We define x_1 , y_1 , z_1 , and v_1 to be the MARVs description at the end of its $\Delta\gamma$ maneuver. If leveling off was achieved, these values are given by eqs (15)-(18), with $\gamma = \gamma_l = 0$ or 180 degrees. If leveling off could not be achieved before z became zero or v became v_f then x_1 and y_1 give the edge of the footprint with $y_1 = 0$ and x_1 given by eq (16) evaluated at γ_l .

For convenience the formulae for Part 1 are given in Appendix A.

Part 2, $\Delta\alpha$.

In this part of the trajectory, the MARV exerts just enough lift to maintain horizontal flight ($\gamma = \gamma_l = 0$ or 180 degrees) and turns maximally through an angle α_T . At the end of the maneuver, the

MARV is at x_2, y_2, z_2 , which we determine. Eqs. (1-3) now become

$$\frac{g \rho v^2}{2 \beta_d} + \dot{v} = 0 \quad (19)$$

$$\frac{g \rho v^2}{2 \beta_\gamma} - g \cos \gamma_l = 0 \quad (20)$$

$$\frac{g \rho v^2}{2 \beta_\alpha} + \dot{\alpha} v \cos \gamma_l = 0 \quad (21)$$

In (19) we neglect gravity since g is still small compared to the drag deceleration. We use gravity in (20) to set the lift required to maintain level flight, i.e. to determine β_γ . The $\cos \gamma_l$ terms above explicitly account for the sign change between lifting and diving trajectories. By our sign convention, a trajectory that lifts to achieve level flight does so with $\beta_\gamma > 0$ ending with $\gamma_l = 0$. Similarly, a trajectory that dives to achieve level flight does so with $\beta_\gamma < 0$, reaching $\gamma_l = 180$. We explicitly use $\cos \gamma_l$ to account conventionally for the sign of gravity, relative to the direction of the lifting force. Eq. (20) re-introduces gravity into eq. (2) with $\dot{\gamma} = 0$. The result is an expression for β_γ required to supply enough lift to maintain level flight, an expression that is required to adjust the drag coefficient. The drag coefficient in (19) then becomes

$$\beta_d = \frac{\beta_{d0}}{1 + c_2 \frac{\beta_{d0}}{|\beta_\alpha|} + 2 c_1 \frac{\beta_{d0}}{\rho v^2}} \quad (22)$$

In level flight, $z_2 = z_1$ and $\rho = \rho(z_1)$, so combining (4) and (21) gives

$$\frac{dx}{d\alpha} = \frac{\dot{x}}{\dot{\alpha}} = \frac{-2 \beta_\alpha \cos \gamma_l \cos \alpha}{g \rho(z_1)}$$

The result is

$$x_2 - x_1 = \frac{2 \beta_\alpha \cos \gamma_l \sin \alpha}{g \rho(z_1)} \quad (23)$$

Similarly, eq. (5) combines with (21) to yield y as a function of α

$$y_2 - y_1 = \frac{2 \beta_\alpha \cos \gamma_l (\cos \alpha - 1)}{g \rho(z_1)} \quad (24)$$

Finally we obtain $v_2(\alpha)$ by combining (19) and (21) using (22), for

$$\int_{\alpha_1=0}^{\alpha_2} \frac{\beta_\alpha}{\beta_{d0}} d\alpha = \int_{v_1}^{v_2} \frac{v dv}{\frac{2 c_1 \beta_{d0}}{\rho(z_1)} + v^2 \left(1 + \frac{c_2 \beta_{d0}}{|\beta_\alpha|}\right)}$$

which yields

$$\frac{A + C v_2^2}{A + C v_1^2} = \exp \left[2C(\alpha_2 - \alpha_1) \frac{\beta_\alpha}{\beta_{d0}} \right] \quad (25)$$

where $C = 1 + c_2 \frac{\beta_{d0}}{|\beta_\alpha|}$ and $A = \frac{2 c_1 \beta_{d0}}{\rho(z_1)}$. With $z_2 = z_1 = \text{constant}$, eqs. (23)-(25) give the position and velocity of the MARV as a function of turn angle completed. If for $\alpha_2 = \alpha_T$, $v_2 > v_f$, then Part 2 is completed and Part 3 begun. Otherwise the maximal x-y extent of the footprint is x_2 and y_2 such that $v_2 = v_f$.

