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# Almost Exact Sum Rules for Nucleon Moments From An Infinite Dimensional Algebra*

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1. Recently there has been a great surge of interest in almost-exact sum rules for the magnetic moments of nucleons.[1-4] (By almost-exact we mean: exact to all orders in the strong couplings but only the lowest order in electromagnetic and weak couplings.) Besides providing a means for calculation of the magnetic moments on the same level as the calculation of $G_A/G_V$ by Adler[5] and Weisberger[6] these sum rules, taken together with the Adler-Weisberger sum rule, constitute a useful tool for investigating the nature of the dynamical approximations that underlie higher symmetry schemes.

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2. The purpose of the present note is to report on a set of sum rules which follow from an infinite dimensional algebra, which contains (and may be regarded as the most natural extension of) an algebra suggested by Gell-Mann.[7] In order to specify the algebra we consider the function

$$M_{\mu\nu}^{\alpha\beta}(k',k)\cdot\frac{1}{(2\pi)^3}\cdot\frac{1}{(4k_o'k_o)^{\frac{1}{2}}}(2\pi)^4\delta^4(p'+k'-p-k)$$

$$=\frac{-i}{(2\pi)^3}\frac{1}{(4k_o'k_o)^{\frac{1}{2}}}\int d^4x d^4y\, e^{i(k'\cdot x-k\cdot y)} < p'|[T\{J_\mu^\alpha(x)J_\nu^\beta(y)\}-i\rho_{\mu\nu}^{\alpha\beta}(x)\delta^4(x-y)]|p> \tag{1}$$

Here $\alpha,\beta$ are isotopic indices, $\mu,\nu$ are Minkowski indices, $k',k$ ($p',p$) are outgoing and incident "photon" (nucleon) momenta and $J_\mu^\alpha$ is the conserved iso-spin current which participates in weak interactions, $\partial^\mu J_\mu^\alpha = 0$.

The second term in the right hand side of Eq. (1) is designed to compensate for the non-covariant nature of the T-product,[8] so that $M_{\mu\nu}^{\alpha\beta}$ is a covariant object. The simplest equal-time commutation relations which ensure this covariance are[8]

$$\left[J_o^\alpha(x),\ J_o^\beta(y)\right]_{x_o=y_o} = i\,\epsilon^{\alpha\beta\gamma}J_o^\gamma(x)\delta^3(x-y) \tag{2}$$

$$\left[J_o^\alpha(x),\ J_n^\beta(y)\right]_{x_o=y_o} = i\,\epsilon^{\alpha\beta\gamma}J_n^\gamma(x)\delta^3(x-y) + i\,\partial_m\left[\rho_{mn}^{\alpha\beta}(x)\delta^3(x-y)\right] \tag{3}$$

m,n: 3-space indices.

Equations (2) and (3) specify the algebra under consideration. Equation (2) by itself is the starting point of several interesting conjectures by Gell-Mann.

3. We note that Eqs. (1)-(3) lead to the divergence conditions

$$k'^{\mu} M^{\alpha\beta}_{\mu\nu} = M^{\alpha\beta}_{\nu\lambda} k^{\lambda} = i\, \epsilon^{\alpha\beta\gamma} < p' | J^{\gamma}_{\nu}(0) | p > \quad . \tag{4}$$

These divergence conditions enable us to derive low energy theorems for the function $M^{\alpha\beta}_{\mu\nu}$, by a straightforward application of techniques invented by F. E. Low in connection with Compton scattering.[9] Complications arising from isotopics do not pose any special problem[10]; without further ado, therefore, we state these theorems.

Define

$$e^2 (2\pi)^3 M^{\alpha\beta}_{mn}(k,k) = \delta^{\alpha\beta} \left\{ S_1(\omega)\delta_{mn} + S_2(\omega)\, \tfrac{1}{2}[\sigma_m, \sigma_n] \right\}$$
$$+ \tfrac{1}{2}[\tau^{\alpha}, \tau^{\beta}] \left\{ A_1(\omega)\delta_{mn} + A_2(\omega)\, \tfrac{1}{2}[\sigma_m, \sigma_n] \right\} \tag{5}$$

where $\omega \equiv k_o$ and Pauli spinors are understood, but not displayed on the right hand side. Also we have specialized to the forward direction in the laboratory frame $\vec{p} = 0$. The theorems are [units: $e^2 = 4\pi\alpha \cong \frac{4\pi}{137}$]

$$S_1(\omega) = \frac{\pi\alpha}{M} + 0(\omega^2) \tag{6}$$

$$S_2(\omega) = \frac{\pi\alpha(\kappa_P-\kappa_N)^2}{2M^2}\,\omega + 0(\omega^2) \tag{7}$$

$$A_1(\omega) = 4\pi\alpha\left\{\frac{(1+\kappa_P-\kappa_N)^2}{8M^2} - \left(\frac{\partial G_E^V(q^2)}{\partial q^2}\right)_{q^2=0} - \frac{1}{8M^2}\right\}\omega + 0(\omega^2) \tag{8}$$

$$A_2(\omega) = 4\pi\alpha\left(\frac{1+\kappa_P-\kappa_N}{4M}\right) + 0(\omega^2) \quad . \tag{9}$$

Here $\kappa_P$ and $\kappa_N$ are the anomalous moments of proton and neutron respectively (in units of nucleon magnetons), M is the nucleon mass and $G_E^V$ is the electric iso-vector Sachs form factor. We have assumed that the neutral component of the conserved iso-spin current is proportional to the iso-vector part of the electromagnetic current, and scaled our amplitudes to correspond to the scattering of iso-vector photons.

