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Y/OWI/SUB-78/22303/10

TECHNICAL MEMORANDUM REPORT RSI-0064

ANALYTICAL APPROXIMATIONS TO THE
THERMOELASTIC BEHAVIOR OF REPOSITORY CONFIGURATIONS

Submitted To

Office of Waste Isolation
Oak Ridge, Tennessee

operated by

Union Carbide Corporation
Nuclear Division

for the

Department of Energy

By

Gary D. Callahan

and

Paul F. Gnirk

January 6, 1978

Revised and Resubmitted on October 13, 1978

This report was prepared by RE/SPEC Inc., under Subcontract No. 89Y-22303C with Union Carbide Corporation, Nuclear Division, under Contract No. W-7405-eng-26 with the Department of Energy. The subcontract was administered by the Office of Waste Isolation, Union Carbide Corporation, Nuclear Division.

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FOREWORD

This report was prepared under a subcontract with Union Carbide Corporation, Nuclear Division, a DOE contractor. The subcontract was administered by the Office of Waste Isolation and is part of the National Waste Terminal Storage (NWTS) Program. The principal objective of the NWTS Program is to provide facilities in various deep geologic formations at multiple locations in the United States which will safely dispose of commercial radioactive waste, which must be delivered to a Federal Repository for terminal storage. Some of the expected wastes produce both heat and radioactivity. This situation leads to many unique problems in rock mechanics. This report addresses a particular problem relative to the Rock Mechanics Program.

The overall objective of the OWI Rock Mechanics Program is to predict the response of a rock mass hosting a waste repository during its construction and operation, as well as the post-operational phase. The operational phase is expected to be approximately 20 years while the post-operational phase will last until the repository no longer poses any potential hazard to mankind, a period that may last several hundred thousand years. The Rock Mechanics Program is concerned with near field effects on mine stability as well as far field effects relative to the overall integrity of the geologic containment of waste.

In order to accomplish the objectives of the Rock Mechanics Program, numerical simulation, laboratory (including bench scale), and field studies are under way. The laboratory and field studies provide input to the numerical simulations and also the opportunity for validation of the predictive capabilities of the computer codes. Ultimately, the computer codes will provide the predicted response of the host rock mass and thereby form an essential part of the overall Rock Mechanics Program.

The study provided herein originated from far field thermomechanical concerns relating to the conceptual storage of radioactive waste and may be viewed as a complementary study to previous analyses. Specifically, analytical solutions to approximate the surface uplift resulting from temperature fields induced by the burial of heat generating radioactive waste are presented. Comparisons with earlier numerical results are also included.

The technical contents of this report have been reviewed by Joe L. Ratigan and John D. Osnes and the report was typed by Ms. Julie S. Annicchiarico.

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January 6, 1978

TECHNICAL MEMORANDUM REPORT RSI-0064

TO: Office of Waste Isolation
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Oak Ridge, TN 37830
Attn.: Dr. James E. Russell
Dr. William C. McClain

FROM: Mr. Gary D. Callahan
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SUBJECT: Selected Analytic Approximations to the Thermoelastic Surface
Displacement of Axially Symmetric Repository Geometries (Union
Carbide Corporation, Nuclear Division Subcontract No. 89Y-22303;
RSI/001000/FY78).

1. INTRODUCTORY REMARKS

The following three sections of this report present three different analytical solutions to approximate the surface uplift of a heated repository. The first solution assumes a temperature distribution which is a function only of the vertical coordinate (z) constant throughout the radius and the lateral strain is assumed to be zero. The second solution assumes a temperature distribution which is a function of the radial coordinate (r) constant throughout the height and assumes no radial displacement at the outer boundary. The third solution is similar to the second solution; the difference being that the outer boundary is assumed to be traction free. The second solution is subsequently derived from the third solution by superposing the elastic solution obtained from imposing a constant radial displacement at the outer boundary of a solid cylinder. The first solution

consistently gives the highest vertical displacements due to the lateral constraints. In Section 5, the solutions are summarized and specific temperature distribution forms are used to formulate expressions for the vertical displacements. Section 6 presents some example problems with comparisons to previous finite element analyses of various repository models (bedded and dome salt)^{1*}. Generally, the first and second solutions appear to bound the finite-element computations of the surface uplift.

Throughout this document, axially symmetric geometry will be assumed with the orthogonal coordinate system defined by r , θ , and z . Representations for stress, strain, and temperature are σ , ϵ , and T , respectively, and symbols identifying Young's modulus, Poisson's ratio, and the thermal expansion coefficient are E , ν , and α , respectively.

* Superscript numbers refer to references cited at the end of the text.

