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BEHAVIOR OF A BI-MATERIAL CYLINDER
INCLUDING THE EFFECTS OF FRACTURE

by

Roy L. Johnson, Jr.

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ABSTRACT

A finite element method is presented which utilizes a three dimensional quadrilateral finite element with a stiffness matrix developed in cylindrical coordinates. The method is used to study the behavior of layered cylinders due to asymmetric static thermal loadings. Two types of bonding, rigid and flexible are considered. Additional examples including a pressurized thick walled cylinder and a cylindrical shell under pressure and thermal loadings are presented.

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The closed-form solutions referred to in the text and discussed in the Appendix are the work of S. W. Key.

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LIST OF SYMBOLS

Following the description of each symbol is the equation number in which the symbol is first used.

a	local oblique coordinate; (1)
b	local oblique coordinate; (1)
$[b]$	strain displacement matrix; (12)
$[C]$	matrix of orthotropic elastic stress constants; (20)
c	local oblique coordinate; (1)
D_{ijrs}	component of anisotropic elastic flexibility tensor; (16)
$[D]$	orthotropic elastic flexibility matrix; (18)
$\det[J]$	determinant of the Jacobian; (8)
$\det[J_{11}]$	determinant of Jacobian submatrix $[J_{11}]$; (8)
$\left. \begin{matrix} E_{rr} \\ E_{\theta\theta} \\ E_{zz} \end{matrix} \right\}$	orthotropic elastic material moduli; (17)
$\{e\}$	vector of total strain; (13)
$\{e_T\}$	vector of thermal strain; (13)
e_{ij}	component of elastic strain tensor; (16)
e_{Tij}	component of thermal strain tensor; (14)
$\{F\}$	vector of generalized nodal forces; (21)
f_i	first order LaGrange interpolation polynomial; (1)
$\left. \begin{matrix} G_{\theta z} \\ G_{\theta r} \\ G_{zr} \end{matrix} \right\}$	orthotropic elastic shear moduli; (17)
$\{h\}$	vector of generalized thermal nodal forces; (21)

H_i	Gauss weight factor; (24)
$[J]$	Jacobian matrix; (5)
$\left. \begin{array}{l} [J_{11}] \\ [J_{21}] \\ [J_{22}] \end{array} \right\}$	submatrices of the Jacobian matrix; (6)
$[k]$	element stiffness matrix; (21)
$M_{\theta\theta}$	circumferential bending moment; (31)
M_{zz}	longitudinal bending moment; (31)
$N_{\theta\theta}$	circumferential membrane force; (31)
N_{zz}	longitudinal membrane force; (31)
$\{P\}$	vector of total generalized nodal forces; (29)
$\left. \begin{array}{l} Q_{\theta r} \\ Q_{zr} \end{array} \right\}$	transverse shear forces; (31)
q	surface traction; (26)
q_r	radial component of a surface traction; (36)
R	radius of shell reference surface; (34)
R_{ave}	radius of cylinder mid-surface; (31)
R_i	radius of cylinder inner surface; (4)
R_o	radius of cylinder outer surface; (4)
r	radial spatial coordinate; (4)
u	circumferential displacement function; (1)
u_i	circumferential nodal displacement; (1)
v	longitudinal displacement function; (1)
v_i	longitudinal nodal displacement; (1)
w	radial displacement function; (1)
w_i	radial nodal displacement; (1)

z	longitudinal spatial coordinate; (4)
z_i	longitudinal nodal spatial coordinate; (4)
α_{ij}	anisotropic material thermal expansion coefficient; (16)
$\left. \begin{matrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{matrix} \right\}$	orthotropic material thermal expansion coefficients; (17)
$\left. \begin{matrix} \gamma_{\theta z} \\ \gamma_{\theta r} \\ \gamma_{zr} \end{matrix} \right\}$	shear strain components; (9)
ΔT	temperature at a point; (14)
ΔT_i	nodal point temperature; (28)
$\{\delta\}$	vector of nodal point displacements; (12)
δ_{ij}	Kroneker Delta; (14)
$\left. \begin{matrix} \epsilon_r \\ \epsilon_\theta \\ \epsilon_z \end{matrix} \right\}$	elastic normal strain components; (12)
$\{\epsilon\}$	vector of elastic strains; (12)
ζ	shell element thickness coordinate; (31)
θ	circumferential spatial coordinate; (4)
θ_i	circumferential nodal spatial coordinate; (4)
$\left. \begin{matrix} \nu_{\theta r} \\ \nu_{r\theta} \\ \nu_{zr} \\ \nu_{rz} \\ \nu_{z\theta} \\ \nu_{\theta z} \end{matrix} \right\}$	orthotropic Poisson's ratio; (19)

σ_{rs}	component of elastic stress tensor; (16)
$\{\sigma\}$	stress vector; (18)
$\left. \begin{matrix} \tau_1 \\ \tau_2 \end{matrix} \right\}$	strain-displacement parameters; (10)
ϕ	strain-displacement parameter; (10)
ψ	strain-displacement parameter; (10)
$\left. \begin{matrix} \omega_1 \\ \omega_2 \end{matrix} \right\}$	strain-displacement parameters; (10)

1. INTRODUCTION

1.1 Purpose

The performance of structures containing cracks has been of interest for years. Almost a half century ago, Griffith [1] formulated a theory of fracture for ideally brittle materials. Subsequent work by Irwin [2] and Orowan [3] extended this theory to fracture of ductile materials. More recently, articles have appeared in the literature describing response of initially cracked structures for which a crack geometry has been specified [4-10]. A continuum theory describing the brittle fracture of initially uncracked structures under multi-axial states of stress or strain is unavailable, however. The purpose of the present study is to examine the effect of cracking on the strength of layered cylindrical structures composed of one or more brittle materials but which are initially uncracked.