Part 3, Cruise.

In this part of the trajectory, the MARV exerts just enough lift to maintain constant altitude and cruises in a straight line until $v = v_f$. The final position is x_3, y_3, z_3 . Hence eqs (1)-(6) become

$$\frac{g \rho v^2}{2 \beta_d} + \dot{v} = 0 \quad (26)$$

$$\frac{g \rho v^2}{2 \beta_\gamma} - g \cos \gamma_l = 0 \quad (27)$$

$$\dot{\alpha} = 0 \quad (28)$$

$$\dot{x} = v \cos \gamma_l \cos \alpha_T \quad (29)$$

$$\dot{y} = v \cos \gamma_l \sin \alpha_T \quad (30)$$

$$\dot{z} = -v \sin \gamma_l = 0 \quad (31)$$

The drag coefficient is again given by (22), so we combine (26) with (29) for

$$\frac{dx}{dv} = \frac{\dot{x}}{\dot{v}} = \frac{-\cos \gamma_l \cos \alpha_T v}{g c_1 \left[1 + \frac{\rho(z_2) v^2}{2 c_1 \beta_{d0}} \right]}$$

resulting in

$$x_3 - x_2 = \frac{-\beta_{d0} \cos \alpha_T \cos \gamma_i}{g \rho(z_2)} \ln \left[\frac{c_1 + \frac{\rho(z_2) v_3^2}{2 \beta_{d0}}}{c_1 + \frac{\rho(z_2) v_2^2}{2 \beta_{d0}}} \right] \quad (32)$$

Similarly, (26) and (30) yield

$$y_3 - y_2 = \frac{-\beta_{d0} \sin \alpha_T \cos \gamma_i}{g \rho(z_2)} \ln \left[\frac{c_1 + \frac{\rho(z_2) v_3^2}{2 \beta_{d0}}}{c_1 + \frac{\rho(z_2) v_2^2}{2 \beta_{d0}}} \right] \quad (33)$$

Setting $v_3 = v_f$ in the above gives our maximal footprint extension for this trajectory.

3.2. Trajectory B

Trajectory B has three ordered parts, denoted $\Delta \alpha$, $\Delta \gamma$, and cruise. Since we have symmetry about the y-axis, we only solve one case, $\alpha \geq 0$, which requires $\beta_\alpha < 0$ below.

Part 1, $\Delta \alpha$

The MARV is incident with initial conditions x_0, y_0, z_0 , etc. It exerts maximal turning force so aligned to rotate its velocity vector about the z axis, through an angle α_T , keeping $\gamma = \gamma_0$. The equations describing this maneuver are (1)-(6) with $\gamma = \gamma_0$ and $\dot{\gamma} = 0$,

$$\frac{g \rho v^2}{2 \beta_d} + \dot{v} = 0 \quad (34)$$

$$\dot{\gamma} = 0 \quad (35)$$

$$\frac{g \rho v^2}{2 \beta_\alpha} + v \dot{\alpha} \cos \gamma_0 = 0 \quad (36)$$

$$\dot{x} = v \cos \gamma_0 \cos \alpha \quad (37)$$

$$\dot{y} = v \cos \gamma_0 \sin \alpha \quad (38)$$

$$\dot{z} = -v \sin \gamma_0 \quad (39)$$

In the above, gravity has been neglected since it is small relative to the drag and turning forces. Combining eqs (34) and (39) gives

$$\frac{g \rho(z) dz}{2 \beta_\alpha \cos \gamma_0 \sin \gamma_0} = d\alpha$$

resulting in

$$\alpha - \alpha_0 = \frac{gH(e^{-kz_0} - e^{-kz})}{2 \beta_\alpha \cos \gamma_0 \sin \gamma_0} \quad (40)$$

For x and y as functions of α we use (37)-(39) for

$$\begin{aligned} \int_{x_0}^{x_1} dx &= -\cot \gamma_0 \int_{z_0}^{z_1} dz \cos(\alpha(z)) \\ \int_{y_0}^{y_1} dy &= -\cot \gamma_0 \int_{z_0}^{z_1} dz \sin(\alpha(z)) \end{aligned}$$