2. THERMOELASTIC SURFACE DISPLACEMENT SOLUTION-I (TSDS-I)

The axially symmetric repository situation described by Figure 2.1 may be approximated by a thin rod of material about the centerline of the repository. Subsequently, the temperature distribution may be represented as a function of z only, i.e.

$$T(r,z,t) \doteq T(z,t)$$

as depicted in Figure 2.2.

Figure 2.1. Axially Symmetric Repository Representation.

Figure 2.2. Assumed Temperature Distribution and Repository Idealization.

The appropriate strain displacement relations are:

$$\epsilon_r = \frac{\partial u}{\partial r}, \quad \epsilon_\theta = \frac{u}{r}, \quad \epsilon_z = \frac{\partial w}{\partial z}, \quad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \quad (2.1)$$

where u , v , and w represent displacements in the r , θ , and z coordinate directions, respectively.

Boundary conditions for the problem are

$$\sigma_z = 0 \quad \text{at} \quad z = H \quad (2.2a)$$

$$w = 0 \quad \text{at} \quad z = 0 \quad (2.2b)$$

Also, we will suppose that, everywhere

$$u = \frac{\partial u}{\partial r} = 0 \quad (2.3)$$

which requires that

$$\epsilon_r = \epsilon_\theta = 0 \quad (2.4)$$

The thermoelastic stress-strain relations are:

$$\epsilon_r - \alpha T = \frac{1}{E} [\sigma_r - \nu (\sigma_\theta + \sigma_z)] \quad (2.5a)$$

$$\epsilon_\theta - \alpha T = \frac{1}{E} [\sigma_\theta - \nu (\sigma_r + \sigma_z)] \quad (2.5b)$$

$$\epsilon_z - \alpha T = \frac{1}{E} [\sigma_z - \nu (\sigma_r + \sigma_\theta)] \quad (2.5c)$$

Where T represents the temperature rise above the assumed geothermal temperature distribution.

Utilizing eqns. (2.5a) and (2.5b) with conditions (2.4), we find:

$$\sigma_r = \sigma_\theta \quad (2.6)$$

Thus, by substitution of eqn. (2.6) and condition (2.2b) into eqn. (2.5a) (or eqn. 2.5b), it follows that:

$$\begin{aligned}
\sigma_r - \nu(\sigma_r + \sigma_z) &= -\alpha ET \\
\sigma_r(1 - \nu) - \nu\sigma_z &= -\alpha ET \\
\sigma_r = \sigma_\theta &= \left(\frac{\nu}{1-\nu}\right)\sigma_z - \frac{\alpha ET}{1-\nu}
\end{aligned} \tag{2.7}$$

Substitution of eqn. (2.7) into eqn. (2.5c) gives:

$$\begin{aligned}
E(\epsilon_z - \alpha T) &= \sigma_z - 2\nu\sigma_r \\
\sigma_z &= E(\epsilon_z - \alpha T) + 2\nu\left(\frac{\nu}{1-\nu}\right)\sigma_z - \left(\frac{2\nu}{1-\nu}\right)\alpha ET \\
\sigma_z \left\{1 - \frac{2\nu^2}{1-\nu}\right\} &= E\epsilon_z - \alpha ET \left\{1 + \frac{2\nu}{1-\nu}\right\} \\
\sigma_z \left\{\frac{1-\nu-2\nu^2}{1-\nu}\right\} &= E\epsilon_z - \left(\frac{1+\nu}{1-\nu}\right)\alpha ET
\end{aligned} \tag{2.8}$$

Substitution of eqn. (2.1) into eqn. (2.8) gives:

$$\sigma_z \left\{\frac{1-\nu-2\nu^2}{1-\nu}\right\} = E \frac{dw}{dz} - \left(\frac{1+\nu}{1-\nu}\right)\alpha ET \tag{2.9}$$

Furthermore, since this solution involves linear elasticity, we may assume that the term on the left hand side of eqn. (2.9) is zero and subsequently superpose this value for the final solution. However, in this case, we are interested in strains and displacements resulting from increases in temperature, whereas, the effect of σ_z results in gravitational displacements which have already occurred. Therefore the term on the left hand side of eqn. (2.9) will be permanently neglected and eqn. (2.9) becomes

$$\epsilon_z = \frac{dw}{dz} = \left(\frac{1+\nu}{1-\nu}\right)\alpha T \tag{2.10}$$

Since we have assumed the temperature distribution to be a function of z , the vertical displacement at any time t is given by integration of eqn. (2.10) yielding

$$w(z) = \left(\frac{1+\nu}{1-\nu}\right) \alpha \int_0^z [T(z',t) - T_0(z')] dz' \quad (2.11)$$

where $T(z')$ is the initial geothermal temperature distribution, and $w(0)$ is assumed to be zero.