1.2 Scope of This Report

There are three distinct stages into which the study divides:

1. Development of a method of analysis capable of adequately determining stress and strain fields at all points in the structure.
2. Imposition of a fracture criterion relating computed stress and strain fields to the fracture strengths of the materials in order to predict crack initiation.

3. Examination of the post fracture behavior of the structure as the crack or cracks propagate in the structure.

It is desirable at the present time to develop a method of analysis which is independent of any particular fracture theory. This circumvents some of the difficulties mentioned above and facilitates the inclusion of suitable criteria as they become available in the future. The scope of this report has been limited to a discussion of stage 1. Current effort is being directed to stages 2 and 3 with the major consideration being that of developing a means of studying post fracture behavior of the cylinder.

1.3 Description of Research

The analysis discussed in this report employs the finite element method in which a three dimensional element is utilized with a stiffness matrix developed in cylindrical coordinates using the three dimensional field equations of elasticity. A trilinear displacement field using first order Lagrange interpolation polynomials is assumed which satisfies interelement compatibility, and for which polynomial functions complete with respect to first derivatives are specified.

The analysis is based upon the following assumptions:

1. Displacements and strains are sufficiently small so that linear elasticity is applicable.
2. Perfect contact exists between adjacent layers in the uncracked structure where the bond is treated as a distinct layer.

3. The materials of each layer are linearly elastic but may possess cylindrical orthotropy.

Shearing deformations of each layer are included and relative normal displacements in each layer are permitted, the latter allowing "pinching" of the layers.

A computer program has been written to carry out the static analysis of a layered cylinder subject to asymmetric pressure and thermal loadings. Normal and tangential surface tractions are permitted to vary bilinearly over the loaded surface. Temperatures are allowed to vary trilinearly through the element. Obtained from the analysis are stress and strain components, stress resultants and stress couples, as well as displacements at discrete points in the cylinder.

Included in this report are several numerical examples. Results are presented for a thick-walled cylinder subjected to internal and external pressure. This problem is principally an application of axisymmetric elasticity and has been presented in order to compare results of the finite element method with an exact solution. Four other examples are presented, all of which are problems involving bending of moderately thick shells.

2. ELEMENT FORMULATION

2.1 Coordinate Systems

The element occupies a region in space described by cylindrical coordinates r , θ and z as shown in Figure 1. These spatial coordinates are global coordinates to which all displacements and stress components are referred.

A local, non-dimensional, oblique, curvilinear coordinate system a , b and c is prescribed for the element such that the region R occupied by the element in the global system is mapped into a unit cube occupying a region S of the local system.

Referring to Figure 1 the inner and outer radial surfaces 1234* and 5678 of the cylindrical element form concentric surfaces, however, when developed in the θ - z plane as shown in Figure 2, they may be quadrilaterals. Hence, the tangential surfaces 1485 and 2376, and the longitudinal surfaces 1562 and 4873 may be arbitrary planes.

2.2 Displacement Functions

The global displacements u , v and w will be taken to vary trilinearly in the local coordinate system a , b and c as follows:

$$\begin{aligned} u &= u(a,b,c) = f_1 u_1 + f_2 u_2 + \dots + f_8 u_8 \\ v &= v(a,b,c) = f_1 v_1 + f_2 v_2 + \dots + f_8 v_8 \\ w &= w(a,b,c) = f_1 w_1 + f_2 w_2 + \dots + f_8 w_8 \end{aligned} \quad (1)$$

* Numbers refer to nodal points joined by that surface.

where, the u_i , v_i and w_i ($i=1,2,\dots,8$) are values of the displacements at their respective nodal points as the nodes are numbered in Figure 1.

The f_i ($i=1,2,\dots,8$) are interpolation functions, specifically first order Lagrange polynomials, which at a given node i possess the property

$$\begin{aligned} f_i(a_j, b_j, c_j) &= 0 & i \neq j \\ f_i(a_j, b_j, c_j) &= 1 & i = j \end{aligned} \quad (2)$$

where (a_i, b_i, c_i) are the local coordinates of node i . Use of these interpolation formulae guarantees compatibility of displacements on the boundaries of adjoining elements since displacements on a common boundary are functions only of the displacements at the four nodal points defining that boundary.

The set of interpolation functions for the nodal numbering sequence shown in Figure 1 is given by

$$\begin{aligned} f_1 &= \frac{1}{8} (1-a)(1-b)(1-c) \\ f_2 &= \frac{1}{8} (1+a)(1-b)(1-c) \\ f_3 &= \frac{1}{8} (1+a)(1+b)(1-c) \\ f_4 &= \frac{1}{8} (1-a)(1+b)(1-c) \\ f_5 &= \frac{1}{8} (1-a)(1-b)(1+c) \\ f_6 &= \frac{1}{8} (1+a)(1-b)(1+c) \\ f_7 &= \frac{1}{8} (1+a)(1+b)(1+c) \\ f_8 &= \frac{1}{8} (1-a)(1+b)(1+c) \end{aligned} \quad (3)$$

2.3 Coordinate Transformation

The global, spatial coordinates r , θ and z may be expressed exactly as functions of the local, oblique coordinates a , b and c by

$$\begin{aligned}\theta &= \theta(a,b,c) = f_1\theta_1 + f_2\theta_2 + \dots + f_8\theta_8 \\ z &= z(a,b,c) = f_1z_1 + f_2z_2 + \dots + f_8z_8 \\ r &= r(c) = \frac{1}{2} [(1-c)R_i + (1+c)R_o]\end{aligned}\tag{4}$$

where the θ_i and z_i ($i=1,2,\dots,8$) are the θ and z coordinates of their respective nodes as numbered in Figure 1; R_i and R_o are the inner and outer radii of the element. The f_i ($i=1,2,\dots,8$) are as defined by equation 3.