Using (40) in the above integrals and manipulating a little yields (via Gradshteyn & Ryzhik, 1980, 2.641),

$$x_1 - x_0 = H \cot \gamma_0 \left[\cos(S_0) \text{ci}(S) + \sin(S_0) \text{si}(S) \right]_{S_0}^S \quad (41)$$

$$y_1 - y_0 = -H \cot \gamma_0 \left[\sin(S_0) \text{ci}(S) - \cos(S_0) \text{si}(S) \right]_{S_0}^S \quad (42)$$

where

$$S = \alpha + S_0 = \frac{gH \rho_0 e^{-kz}}{2 \beta_\alpha \cos \gamma_0 \sin \gamma_0}$$

and $S_0 = S(z_0) = \frac{gH \rho_0 e^{-kz_0}}{2 \beta_\alpha \cos \gamma_0 \sin \gamma_0}$. The sine and cosine integrals in (41) and (42) are widely tabulated and available in most mainframe computing software libraries (See e.g. Gradshteyn and Ryzhik,

1980, p. 928 for definitions of si and ci.)

To complete the $\Delta\alpha$ solution, we solve for $v(\alpha)$ using eqs. (31)-(33),

$$v(\alpha) = v_0 \exp \left[(\alpha_0 - \alpha) \frac{\beta_\alpha}{\beta_d} \right] \quad (43)$$

where $\beta_d = \beta_{d0} / (1 + \frac{\beta_{d0}}{|\beta_\alpha|})$. In Part 1, the MARV turns until $\alpha = \alpha_T$, at which time if $v > v_f$ the MARV begins Part 2, lifting.

Part 2, $\Delta\gamma$.

In Part 2, the MARV exerts pure lifting force, keeping α constant. As in the pure lift part of Trajectory A, our dynamics are given by eqs (1) and (2), and (4)-(6) are now

$$\dot{x} = v \cos \gamma \cos \alpha_T \quad (44)$$

$$\dot{y} = v \cos \gamma \sin \alpha_T \quad (45)$$

$$\dot{z} = -v \sin \gamma. \quad (46)$$

Since (46) is unchanged from (6), equation (15) again relates γ and z . Inspecting (44), we observe that $x(\gamma)$ will be given here by the result (16) as in Trajectory A, Part 1, multiplied by $\cos \alpha_T$, and $y(\gamma)$ will be given by the result (16) multiplied by $\sin(\alpha_T)$. Further, the speed $v(\gamma)$ is given again by eq. (17).

Part 3, Cruise.

Part 3 of this trajectory is identical in form to Part 3 of Trajectory A. The MARV cruises at constant altitude and constant $\alpha = \alpha_T$ until its speed is reduced to v_f . The result is that equations (33) and (34) again give the x and y extension of the footprint resulting from the cruise phase of the trajectory.

Complete expressions for the footprint boundary values are collected in the Appendix.

REFERENCES

1. I. S. Gradshteyn and I. M. Ryzhik, "Table of Integrals, Series and Products," Academic Press, 1980.
2. John J. Martin, "Atmospheric Reentry," Prentice-Hall, 1966.

Appendix

1. Trajectory A and C

For trajectory A, γ ranges from γ_0 to $\gamma_1 = 0$, and $\beta_\gamma > 0$. For Trajectory C, γ ranges from γ_0 to $\gamma_1 = 180^\circ$, and $\beta_\gamma < 0$. Here $k = (\text{scale height } H)^{-1}$, $\rho_0 = \rho(z = 0)$, $\eta = \frac{gH\rho_0 e^{-kz_0}}{2\beta_\gamma}$, and $B_0 = \cos \gamma_0 - \eta$.