Assuming H to be the depth of temperature perturbation, the vertical surface displacement (i.e. at $z = H$) is then

$$w(H) = \left(\frac{1+\nu}{1-\nu}\right) \alpha \int_0^H [T(z,t) - T_0(z)] dz \quad (2.12)$$

3. THERMOELASTIC SURFACE DISPLACEMENT SOLUTION-II (TSDS-II)

As an approximate solution to the surface displacements (thermoelastic) of a repository model, consider a solid circular disk with outer radius b and height h as shown in Figure 3.1.

Figure 3.1. Circular Disk Geometry.

Conditions of plane stress (z -direction) will be assumed with the upper and lower surfaces of the disk being traction free. In reality, the plane stress conditions cannot be satisfied² unless the temperature distribution is a function of the z -direction only. However, when the temperature does not vary over the thickness of the disk, we may assume that the stress and displacement due to heating also do not vary over the thickness.³ Consistent with the numerical modeling procedures, the radial displacement, u , is specified as zero at the centerline and at $r = b$.

Observing the temperature profiles shown in Appendix A of Reference 1, one sees that the temperature distribution along the centerline (i.e. through the height of the disk) is perturbed only over a few thousand feet (at least for short times and the lower thermal loadings). If one is to assume a temperature perturbation over 2,000 feet (effectively the height of the disk), then the ratio of h/b is $1/5$ for the bedded salt model and $1/8$ for the dome salt model. Therefore, the disk approximation is not totally unreasonable.

Also viewing the mid-plane temperature distributions, one sees a relatively constant temperature above the repository that drops down at the outer edge and becomes fairly constant again toward the outer boundary of the model. Therefore, a ramp temperature distribution as shown in Figure 3.2, constant throughout the height of the disk, will be used to simulate the various temperature rises experienced in the finite-element models.

Figure 3.2. Radial Temperature Distribution.

The solution to this problem may then be derived from those given by Timoshenko and Goodier³ or Boley and Weiner².

The stresses σ_r and σ_θ satisfy the equation of equilibrium

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \quad (3.1)$$

and the radial displacement $u(r)$ is subject to the boundary conditions

$$u(0) = 0 \quad (3.2a)$$

$$u(b) = 0 \quad (3.2b)$$

The temperature distribution as a function of r is described as follows:

$$T(r) = \begin{cases} T_0 & 0 \leq r \leq r_1 \\ \frac{T_0(r_2 - r)}{r_2 - r_1} & r_1 < r < r_2 \\ 0 & r_2 \leq r \leq b \end{cases} \quad (3.3)$$

where T_0 would represent a temperature rise above the initial geothermal distribution.

The thermoelastic stress-strain relations are given by eqn. (2.5).

Rewriting in terms of the strains, eqns. (2.5) become:

$$\sigma_r = \frac{E}{1-\nu^2} [\epsilon_r + \nu \epsilon_\theta - (1 + \nu) \alpha T] \quad (3.4a)$$

$$\sigma_\theta = \frac{E}{1-\nu^2} [\epsilon_\theta + \nu \epsilon_r - (1 + \nu) \alpha T] \quad (3.4b)$$

Upon substitution of eqns. (3.4) and (2.1) into eqn. (3.1), we have

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} = (1 + \nu) \alpha \frac{dT}{dr} \quad (3.5)$$

which may be written

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d(ru)}{dr} \right] = (1 + \nu) \alpha \frac{dT}{dr} \quad (3.6)$$

Integration of equation (3.6) yields

$$u = (1 + \nu) \alpha \frac{1}{r} \int_0^r T r dr + C_1 r + \frac{C_2}{r} \quad (3.7)$$

Using eqn. (3.7) and noting that

$$\lim_{r \rightarrow 0} \frac{1}{r} \int_0^r T r dr = 0$$

we see that $C_2 = 0$ in order to satisfy (3.2a).

Using eqns. (3.2b) and (3.7), C_1 becomes

$$C_1 = - (1 + \nu) \frac{\alpha}{b^2} \int_0^b T r dr \quad (3.8)$$

and eqn. (3.7) becomes

$$u = (1 + \nu) \frac{\alpha}{r} \int_0^r T r dr - (1 + \nu) \frac{\alpha r}{b^2} \int_0^b T r dr \quad (3.9a)$$

Using eqn. (3.3) and rewriting, eqn. (3.9a) becomes

$$u = (1 + \nu) \frac{\alpha}{r} \int_0^r T r dr - (1 + \nu) \frac{\alpha T_o r}{b^2} \left\{ \frac{r_1^2}{2} + \frac{1}{r_2 - r_1} \left(\frac{r_2(r_2^2 - r_1^2)}{2} - \frac{r_2^3 - r_1^3}{3} \right) \right\} \quad (3.9b)$$

Note that the term in brackets may be written as

$$r_1^2/2 + (r_2^2 + r_2 r_1 - 2r_1^2)/6 \quad (3.9c)$$

Equation (3.9b) describes the radial displacement at any radii resulting from the heating described by eqn. (3.3).