Applying the chain rule for partial differentiation, derivatives with respect to the local coordinates a , b and c are given by

$$\begin{bmatrix} \frac{\partial}{\partial a} \\ \frac{\partial}{\partial b} \\ \frac{\partial}{\partial c} \end{bmatrix} = [J] \begin{bmatrix} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial z} \\ \frac{\partial}{\partial r} \end{bmatrix} = \begin{bmatrix} J_{11} & | & 0 \\ \hline J_{21} & | & J_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial z} \\ \frac{\partial}{\partial r} \end{bmatrix}\tag{5}$$

where $[J_{11}]$, $[J_{21}]$ and $[J_{22}]$ are submatrices of the Jacobian defined by

$$[J_{11}] = \begin{bmatrix} \frac{\partial \theta}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial \theta}{\partial b} & \frac{\partial z}{\partial b} \end{bmatrix}$$

$$[J_{21}] = \begin{bmatrix} \frac{\partial \theta}{\partial c} & \frac{\partial z}{\partial c} \end{bmatrix}$$

$$[J_{22}] = \frac{\partial r}{\partial c} \quad (6)$$

The required coordinate transformation expressing the derivatives with respect to the global system of coordinates in terms of derivatives with respect to the local system is then

$$\begin{bmatrix} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial z} \\ \frac{\partial}{\partial r} \end{bmatrix} = [J]^{-1} \begin{bmatrix} \frac{\partial}{\partial a} \\ \frac{\partial}{\partial b} \\ \frac{\partial}{\partial c} \end{bmatrix} \quad (7)$$

The inverse of the Jacobian may be easily evaluated algebraically and is found to be

$$[J]^{-1} = \left[\begin{array}{cc|c} \frac{1}{\det[J_{11}]} \frac{\partial z}{\partial b} & \frac{-1}{\det[J_{11}]} \frac{\partial z}{\partial a} & 0 \\ \frac{-1}{\det[J_{11}]} \frac{\partial \theta}{\partial b} & \frac{1}{\det[J_{11}]} \frac{\partial \theta}{\partial a} & 0 \\ \hline \frac{1}{\det[J]} \left(\frac{\partial z}{\partial c} \frac{\partial \theta}{\partial b} - \frac{\partial \theta}{\partial c} \frac{\partial z}{\partial b} \right) & \frac{1}{\det[J]} \left(\frac{\partial z}{\partial a} \frac{\partial \theta}{\partial c} - \frac{\partial \theta}{\partial a} \frac{\partial z}{\partial c} \right) & \frac{1}{\frac{\partial r}{\partial c}} \end{array} \right] \quad (8)$$

2.4 Strain Displacement Relations

From the theory of elasticity the linearized strain displacement relations for an elastic solid are given in cylindrical coordinates as

$$\begin{aligned}
 \epsilon_r &= \frac{\partial w}{\partial r} \\
 \epsilon_\theta &= \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{w}{r} \\
 \epsilon_z &= \frac{\partial v}{\partial z} \\
 \gamma_{\theta z} &= \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial u}{\partial z} \\
 \gamma_{\theta r} &= \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{u}{r} \\
 \gamma_{zr} &= \frac{\partial w}{\partial z} + \frac{\partial v}{\partial r}
 \end{aligned} \tag{9}$$

Applying the coordinate transformation given by equation 7, the strains, exclusive of thermal or initial strains, may be expressed by the following:

$$\begin{aligned}
 \epsilon_r &= \frac{\partial w}{\partial r} = \sum_{t=1}^8 \left[\frac{1}{\det[J]} \left(\phi \frac{\partial f_t}{\partial a} + \psi \frac{\partial f_t}{\partial b} \right) + \frac{\partial f_t}{\partial c} \right] w_t \\
 \epsilon_\theta &= \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{w}{r} = \frac{1}{r \det[J_{11}]} \sum_{t=1}^8 \left(\omega_1 \frac{\partial f_t}{\partial a} - \tau_1 \frac{\partial f_t}{\partial b} \right) u_t + \frac{1}{r} \sum_{t=1}^8 f_t w_t \\
 \epsilon_z &= \frac{\partial v}{\partial z} = \frac{1}{\det[J_{11}]} \sum_{t=1}^8 \left(\omega_2 \frac{\partial f_t}{\partial b} - \tau_2 \frac{\partial f_t}{\partial a} \right) v_t \\
 \gamma_{\theta z} &= \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial u}{\partial z} = \frac{1}{r \det[J_{11}]} \sum_{t=1}^8 \left(\omega_1 \frac{\partial f_t}{\partial a} - \tau_1 \frac{\partial f_t}{\partial b} \right) v_t \\
 &\quad + \frac{1}{\det[J_{11}]} \sum_{t=1}^8 \left(\omega_2 \frac{\partial f_t}{\partial b} - \tau_2 \frac{\partial f_t}{\partial a} \right) u_t
 \end{aligned}$$