Part 1

$$x_1 - x_0 = H(\gamma_0 - \gamma_1) - \frac{B_0 H}{\sqrt{1 - B_0^2}} \log \left[\frac{\eta}{(\cos \gamma - B_0)} \frac{(1 - B_0 \cos \gamma + \sin \gamma \sqrt{1 - B_0^2})}{(1 - B_0 \cos \gamma_0 + \sin \gamma_0 \sqrt{1 - B_0^2})} \right]$$

for $B_0^2 < 1$ and

$$x_1 - x_0 = H(\gamma_0 - \gamma_1) - \frac{HB_0}{\sqrt{B_0^2 - 1}} \left[\arcsin \left[\frac{1 - B_0 \cos \gamma_1}{\cos \gamma_1 - B_0} \right] - \arcsin \left[\frac{1 - B_0 \cos \gamma_0}{\cos \gamma_0 - B_0} \right] \right]$$

for $B_0^2 > 1$.

$$y_1 - y_0 = 0.$$

$$z_1 = -H \log \left[\frac{2\beta_\gamma}{g\rho_0 H} \left[\frac{g\rho_0 H e^{-kz_0}}{2\beta_\gamma} + \cos \gamma - \cos \gamma_0 \right] \right]$$

$$v_1 = v_0 \exp \left[\frac{\beta_\gamma}{\beta_d} (\gamma - \gamma_0) \right].$$

Part 2

$$x_2 - x_1 = \frac{-2 \beta_\alpha \cos \gamma_l \sin \alpha}{g \rho(z_1)}$$

$$y_2 - y_1 = \frac{2 \beta_\alpha \cos \gamma_l (\cos \alpha - 1)}{g \rho(z_1)}$$

$$z_2 - z_1 = 0$$

v_2 obtained from:

$$\frac{A + C v_2^2}{A + C v_1^2} = \exp \left[2C (\alpha_T - \alpha_0) \frac{\beta_\alpha}{\beta_{d0}} \right]$$

$$\text{where } C = 1 + c_2 \frac{\beta_{d0}}{|\beta_\alpha|} \text{ and } A = \frac{2 c_1 \beta_{d0}}{\rho(z_1)}$$

Part 3

$$x_3 - x_2 = \frac{-\beta_{d0} \cos \alpha_T \cos \gamma_l}{g \rho(z_2)} \log \left[\frac{c_1 + \frac{\rho(z_2) v_3^2}{2 \beta_{d0}}}{c_1 + \frac{\rho(z_2) v_2^2}{2 \beta_{d0}}} \right]$$

$$y_3 - y_2 = \frac{-\beta_{d0} \sin \alpha_T \cos \gamma_l}{g \rho(z_2)} \log \left[\frac{c_1 + \frac{\rho(z_2) v_3^2}{2 \beta_{d0}}}{c_1 + \frac{\rho(z_2) v_2^2}{2 \beta_{d0}}} \right]$$

$$z_3 - z_2 = 0$$

$$v_3 = v_f$$

2. Trajectory B

Here $\beta_\gamma > 0$ and $\alpha_T > 0$.

Part 1

$$x_1 - x_0 = H \cot \gamma_0 \left[\cos(S_0) ci(S) + \sin(S_0) si(S) \right]_{S_0}^S$$

$$y_1 - y_0 = -H \cot \gamma_0 \left[\sin(S_0) ci(S) - \cos(S_0) si(S) \right]_{S_0}^S$$

where

$$S = \alpha + S_0 = \frac{gH \rho_0 e^{-kz}}{2 \beta_\alpha \cos \gamma_0 \sin \gamma_0}$$

$$\text{and } S_0 = S(z_0) = \frac{gH \rho_0 e^{-kz_0}}{2 \beta_\alpha \cos \gamma_0 \sin \gamma_0}$$

$$z_1 = -H \log \left[\frac{2 \beta_\alpha \cos \gamma_0 \sin \gamma_0}{gH \rho_0} (\alpha_0 - \alpha_T) + e^{-kz_0} \right]$$

$$v_1(\alpha) = v_0 \exp \left[(\alpha_T - \alpha_0) \cos \gamma_0 \frac{\beta_\alpha}{\beta_d} \right]$$

Part 2

$$x_2 - x_1 = \cos(\alpha_T - \alpha_0) \left[H(\gamma_0 - \gamma) - \frac{B_1 H}{\sqrt{1 - B_1^2}} \log \left[\frac{\cos \gamma \eta_1}{1 - B_1 \cos \gamma + \sin \gamma \sqrt{1 - B_1^2}} \right] \right]$$

when $B_1^2 < 1$.