Substituting eqn. (3.10) into eqns. (3.4) using eqns. (2.1) yields the following stress values:

$$\sigma_r = -\alpha E \left[\frac{1}{r^2} \int_0^r T r dr + \frac{1 + \nu}{(1 - \nu)b^2} \int_0^b T r dr \right] \quad (3.10a)$$

$$\sigma_\theta = \alpha E \left[-T + \frac{1}{r^2} \int_0^r T r dr - \frac{(1 + \nu)}{(1 - \nu)b^2} \int_0^b T r dr \right] \quad (3.10b)$$

The vertical strain, ϵ_z , may now be evaluated using eqns. (3.10) with eqn. (2.5c) which results in

$$\epsilon_z = (1 + \nu)\alpha \left[T + \frac{2\nu}{(1 - \nu)b^2} \int_0^b T r dr \right] \quad (3.11)$$

From eqns. (2.1) and (3.11)

$$\epsilon_z = \frac{\partial w}{\partial z} = (1 + \nu)\alpha \left[T + \frac{2\nu}{(1 - \nu)b^2} \int_0^b T r dr \right]$$

and upon integration with respect to z , we have

$$w(r,z) = (1 + \nu) \alpha z \left[T + \frac{2\nu}{(1-\nu)b^2} \int_0^b T r dr \right] + f(r) \quad (3.12a)$$

where $f(r)$ is an integration function of r . Since we are interested in the net vertical displacement, $w(r,0)$ is arbitrarily fixed at zero. This requires that $f(r) = 0$ and the vertical displacement at $z = h$ becomes

$$w(r,h) = (1 + \nu) \alpha h \left[T + \frac{2\nu}{(1-\nu)b^2} \int_0^b T r dr \right] \quad (3.12b)$$

Using the prescribed temperature distribution given in eqn. (3.3), eqn. (3.12b) becomes

$$w(r,h) = (1 + \nu) \alpha h \left[T + \frac{\nu T_o}{(1-\nu)b^2} \left(r_1^2 + \frac{1}{3} (r_2^2 + r_2 r_1 - 2r_1^2) \right) \right] \quad (3.12c)$$

4. THERMOELASTIC SURFACE DISPLACEMENT SOLUTION-III (TSDS-III)

The formulation of the solution in this section is identical to the solution derived in Section 3 except for a change in the boundary conditions. In this case, we desire the outer disk surface ($r=b$) to be traction free. Thus eqns. (3.2) become

$$\sigma_r = 0 \quad \text{at} \quad r = b \quad (4.1a)$$

$$u = 0 \quad \text{at} \quad r = 0 \quad (4.1b)$$

Equation (3.7) remains valid and C_2 is again zero. Using eqn. (4.1a) in eqn. (3.7) we find

$$C_1 = (1 - \nu) \frac{\alpha}{b^2} \int_0^b T r dr \quad (4.2)$$

which upon substitution into eqn. (3.7) yields

$$u = (1 + \nu) \frac{\alpha}{r} \int_0^r T r dr + (1 - \nu) \frac{r\alpha}{b^2} \int_0^b T r dr \quad (4.3)$$

Using eqn. (4.3) with eqns. (2.1) and (3.4) the stresses become

$$\sigma_r = \alpha E \left(\frac{1}{b^2} \int_0^b T r dr - \frac{1}{r^2} \int_0^r T r dr \right) \quad (4.4a)$$

$$\sigma_\theta = \alpha E \left(-T + \frac{1}{b^2} \int_0^b T r dr + \frac{1}{r^2} \int_0^r T r dr \right) \quad (4.4b)$$

Using eqns. (4.4) with eqn. (2.5c) yields the vertical strain, viz

$$\epsilon_z = \frac{\partial w}{\partial z} = (1 + \nu) \alpha T - \frac{2\nu\alpha}{b^2} \int_0^b T r dr \quad (4.5)$$

Similar to the process used to derive eqns. (3.12), integration of eqn. (4.5) yields

$$w(r,z) = (1 + \nu) \alpha T z - \frac{2\nu\alpha z}{b^2} \int_0^b T r dr \quad (4.6a)$$

and for the prescribed temperature distribution given in eqn. (3.3) becomes

$$w(r,h) = \alpha h \left[(1 + \nu) T - \frac{\nu T_0}{b^2} \left(r_1^2 + \frac{1}{3} (r_2^2 + r_2 r_1 - 2r_1^2) \right) \right] \quad (4.6b)$$