$$\begin{aligned}
\gamma_{\theta r} &= \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{u}{r} = \sum_{t=1}^8 \left[\frac{1}{\det[J]} \left(\phi \frac{\partial f_t}{\partial a} + \psi \frac{\partial f_t}{\partial b} \right) + \frac{\frac{\partial f_t}{\partial c}}{\frac{\partial r}{\partial c}} \right] u_t \\
&\quad + \frac{1}{r \det[J_{11}]} \sum_{t=1}^8 \left(\omega_1 \frac{\partial f_t}{\partial a} - \tau_1 \frac{\partial f_t}{\partial b} \right) w_t - \frac{1}{r} \sum_{t=1}^8 f_t u_t \\
\gamma_{Zr} &= \frac{\partial w}{\partial Z} + \frac{\partial v}{\partial r} = \frac{1}{\det[J_{11}]} \sum_{t=1}^8 \left(\omega_2 \frac{\partial f_t}{\partial a} - \tau_2 \frac{\partial f_t}{\partial b} \right) w_t \\
&\quad + \sum_{t=1}^8 \left[\frac{1}{\det[J]} \left(\phi \frac{\partial f_t}{\partial a} + \psi \frac{\partial f_t}{\partial b} \right) + \frac{\frac{\partial f_t}{\partial c}}{\frac{\partial r}{\partial c}} \right] v_t \tag{10}
\end{aligned}$$

The displacements u_t , v_t and w_t ($t=1,2,\dots,8$) are the three components of displacement occurring at each of the eight nodal points.

The functions ϕ , ψ , ω_1 , τ_1 , ω_2 , τ_2 are given by

$$\begin{aligned}
\phi &= \sum_{s=1}^8 \sum_{r=1}^8 \frac{\partial f_r}{\partial c} \frac{\partial f_s}{\partial b} (Z_r \theta_s - \theta_r Z_s) \\
\psi &= \sum_{s=1}^8 \sum_{r=1}^8 \frac{\partial f_r}{\partial a} \frac{\partial f_s}{\partial c} (Z_r \theta_s - \theta_r Z_s) \\
\omega_1 &= \sum_{r=1}^8 \frac{\partial f_r}{\partial b} Z_r \\
\omega_2 &= \sum_{r=1}^8 \frac{\partial f_r}{\partial a} \theta_r \\
\tau_1 &= \sum_{r=1}^8 \frac{\partial f_r}{\partial a} Z_r \\
\tau_2 &= \sum_{r=1}^8 \frac{\partial f_r}{\partial b} \theta_r \tag{11}
\end{aligned}$$

Equations 10 are the strain displacement relations in terms of nodal point displacements. In matrix notation these equations may be written as

$$\{\epsilon\} = [b]\{\delta\} \quad (12)$$

where $\{\epsilon\}$ is a vector of elastic strains, $\{\delta\}$ is a vector of nodal displacements and $[b]$ is a matrix of coefficients which are functions of a , b and c relating the nodal displacements to the elastic strains.

If thermal strains are included then the total strains can be written as the sum of the elastic and thermal strains. Thus,

$$\{e\} = \{\epsilon\} + \{e_T\} \quad (13)$$

where $\{e\}$ is a vector of total strains and $\{e_T\}$ is a vector of thermal strains whose components may be expressed as

$$e_{T_{ij}} = \delta_{ij} \alpha \Delta T \quad (i, j = x, y, z) \quad (14)$$

in which δ_{ij} is the kroneker delta given by

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad (15)$$

α is the coefficient of thermal expansion of the material and ΔT is the temperature differential measured from a given datum.

2.5 Constitutive Equations

For an anisotropic material the stress-strain temperature relationship is given by

$$e_{ij} = D_{ijrs} \sigma_{rs} + \alpha_{ij} \Delta T \quad (16)$$

If the material possesses cylindrical orthotropy, equation 16 reduces to

$$\begin{bmatrix} e_{rr} \\ e_{\theta\theta} \\ e_{zz} \\ 2e_{\theta z} \\ 2e_{r\theta} \\ 2e_{rz} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_{rr}} & \frac{-\nu_{\theta r}}{E_{\theta\theta}} & \frac{-\nu_{zr}}{E_{zz}} \\ \frac{-\nu_{r\theta}}{E_{rr}} & \frac{1}{E_{\theta\theta}} & \frac{-\nu_{z\theta}}{E_{zz}} \\ \frac{-\nu_{rz}}{E_{rr}} & \frac{-\nu_{\theta z}}{E_{\theta\theta}} & \frac{1}{E_{zz}} \\ & & \frac{1}{G_{\theta z}} \\ & & \frac{1}{G_{r\theta}} \\ & & \frac{1}{G_{rz}} \end{bmatrix} \begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{\theta z} \\ \sigma_{r\theta} \\ \sigma_{rz} \end{bmatrix} + \Delta T \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (17)$$

or, more briefly,

$$\{e\} = [D]\{\sigma\} + \{e_T\} \quad (18)$$

The elastic flexibility matrix $[D]$ possesses the following symmetries:

$$\frac{\nu_{\theta r}}{E_{\theta\theta}} = \frac{\nu_{r\theta}}{E_{rr}}; \quad \frac{\nu_{zr}}{E_{zz}} = \frac{\nu_{rz}}{E_{rr}}; \quad \frac{\nu_{z\theta}}{E_{zz}} = \frac{\nu_{\theta z}}{E_{\theta\theta}} \quad (19)$$