Exactly analogous considerations can be advanced for the (simpler) case of iso-scalar photons. We start with the conserved hypercharge current and derive the theorems

$$S_1(\omega)^Y = \frac{\pi\alpha}{M} + 0(\omega^2) \tag{10}$$

$$S_2(\omega)^Y = \frac{\pi\alpha(\kappa_P+\kappa_N)^2}{2M^2}\,\omega + 0(\omega^2) \quad , \tag{11}$$

where we have used an obvious notation for the hyper-photon amplitudes.

4. The above low frequency theorems may be converted into sum rules by postulating unsubtracted dispersion relations for the relevant amplitudes.[11] The dispersion relations are easily written down by following the procedure of Gell-Mann, Goldberger and Thirring.[12] One finds that the amplitudes $S_1(\omega)$ and $S_1(\omega)^Y$ can not satisfy unsubtracted dispersion relations for exactly the same reasons as in ordinary Thompson scattering.[4] The other amplitudes yield the following sum rules:

$$\frac{\alpha(\kappa_P+\kappa_N)^2}{2M^2} = \frac{1}{\pi^2}\int_0^\infty \frac{\sigma_P^S-\sigma_A^S}{\omega}\, d\omega \tag{12}$$

$$\frac{\alpha(\kappa_P-\kappa_N)^2}{2M^2} = \frac{1}{\pi^2}\int_0^\infty \frac{\sigma_P^V-\sigma_A^V}{\omega}\, d\omega \tag{13}$$

$$\alpha\left\{\frac{(1+\kappa_P-\kappa_N)^2}{4M^2} - 2\left(\frac{\partial G_E^V(q^2)}{\partial q^2}\right)_{q^2=0} - \frac{1}{4M^2}\right\} = \frac{1}{4\pi^2}\int_0^\infty \frac{(\sigma_3-2\sigma_1)_A^V + (\sigma_3-2\sigma_1)_P^V}{\omega}\, d\omega \tag{14}$$

$$\alpha\left\{\frac{1+\kappa_P-\kappa_N}{2M}\right\} = \frac{1}{4\pi^2}\int_0^\infty [(\sigma_3-2\sigma_1)_A^V-(\sigma_3-2\sigma_1)_P^V]\, d\omega \tag{15}$$

All the cross-sections in Eqs. (12)-(15) are absorption cross-sections for photons on protons, the superscripts S and V indicate whether the photon is iso-scalar or iso-vector, the subscripts A and P indicate whether the helicity of the photon is anti-parallel or parallel to the proton spin, and the sub-scripts 3 and 1 imply that the cross-section is a partial cross-section for absorption in iso-spin states $\frac{3}{2}$ or $\frac{1}{2}$.

Equation (14) is the well known sum rule of Cabibbo and Radicati, and has been thoroughly discussed in the literature. Some interesting features of the other sum rules are discussed below.

5. Remarks:

(a) If one adds Eqs. (12) and (13), one obtains a sum rule for $\kappa_P^2+\kappa_N^2$. This sum rule agrees with the sum rule one would infer for this quantity from the work of Drell and Hearn.[4] Note that we were able to obtain separate sum rules for iso-scalar and iso-vector moments because of our assumption that the iso-spin and hypercharge were separately constants of the motion. Drell and Hearn only assumed the conservation of total charge.

(b) If one assumes that the cross-sections are dominated by the (3,3) resonance [isobar model], Eq. (12) leads to the relation $\kappa_P = -\kappa_N$, which is in fair agreement with experiment.[13] Substituting this relation in Eq. (13), and staying within the framework of the isobar model, one finds that Eq. (13) is also well satisfied. Note that the numerics here reduces to that in the paper of Drell and Hearn.

These successes of the isobar model should <u>not</u>, however, lead one to infer that the spin $\frac{3}{2}$ resonances play a decisive role in determining the static properties of baryons.

(c) We find it remarkable that Eq. (13), derived by us as an almost-exact sum rule in its own right, appears to follow from a sum rule of Fubini, Segrè and Walecka [14] [based on the algebra of U(12)] if one makes the empirical approximation $\kappa_P + \kappa_N = 0$. Clearly both sum rules can not simultaneously be regarded as almost-exact. Either the difference is of trivial origin, having its roots in the subtraction ambiguities which worried Fubini, Segrè and Walecka, or it points towards some deep relationship between the algebra of U(12) and the infinite dimensional algebra considered by us. We do not, at the moment, have a clear cut answer.

(d) It has been noted elsewhere [10] that the sum rules (14) and (15) do not admit of even approximate saturation by the (3,3) resonance. We feel, therefore, that they provide a good testing ground for representation mixing theories of the type recently proposed.[15]

(e) By evaluating the above sum rules and their U-spin and V-spin counterparts, in an approximation suggested by Alessandrini, Bég and Brown [16] it is possible to derive many of the attractive results of SU(6). We hope to report on this in the near future.

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