Now we desire to reformulate the first solution (TSDS-I) by superposing the elastic solution which requires that the radial displacement at $r = b$ be zero. Clearly from eqn. (4.3), the required displacement is

$$u(b) = \frac{2\alpha}{b} \int_0^b T r dr = \kappa = \text{constant} \quad (4.7)$$

Therefore superposition of the displacements and stresses derived from the application of the displacement conditions

$$u(b) = -\kappa \quad (4.8a)$$

$$u(0) = 0 \quad (4.8b)$$

with the stresses and displacements derived from (TSDS-III) should yield the solution given in (TSDS-II).

Equation (3.7) remains valid (again $C_2 = 0$) and neglecting temperature effects results in

$$u = C_3 r$$

Using (4.8a) $C_3 = -\kappa/b$ and thus

$$u = -\frac{\kappa r}{b} \quad (4.9)$$

with κ given by eqn. (4.7).

From eqns. (2.1) and (4.9) we find that

$$\epsilon_r = \epsilon_\theta = -\kappa/b \quad (4.10)$$

Using eqns. (4.10) in eqns. (2.5) and neglecting temperature effects yields

$$\sigma_r = \sigma_\theta = - \frac{\kappa E}{b(1-\nu)} \quad (4.11)$$

Substituting eqns. (4.11) into eqn. (2.5c) with $T = 0$ gives

$$\epsilon_z = \left(\frac{\nu}{1-\nu} \right) \frac{2\kappa}{b} \quad (4.12)$$

Using eqn. (2.1) and integrating eqn. (4.12), we have

$$w(z) = \left(\frac{\nu}{1-\nu} \right) \frac{2\kappa}{b} z + C_4$$

We may eliminate the vertical translation by arbitrarily specifying the vertical displacement to be zero at $z = 0$ which requires that $C_4 = 0$. Thus the net displacement without rigid body motion is

$$w(z) = \left(\frac{\nu}{1-\nu} \right) \frac{2\kappa}{b} z \quad (4.13)$$

Superposing eqns. (4.3) and (4.9) and utilizing (4.7) the radial displacement is

$$u = (1 + \nu) \frac{\alpha}{r} \int_0^r T r dr + (1 - \nu) \frac{r\alpha}{b^2} \int_0^b T r dr - \frac{2r\alpha}{b^2} \int_0^b T r dr \quad (4.14a)$$

$$u = (1 + \nu) \frac{\alpha}{r} \int_0^r T r dr - (1 + \nu) \frac{r\alpha}{b^2} \int_0^b T r dr \quad (4.14b)$$

Equation (4.14b) is seen to be identical to eqn. (3.9a).

In order to find the vertical displacement, we superpose eqns. (4.6a) and (4.13) which yields:

$$\begin{aligned}
 w(r,z) &= \int_0^z \left[(1+\nu) \alpha T - \frac{2\nu\alpha}{b^2} \int_0^b T r dr \right] dz + \frac{4\nu\alpha z}{(1-\nu)b^2} \int_0^b T r dr \\
 &= (1+\nu) \alpha T z + \frac{2\nu(1+\nu)\alpha z}{(1-\nu)b^2} \int_0^b T r dr \\
 w(r,z) &= (1+\nu) \alpha z \left[T + \frac{2\nu}{(1-\nu)b^2} \int_0^b T r dr \right] \quad (4.15)
 \end{aligned}$$

Equation 4.15 is seen to be identical to eqn. (3.12a).

In evaluating the stresses we superpose eqns. (4.4) and (4.11) and use eqn. (4.7) which gives

$$\begin{aligned}
 \sigma_r &= \alpha E \left(\frac{1}{b^2} \int_0^b T r dr - \frac{1}{r^2} \int_0^r T r dr - \frac{2}{b^2(1-\nu)} \int_0^b T r dr \right) \\
 \sigma_\theta &= \alpha E \left(-T + \frac{1}{b^2} \int_0^b T r dr + \frac{1}{r^2} \int_0^r T r dr - \frac{2}{b^2(1-\nu)} \int_0^b T r dr \right)
 \end{aligned}$$

Simplifying, we have

$$\sigma_r = -\alpha E \left(\frac{1+\nu}{(1-\nu)b^2} \int_0^b T r dr + \frac{1}{r^2} \int_0^r T r dr \right) \quad (4.16a)$$

$$\sigma_\theta = \alpha E \left(-T - \frac{1+\nu}{(1-\nu)b^2} \int_0^b T r dr + \frac{1}{r^2} \int_0^r T r dr \right) \quad (4.16b)$$

Equations (4.16) are seen to be identical to eqns. (3.10).