The inverse relationship expressing stresses in terms of strains is obtained from

$$\{\sigma\} = [D]^{-1}\{e - e_T\} = [C]\{e - e_T\} \quad (20)$$

2.6. Element Stiffness Matrix

A number of alternative methods are available for generating the force-displacement relationships describing the stiffness characteristics of structural elements (stiffness matrices). It can be shown that this force-displacement relationship is

$$\{F\} = [k]\{\delta\} + \{h\} \quad (21)$$

where $\{F\}$ is a vector of generalized nodal forces due to externally applied loads; $\{\delta\}$ is a vector of nodal displacements; and

$$[k] = \int_V [b]^T [C] [b] dv \quad (22)$$

$$\{h\} = \int_V [b]^T [C] \alpha \Delta T dv \quad (23)$$

The integrations indicated in equations 22 and 23 are, in general, very difficult to evaluate in closed form. If the integrations are done in the local coordinate system a , b and c these integrals may be evaluated numerically by means of the Gaussian Quadrature numerical integration scheme. For example, the integration of equation 22 proceeds as follows:

$$\int_v [b^T C b] dv = \sum_i^n \sum_j^n \sum_k^n H_i H_j H_k [b^T C b]_{ijk} (r |\det[J]|)_{ijk} \quad (24)$$

where

n = order of integration

H_i = Gauss Weight factor

$\det[J]$ = determinant of the Jacobian

r = r coordinate of the integration point

The subscripts i , j and k indicate that the functions are evaluated successively at the points $(a=a_i)$, $(b=b_j)$ and $(c=c_k)$.

Similarly, equation 23 is evaluated by

$$\int_v [b^T C \alpha \Delta T] dv = \sum_i^n \sum_j^n \sum_k^n H_i H_j H_k [b^T C \alpha \Delta T]_{ijk} (r |\det[J]|)_{ijk} \quad (25)$$

A three point integration ($n=3$) is used to evaluate the integrals associated with generalized nodal forces, boundary stresses, and element stress resultants (see Art. 2.7 and 2.8). In order to improve the bending characteristics of the element for use in shell problems, certain terms in equations 24 and 25 are integrated using a one point integration. For instance,

the terms in equations 24 and 25 containing the shear strain $\gamma_{r\theta}$ are integrated three times with respect to the coordinates b and c , but only once with respect to a . The effect on the element is to remove the strain energy associated with the symmetric shearing mode and to impose a constant shear strain across the element in the θ direction. This reduces the stiffness of the element in circumferential bending. Terms associated with the antisymmetric or pure shearing mode remain and are integrated exactly using this process. This modification is in accordance with classical bending theories which include shearing strains due to shear, but which do not include shearing strains due to bending.

2.7 Generalized Nodal Forces

In general it is necessary to compute the nodal forces, not by intuition, but by a scheme which is consistent with the displacement assumptions for the element. In the work presented here applied loads are specified as stress intensities constant over a surface as in the case of boundary stresses, or varying bilinearly among the four nodes defining a loaded surface.

From Betti's reciprocal theorem the work done by the generalized forces going through a set of displacements $\{\delta\}$ must be equal to the work done by the real loads going through displacements consistent with $\{\delta\}$. Thus,

$$\sum_{r=1}^{12} F_r^{(\delta)} \delta_r = \int_S q^{(\delta)}(a,b) \delta(a,b) r |\det[J_{11}]| da db \quad (26)$$

is a statement of statical equivalence of the generalized forces $\{F\}$ and the distributed load q . Here, the integration is over a radial surface, but it may be done over any of the surfaces of the element.

If $\{\delta\}$ is allowed to have one non-zero component of unity ($\delta_s=1$) and a numerical integration is performed on the right hand side of equation (26) then the generalized nodal force F_s can be computed from

$$F_s^{(\delta)} = \sum_i^n \sum_j^n H_i H_j q_{ij} \delta_{sij} (r |\det[J_{11}]|)_{ij} \quad (27)$$

where $F_s^{(\delta)}$ is the generalized nodal force at coordinate s in the direction of δ_s and δ_{sij} is the displacement at an integration point due to a unit displacement at coordinate s . δ_{sij} is assumed to vary bilinearly over the surface.

Generalized nodal forces due to thermal loading are evaluated directly by use of equation 25. The temperature at a point in the interior of an element is given by

$$\Delta T = \Delta T(a,b,c) = f_1 \Delta T_1 + f_2 \Delta T_2 + \dots + f_8 \Delta T_8 \quad (28)$$

The f_i ($i=1,2,\dots,8$) are defined by equation 13 and the ΔT_i ($i=1,2,\dots,8$) are nodal point temperatures.

If $\{P\}$ is used to designate generalized nodal forces due to combined pressure and thermal effects, then from equation 21

$$\{P\} = \{F\} - \{h\} \quad (29)$$

2.8 Stresses and Stress Resultants

Element stresses are determined from nodal displacements by substitution of equations 12 and 14 into equation 20. Thus,

$$\{\sigma\} = [C][b]\{\delta\} - [C]\{e_T\} \quad (30)$$

Stresses are computed at the center of the element, and for certain components, at the centers of the faces of an element. Specifically, $\sigma_{\theta\theta}$ and σ_{zz} are evaluated at the center of the element and at the centers of the radial faces. These two stress components indicate directly the degree of bending at a point in a shell and are used to compute moments $M_{\theta\theta}$ and M_{zz} .