$$x_2 - x_1 = \cos(\alpha_T - \alpha_0) \left[H(\gamma_0 - \gamma) - \frac{B_1 H}{\sqrt{B_1^2 - 1}} \arccos \left[\frac{\cos \gamma}{\eta_1} (1 - B_1 \cos \gamma) \right] \right]$$

when $B_1^2 > 1$.

$$y_2 - y_1 = \tan(\alpha_T - \alpha_0) [x_2 - x_1]$$

$$z_2 - z_1 = 0$$

$$v_2 = v_1 \exp \left[-\gamma_0 \frac{\beta_\gamma}{\beta_{d0}} \left(1 + c_1 \frac{\beta_{d0}}{\beta_\gamma} \right) \right]$$

Part 3

$$x_3 - x_2 = -\cos(\alpha_T - \alpha_0) \frac{\beta_d}{g \rho(z_2)} \log \left[\frac{c_1 + \frac{\rho(z_2) v_f^2}{2 \beta_d}}{c_1 + \frac{\rho(z_2) v_2^2}{2 \beta_d}} \right]$$

$$y_3 - y_2 = \tan(\alpha - \alpha_0) [x_3 - x_2]$$

$$z_3 - z_2 = 0$$

$$v_3 = v_f$$

MARV FOOTPRINT

Figure 1

TRAJECTORIES: TOP VIEW

Downrange (arbitrary scale)
All start at (0,0,20) with same velocity and gamma0

Figure 2

TRAJECTORIES: SIDE VIEW

Downrange (arbitrary scale)
All start at (0,0,z0) with same velocity and gamma0

Figure 3

TYPICAL MARV FOOTPRINT

Downrange Displacement in Kft.
 $z_0=100\text{Kft.}$ $\gamma=0-20\text{deg.}$ $\beta_{tag}=-\beta_{taa}=\beta_{ad}/4=250.$ $c_1=c_2=0.5$

Figure 4

COMPUTED FOOTPRINTS AND ELLIPTICAL FITS

GAMMA-ZERO = 15 DEGREES

GAMMA-ZERO = 20 DEGREES

Figure 5

$z_0=100\text{Kft.}$ $\text{betag}=-\text{betaa}=\text{betad}/4=250.$ $c_1=c_2=0.5.$ $v_0=22\text{Kfps}$

COMPUTED FOOTPRINTS AND ELLIPTICAL FITS

GAMMA-ZERO = 5 DEGREES

GAMMA-ZERO = 10 DEGREES

Figure 6

$z_0=100\text{Kft.}$ $\text{beta}_g=-\text{beta}_a=\text{beta}_d/4=250.$ $c_1=c_2=0.5.$ $v_0=22\text{Kfps}$

COMPUTED FOOTPRINTS AND ELLIPTICAL FITS

GAMMA-ZERO = 25 DEGREES

GAMMA-ZERO = 30 DEGREES

Figure 7

z0=100Kft, betag=-betaa=betad/4=250, c1=c2=0.5, v0=22Kfps

COMPUTED FOOTPRINTS AND ELLIPTICAL FITS

GAMMA-ZERO = 45 DEGREES

GAMMA-ZERO = 60 DEGREES

Figure 8

z0=100Kft. betag=-betaa=betad/4=250. c1=c2=0.5. v0=22Kfps

GAMMA-ZERO = 85 DEGREES

Downrange Kft.
z0=100Kft, betag=-betaa-betad/4=250, c1=c2=0.5, v0=22Kfps

Figure 9

ELLIPTICAL MODEL PROPERTIES

Gamma-Zero & Eccentricity

Gamma-Zero & Footprint Area

Figure 10

ALTITUDE AND FOOTPRINT AREA

Velocities are reduced by drag from 22 Kft/s at 100 Kft.

Figure 11

ALTITUDE AND ECCENTRICITY

Initial Altitude in Kft
Line labels indicate different gamma-zero values in degrees.

Figure 12

FOOTPRINT EVOLUTION DURING FLIGHT

Downrange Kft.

Footprints 1. 2. 3. 4 are successively later. + 's show aimpoint evolution.

Figure 13

LIFT COEFFICIENT AND FOOTPRINT AREA

Log(Lift/Drag)
Solid line is a linear regression.

Figure 14