5. SUMMARIZATION OF SOLUTIONS

All solutions presented are for axially-symmetric geometry and neglect any vertical stresses. The first solution (TSDS-I) assumes the temperature distribution to be spacially dependent on z only with negligible lateral strain. The second (TSDS-II) and third (TSDS-III) solutions are approximations for a thin disk with the temperature distribution being spacially dependent on r only. TSDS-II assumes the outer boundary of the disk to be fixed against radial displacement and TSDS-III assumes the outer boundary of the disk to be stress free. A supplemental elastic solution (SUPP-I) superimposed on TSDS-III negating the radial displacement at the outer boundary yields TSDS-II.

The specific stress and displacement results for each solution will be summarized below. In addition, the results will be further summarized for prescribed temperature distributions.

5.1. TSDS-I Summary

$$u = 0 \quad (5.1)$$

$$w(z) = \left(\frac{1+\nu}{1-\nu} \right) \alpha \int_0^z [T(z,t) - T_o(z)] dz \quad (5.2)$$

$$\sigma_r = \sigma_\theta = - \frac{\alpha E T}{1-\nu} \quad (5.3)$$

5.2. TSDS-II Summary

$$u(r) = (1 + \nu) \alpha \left[\frac{1}{r} \int_0^r T r dr - \frac{r}{b^2} \int_0^b T r dr \right] \quad (5.4)$$

$$w(r,z) = (1 + \nu) \alpha z \left[T + \frac{2\nu}{(1-\nu)b^2} \int_0^b T r dr \right] \quad (5.5)$$

$$\sigma_r = - \alpha E \left[\frac{1}{r^2} \int_0^r T r dr + \frac{(1+\nu)}{(1-\nu)b^2} \int_0^b T r dr \right] \quad (5.6)$$

$$\sigma_{\theta} = \alpha E \left[-T + \frac{1}{r^2} \int_0^r T r dr - \frac{1+\nu}{(1-\nu)b^2} \int_0^b T r dr \right] \quad (5.7)$$

5.3. TSDS-III Summary

$$u(r) = (1+\nu) \alpha \left[\frac{1}{r} \int_0^r T r dr + \left(\frac{1-\nu}{1+\nu} \right) \frac{r}{b^2} \int_0^b T r dr \right] \quad (5.8)$$

$$w(r,z) = (1+\nu) \alpha z \left[T - \frac{2\nu}{(1+\nu)b^2} \int_0^b T r dr \right] \quad (5.9)$$

$$\sigma_r = -\alpha E \left[\frac{1}{r^2} \int_0^r T r dr - \frac{1}{b^2} \int_0^b T r dr \right] \quad (5.10)$$

$$\sigma_{\theta} = \alpha E \left[-T + \frac{1}{r^2} \int_0^r T r dr + \frac{1}{b^2} \int_0^b T r dr \right] \quad (5.11)$$

5.4. SUPP-I Summary

$$\kappa = \frac{2\alpha}{b} \int_0^b T r dr$$

$$u(r) = -\frac{\kappa r}{b} \quad (5.12)$$

$$w(z) = \left(\frac{\nu}{1-\nu} \right) \frac{2\kappa}{b} z \quad (5.13)$$

$$\sigma_r = \sigma_{\theta} = -\frac{E\kappa}{b(1-\nu)} \quad (5.14)$$

5.5. Vertical Surface Displacement Summary for Prescribed Temperature Distributions

Prescribed temperature distributions will be applied to eqns. (5.2), (5.5), and (5.9) in order to facilitate evaluation in the example problems presented in Section 6. For TSDS-I, the temperature distribution will be

assumed to be linear as shown in Figure 5.1 merely to allow ease in integrating. For TSDS-I and TSDS-II, the temperature distribution will be as given by eqn. (3.3).

Figure 5.1. Assumed Temperature Distribution for TSDS-I.