For shell problems the radial stress σ_{rr} is uncoupled from the other direct stress components. This is done to approximate the two dimensional plane stress assumption normally made in shell theories. Further, this modification gives a constant state of radial stress through the thickness of the shell and has a minor softening effect on the stiffness of the element in bending.

In order to check results obtained from classical shell theories six stress resultants are computed. These are two membrane forces, two transverse shears, and two bending moments given by the following:

$$N_{\theta\theta} = \int_{t^-}^{t^+} \sigma_{\theta\theta} d\zeta$$
$$N_{zz} = \int_{t^-}^{t^+} \sigma_{zz} \left(1 + \frac{\zeta}{R_{ave}}\right) d\zeta$$

$$Q_{\theta r} = \int_{t^-}^{t^+} \sigma_{\theta r} d\zeta$$

$$Q_{zr} = \int_{t^-}^{t^+} \sigma_{zr} \left(1 + \frac{\zeta}{R_{ave}}\right) d\zeta$$

$$M_{\theta\theta} = \int_{t^-}^{t^+} \sigma_{\theta\theta} \zeta d\zeta$$

$$M_{zz} = \int_{t^-}^{t^+} \sigma_{zz} \left(1 + \frac{\zeta}{R_{ave}}\right) d\zeta \quad (31)$$

where R_{ave} is the radius to the mid-surface of the shell and ζ is a coordinate to a point in the shell measured from the mid-surface; t is the half thickness of the shell. The above integrals are evaluated numerically using a 3 point Gaussian quadrature.

3. NUMERICAL EXAMPLES

3.1 Thick-Walled Cylinder

A cylinder with the dimensions shown in Figure 3 and subjected to an internal pressure of 1000 psi and an external compression of 2000 psi was analyzed using both the theory of elasticity and the finite element method. The problem was axisymmetric; hence, four planes of symmetry were imposed and the finite element mesh was reduced to a row of elements through the thickness of the cylinder wall.

In Figure 4 radial displacements are plotted for various points in the cylinder and for varying numbers of elements. The plotted points are nodal displacements obtained from the finite element solution. The heavy line is the exact solution.

Figures 5 and 6 show the distribution of radial and circumferential stresses, respectively, through the wall thickness. The finite element solution was obtained for varying numbers of elements, and the plotted points are stresses computed at the centers of the elements. The heavy line is the exact solution. Note that there is little difference between the solutions for one or ten elements, and the exact solution.

3.2 Cylinder with Half-Cosine Pressure

An infinite cylinder with a moderately thick wall ($R/t=20$) was analyzed for the loading shown in Figure 7. The loading was radial and varied as a half-cosine wave along the half circumference of the cylinder. Four planes of symmetry existed;

hence, the finite element mesh extended over a quarter ring. A typical mesh is shown in Figure 8.

A plot of the radial displacement w at $\theta=0$ is shown in Figure 9 for various numbers of elements taken circumferentially. The solid line represents convergence of solutions obtained using a constant radial stress through the thickness of the element. The dashed line shows convergence prior to this modification. The exact value is the displacement of the middle surface of the shell obtained from a closed-form solution which includes the effects of shear deformation (see Appendix for discussion of shell equations). Note that the finite element solution converges monotonically to this value as the number of elements is increased.

Figure 10 is a plot of the radial displacement w as a function of the circumferential angle θ . The heavy line is the closed-form solution for the displacement of the middle surface of the shell. The two dashed lines are displacements of the shell reference surface computed from the closed-form solution when the reference surface is taken at the inner or outer surface of the cylinder. It should be noted that the radial displacement at $\theta=0$ obtained from the closed-form solution for extreme positions of the reference surface differs by approximately 5 percent from the value obtained when the reference surface is the mid-surface of the cylinder.

In Figure 11 the circumferential membrane force $N_{\theta\theta}$ and the circumferential bending moment $M_{\theta\theta}$ are plotted as functions

of the angle θ . The heavy line represents the closed-form solution for these stress resultants referred to the mid-surface of the cylinder. The dashed lines represent closed-form solutions obtained when the shell reference surface is taken at the inner or outer surfaces of the cylinder. Note that closed-form solutions for $M_{\theta\theta}$ differ from the mid-surface value by approximately 15 and 16 percent when the reference surface was taken at the inner or outer surface of the cylinder, respectively. The finite element solution closely follows the solution for the mid-surface of the cylinder.

Figure 12 shows the deformed shape of the cylinder. Radial planes before deformation are shown as dashed lines through the cylinder wall. The solid lines represent these planes after deformation.

3.3 Cylinder with Half-Cosine Temperature Distribution

An infinite cylindrical shell was analyzed for a peak temperature of 200°F varying as a half-cosine wave along the half circumference of the cylinder as shown in Figure 13. The inside surface was unheated and the temperature was assumed to vary linearly through the wall. Three planes of symmetry existed and the finite element mesh extended over one-half the cross section.