With the temperature distribution prescribed by Figure 5.1, the vertical surface displacement resulting from TSDS-I (See eqn. 5.2) is:

$$w(H) = \left(\frac{1+\nu}{1-\nu} \right) \frac{\alpha}{2} (T_4 - T_2) H \quad (5.15)$$

For solutions TSDS-II and TSDS-III, the problem arises of determining the height of the disk to be analyzed. Since the temperature distribution is indeed a function of z , the height is nebulous but very important. Therefore, a reasonable height (over which the temperature is constant) may be chosen such that the integral of this region is nearly equal to the integral of the actual temperature distribution over the height of perturbation. Mathematically we are saying that the height, h , is given by:

$$h = \frac{1}{\Delta T} \int_0^H \left(T(z, t) - T_0(z) \right) dz$$

where H is the height over which the initial temperature field is perturbed and ΔT is the maximum temperature difference reached. Therefore, the disk height for the distribution shown in Figure 5.1 would be

$$h = \frac{H(T_4 - T_2)}{2(T_4 - T_2)} = \frac{H}{2} \quad (5.16)$$

With h defined by eqn. (5.15) and the radial temperature distribution defined by eqn. (3.3), the vertical surface displacement resulting from TSDS-II (see eqn. 5.5) is as follows:

$$\zeta(r_1, r_2) = \frac{1}{T_0} \int_0^b T r dr = \frac{1}{2} (r_1^2 + \frac{1}{3} (r_2^2 + r_2 r_1 - 2r_1^2)) \quad (5.17)$$

$$\underline{0 \leq r \leq r_1}$$

$$w(h) = (1 + \nu) \alpha h T_0 \left[1 + \frac{2\nu}{(1-\nu)b^2} \zeta(r_1, r_2) \right] \quad (5.18a)$$

$$\underline{r_1 \leq r \leq r_2}$$

$$w(r, h) = (1 + \nu) \alpha h T_0 \left[\frac{r_2 - r}{r_2 - r_1} + \frac{2\nu}{(1-\nu)b^2} \zeta(r_1, r_2) \right] \quad (5.18b)$$

$$\underline{r_2 \leq r \leq b}$$

$$w(h) = (1 + \nu) \alpha h T_0 \left[\frac{2\nu}{(1-\nu)b^2} \zeta(r_1, r_2) \right] \quad (5.18c)$$

For TSDS-III, the vertical surface displacement is (see eqn. 5.9)

$$\underline{0 \leq r \leq r_1}$$

$$w(h) = (1 + \nu) \alpha h T_0 \left[1 - \frac{2\nu}{(1+\nu)b^2} \zeta(r_1, r_2) \right] \quad (5.19a)$$

$$\underline{r_1 \leq r \leq r_2}$$

$$w(r, h) = (1 + \nu) \alpha h T_0 \left[\frac{r_2 - r}{r_2 - r_1} - \frac{2\nu}{(1+\nu)b^2} \zeta(r_1, r_2) \right] \quad (5.19b)$$

$$\underline{r_2 \leq r \leq b}$$

$$w(h) = -(1 + \nu) \alpha h T_0 \left[\frac{2\nu}{(1+\nu)b^2} \zeta(r_1, r_2) \right] \quad (5.19c)$$

6. EXAMPLE PROBLEMS

Examples considered herewith specifically refer to computations from Reference 1. With this notion, several figures from that report are included herein and referred to as labeled in that report. The material properties used in these examples are those of salt (i.e. $\nu = 0.4$, $\alpha = 22 \times 10^{-6}/^{\circ}\text{F}$).

6.1. Example Problem 1

Consider the bedded salt model at 100 years with a gross thermal loading (GTL) of 75 kW/acre of spent fuel 10 years out of a fresh UO_2 fueled pressurized water reactor (PWR). From Figure A.1.5¹ $H=1,300$ feet, $T_2 = 87^{\circ}\text{F}$, and $T_4 = 197^{\circ}\text{F}$. From eqn. (5.12) TS_{DS} - I yields

$$w(H) = \left(\frac{1+.4}{1-.4} \right) \frac{22 \times 10^{-6}}{2} (197-87) (1300 \times 12) = 44.04 \text{ inches}$$

From eqn. (5.16)

$$h = 650 \text{ feet}$$

From Fig. A.1.6¹, $r_1 = 2,800$ feet, $r_2 = 3,800$ feet and $b = 10,000$ feet. Then using eqns. (5.17) and (5.18), TS_{DS}-II with $T_0 = T_4 - T_2$ yields

$$\begin{aligned} \zeta(2800, 3800) &= \frac{1}{2} \left(2800^2 + \frac{1}{3} (3800^2 + (3800)(2800) - 2(28000^2)) \right) \\ &= 5.487 \times 10^6 \text{ ft.}^2 \end{aligned}$$

$$\underline{0 \leq r \leq r_1}$$

$$\begin{aligned} w(h) &= (1 + .4) (22 \times 10^{-6}) (650 * 12) (110) \left[1 + \frac{2(.4)}{(1-.4)(10000)^2} (5.487 \times 10^6) \right] \\ &= 26.43 \left(1 + 1.333 \times 10^{-8} (5.487 \times 10^6) \right) \\ &= 28.36 \text{ inches} \end{aligned}$$