In Figure 14, the radial displacement w at $\theta=0$ is shown plotted as a function of the number of equally sized elements into which the half cylinder was divided. The exact value shown is the displacement of the mid-surface of the shell

obtained using the closed-form solution discussed in 3.2 above and in the Appendix. The finite element solution converges monotonically to this exact value as the number of elements is increased.

The deformed cylinder is shown in Figure 15.

3.4 Two-Layered Cylinder with Rigid Bond and Half-Cosine Temperature Distribution

A two-layered cylinder with the dimensions shown in Figure 16 was analyzed for the temperature distribution discussed in 3.3. Temperature was assumed to vary linearly through the thickness of the ablator with the inner surface of the ablator and the substructure unheated. Perfect bonding was assumed to exist at the ablator substructure interface; therefore, displacements on this interface were constrained to be identical.

The deformed shape of the composite cylinder is shown in Figure 17.

3.5 Two-Layered Cylinder with Flexible Bond and Half-Cosine Temperature Distribution

Figure 18 shows a half cross section of a layered cylinder consisting of two structural layers connected by a flexible bond layer. This composite cylinder was analyzed for the same temperature distribution discussed in the previous two examples. In the finite element analysis the bond layer was treated as a distinct layer whose deformations permitted relative normal displacements or "pinching" and relative shearing displacements or "slip" of the two structural layers.

The deformed shape of the composite cylinder is shown in Figure 19. It can be seen that appreciable slip of the layers is indicated by warping of the radial planes through the thicknesses. Some separation of the layers also occurs.

4. DISCUSSION

The results of the numerical examples have shown that the finite element developed in this report can be used successfully to solve problems of thick-walled cylinders and layered cylindrical shells. From the results of example 3.1 it can be seen that excellent stresses can be obtained with few elements if the centroidal values of stress are used for each element. More elements are required to obtain satisfactory displacements than are needed for stresses, however. For the example presented, a maximum error of -0.44 percent in radial displacement was obtained by using five elements through the wall thickness.

Examples 3.2 and 3.3 were presented in order to illustrate the effectiveness of the element in solving problems of shells with moderately thick walls. Monotonic convergence of displacements was obtained for both pressure and thermal loadings, although a large number of elements was required for convergence of the solution.

The structural response of bi-material cylinders with two different bounding assumptions was presented in examples 3.4 and 3.5. For comparison, displacements of the ablator from examples 3.3, 3.4 and 3.5 were superposed in Figure 20, where radial displacements are plotted as a function of θ . The three curves show the radial displacement of the ablator alone, the ablator and substructure with a rigid bond, and ablator and substructure with a flexible bond. It can be

seen that the displacements of the composite cylinders are smaller than those of the ablator acting alone. This is to be expected because the substructure provides elastic constraint to the deformation of the ablator. The displacements plotted are the average of the displacements of the inner and outer surfaces of the ablator, in each case.

It is interesting to note, however, that the radial displacements of the composite cylinder with flexible bond are smaller than the corresponding displacements of the composite cylinder with rigid bond. An intuitive explanation of this phenomenon is not readily apparent. Some insight into the complex interaction of the substructure and ablator is obtained from examination of Figures 17 and 19. It can be seen that the flexible bond permits slip of the layers relative to one another. Constraining this movement, as in the case of rigid bond, results in additional strains in the substructure. Further, because of the low elastic modulus of the bonding material, the load transmitted to the substructure through the flexible bond is smaller than that transmitted through the rigid bond. These statements can be substantiated by examining the strain and stress distributions through the thickness of the cylinder.

Figures 21 and 22 show the circumferential strain and stress distributions through the wall thickness at $\theta=0$, 45, and 90 degrees for the composite cylinder with rigid bond and with flexible bond, respectively. It can be seen from these figures that when a flexible bond is used the strain at

the ablator/bond interface is greater than the corresponding strain when a rigid bond exists. Conversely, when a flexible bond is used, the strain at the substructure/bond interface is smaller than the corresponding strain for the case of rigid bond. These interface strains are identical when the layers are rigidly bonded.

It can also be seen from the stress distributions that negligible circumferential stresses are transmitted between layers when a flexible bond is used. Coupling is primarily through radial and shearing stresses in the bond layer.

In Figure 23, the circumferential moment $M_{\theta\theta}$ developed in the ablator is plotted as a function of θ for the three cases above. It can be seen that $M_{\theta\theta}$ is a maximum when a flexible bond is used.

The circumferential moment developed in the substructure is shown in Figure 24 for the conditions of rigid and flexible bond. It can be seen that peak values of $M_{\theta\theta}$ in the substructure are lower when a flexible bond rather than a rigid bond is used.

The radial stress developed in the bond layer is plotted as a function of θ in Figure 25. Note that the radial stress is always tensile. Computation of this stress component makes possible a study of delamination of the structural layers because of failure of the bond.

Data presented in Figures 20 through 25 were obtained using the same finite element mesh. Because of program limitations on problem size a mesh of 90 elements for each layer

was used. Data presented earlier for displacements of the ablator only were obtained by using 120 elements. Based on the convergence curve of Figure 14 it is estimated that true displacements may be 17 percent greater than those shown in Figure 20.

5. SUMMARY AND CONCLUSIONS

5.1 Summary of Work Completed

A finite element method has been developed which has the capability of solving problems of layered cylinders subject to static pressure and thermal loadings. Use is made of a three dimensional finite element with a stiffness matrix developed in cylindrical coordinates using the three dimensional equations of elasticity.

A computer program has been written for the CDC 6600 computer. Numerical examples have been solved by use of this program and results have been presented which illustrate the applicability of the method to problems of both thick walled cylinders and layered cylinders with moderately thick walls.

5.2 Future Work

Future effort will be directed toward completion of stages 2 and 3, the incorporation of fracture criteria, and the study of the nonlinear behavior of a layered cylinder with a propagating crack. Concurrently, studies will be made of methods to improve the bending characteristics of the present element to reduce the number of elements required for a satisfactory solution.

5.3 Conclusions

A limited study has been made of the structural response of layered cylinders due to asymmetric thermal loading.

Consideration has been given to the effects of rigid and flexible bonding of the ablator to the substructure. It was found that displacements of the ablator for the layered cylinder with flexible bond were smaller than the corresponding displacements of the composite cylinder with rigid bond. Both, however, were smaller than those of the ablator acting alone. Ablator moments in the composite cylinder with flexible bond were found to be higher than those of the cylinder with rigid bond, and those of the ablator acting alone. Substructure moments were found to be lower for the case of flexible bond.

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Figure 1. Local and Global Coordinate Systems

Figure 2. Quadrilateral Surfaces in Oblique Coordinates

Figure 3. Thick Walled Cylinder of Example 3.1

Figure 4. Radial Displacements of Thick Walled Cylinder

Figure 5. Radial Stresses of Thick Walled Cylinder

Figure 6. Circumferential Stresses of Thick Walled Cylinder

Figure 7. Cylindrical Shell of Example 3.2

Figure 8. Typical Coarse Mesh Pattern

Figure 9. Convergence of Solution for a Pressure Loaded Cylinder

Figure 10. Radial Displacement of a Pressure Loaded Cylinder

Figure 11. Stress Resultants in a Pressure Loaded Cylinder

Figure 12. Deformed Cylinder of Example 3.2

Figure 13. Cylindrical Shell of Example 3.3

Figure 14. Convergence of Solution for a Thermal Loaded Cylinder

Figure 15.. Deformed Cylinder of Example 3.3

Figure 15 continued

Figure 16. Composite Cylinder of Example 3.4

Figure 17. Deformed Cylinder of Example 3.4

Figure 17 continued

Figure 18. Composite Cylinder of Example 3.5

Figure 19. Deformed Cylinder of Example 3.5

Figure 19 continued

Figure 20. Comparison of Ablator Displacements

Figure 21. Strain and Stress Distributions in a Rigid Bonded Composite Cylinder

Figure 22. Strain and Stress Distributions in Flexible Bonded Composite Cylinder

Figure 23. Comparison of Ablator Moments

Figure 24. Comparison of Substructure Moments

Figure 25. Radial Stress Distribution

Figure 26. Shell Coordinates and Displacements

APPENDIX
THIN SHELL THEORY
INCLUDING SHEAR DEFORMATION

The following are equations of bending of a thin elastic cylindrical shell of infinite length which include the effect of shear deformation. A linear variation of temperature through the thickness of the shell is assumed. Small displacements are also assumed and pinching deformation of the shell is neglected.

Referring to Figure 26 displacements allowed in the shell are of the form

$$\begin{aligned}u_{\theta}(\theta) &= u(\theta) + \zeta f(\theta) \\u_r(\theta) &= w(\theta)\end{aligned}\tag{32}$$

where ζ is the radial coordinate of a point in the shell measured from the reference surface of the shell.

The resulting strains are

$$\begin{aligned}e_{\theta\theta} &= e + \zeta K \\ \gamma_{\theta r} &= \gamma\end{aligned}\tag{33}$$

where

$$\begin{aligned}e &= \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{w}{R} \\ K &= \frac{1}{r} \frac{\partial f}{\partial \theta} \\ \gamma &= f + \frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{u}{R}\end{aligned}\tag{34}$$

The constitutive equations for conditions of plane stress and plane strain are given by

$$\begin{aligned}\sigma_{\theta\theta} &= \frac{E}{1-\nu^2} e_{\theta\theta} - \frac{E}{1-\nu} \alpha \Delta T \\ \sigma_{zz} &= \frac{-E}{1-\nu} \alpha \Delta T \\ \sigma_{r\theta} &= G \gamma_{r\theta} = \frac{E}{2(1+\nu)} \gamma_{r\theta}\end{aligned}\tag{35}$$

The equations of equilibrium in terms of stress resultants are given by

$$\begin{aligned}\frac{1}{r} \frac{\partial N}{\partial \theta} + \frac{Q}{a} &= 0 \\ \frac{1}{r} \frac{\partial Q}{\partial \theta} - \frac{N}{a} &= -q_r \\ \frac{1}{r} \frac{\partial M}{\partial \theta} - Q &= 0\end{aligned}\tag{36}$$

where

$$\begin{aligned}N_{\theta\theta} &= \int_{t^-}^{t^+} \sigma_{\theta\theta} d\zeta \\ Q_{r\theta} &= \int_{t^-}^{t^+} \sigma_{r\theta} d\zeta \\ M_{\theta\theta} &= \int_{t^-}^{t^+} \sigma_{\theta\theta} \zeta d\zeta\end{aligned}\tag{37}$$

In the usual manner, substitution of Equations 33, 34, 35 and 37 into the equations of equilibrium 36 results in three simultaneous ordinary differential equations in terms of the displacements. Solution of these equations is by expansion of the displacements and loads in terms of Fourier series in the circumferential coordinate θ .