$$\underline{r_1 \leq r \leq r_2}$$

$$w(r, h) = 26.43 \left(\frac{3800-r}{1000} + 0.0732 \right)$$

$$\underline{r_2 \leq r \leq b}$$

$$w(h) = 26.43 (0.0732)$$

$$= 1.9 \text{ inches}$$

From eqns. (5.19), the resulting displacements for TSDS-III are readily found to be:

$$\underline{0 \leq r \leq r_1}$$

$$w(h) = 26.43 (1 - 0.0314) = 25.6 \text{ inches}$$

$$\underline{r_1 \leq r \leq r_2}$$

$$w(r, h) = 26.43 \left(\frac{3800 - r}{1000} - 0.0314 \right) \text{ inches}$$

$$\underline{r_2 \leq r \leq b}$$

$$w(h) = 26.43 (-0.0314) = -0.83 \text{ inches}$$

Note that the difference in displacement values between TSDS-II and TSDS-III is given by SUPP-I eqn. (5.13). In this case, we have

$$\kappa = \frac{T_o}{b} \frac{2\alpha}{\zeta(r_1, r_2)}$$

and

$$w(h) = \left(\frac{\nu}{1-\nu} \right) \frac{4\alpha T_o}{b^2} \zeta(r_1, r_2) h$$

$$= 2.76 \text{ inches}$$

Figure 6.1 compares the surface displacements from the analytical thermoelastic solutions with those computed in Reference 1 using the finite element method.

6.2. Example Problem 2

As a second example, consider the dome salt model at 500 years with a gross thermal loading of 150 kW/acre of high level waste from a fresh UO_2 fueled PWR 10 years after reprocessing. From Fig. A.4.9.¹ $H = 3200 \text{ ft.}$,

$T_2 = 103^\circ\text{F}$, and $T_4 = 166^\circ\text{F}$. From eqn. (5.12), the vertical surface displacement for TSDS-I is

$$w(H) = \left(\frac{1+.4}{1-.4}\right) \frac{22 \times 10^{-6}}{2} (166-103) (3200 \times 12) = 60.15 \text{ inches}$$

From eqn. (5.16)

$$h = 1550 \text{ feet}$$

From Fig. A.1.10¹ r_1 , r_2 , and b are chosen as 2000, 5625, and 16,000 feet, respectively. Using eqns. (5.17) and (5.18), the vertical surface displacement resulting from TSDS-II is:

$$\begin{aligned} \zeta(2000, 5625) &= \frac{1}{2} \left(2000^2 + \frac{1}{3} (5625^2 + (5625)(2000) - 2(2000)^2) \right) \\ &= 7.815 \times 10^6 \text{ ft.}^2 \end{aligned}$$

$$\underline{0 \leq r \leq r_1}$$

$$\begin{aligned} w(h) &= (1.4) (22 \times 10^{-6}) (1550 \times 12) (63) \left[1 + \frac{2(.4)}{(1-.4)(16000)^2} (7.815 \times 10^6) \right] \\ &= 36.09 \left[1 + 5.208 \times 10^{-9} (7.815 \times 10^6) \right] \\ &= 37.6 \text{ inches} \end{aligned}$$

$$\underline{r_1 \leq r \leq r_2}$$

$$w(r, h) = 36.09 \left[\frac{5625-r}{5625-2000} + 0.0407 \right]$$

$$\underline{r_2 \leq r \leq b}$$

$$w(h) = 36.09 (0.0407) = 1.47 \text{ inches}$$

Similarly the vertical surface displacement for TSDS-III is readily found to be:

$$\underline{0 < r \leq r_1}$$

$$w(h) = 36.09 \left[1 - 0.0174 \right] = 35.5 \text{ inches}$$

$$\underline{r_1 \leq r \leq r_2}$$

$$w(r, h) = 36.09 \left[\frac{5625-r}{5625-2000} - 0.0174 \right]$$

$$\underline{r_2 \leq r \leq b}$$

$$w(h) = 36.09 \left[- 0.0174 \right] = -0.63 \text{ inches}$$

Again the displacement difference between TSDS-II and TSDS-III is given by eqn. (5.13) and is

$$w(h) = \left(\frac{\nu}{1-\nu} \right) \frac{4\alpha T_o}{b^2} \zeta(r_1, r_2)$$

$$= 2.1 \text{ inches}$$

Figure 6.2 compares the analytical and finite element thermoelastic surface displacements for this example problem.

(RSI-0064)

Figure A.1.5. Bedded Salt Repository Model Centerline Temperatures Considering a Gross Thermal Loading of 75 kW/Acre of Spent Fuel 10 Years Out of a Fresh UO_2 Fueled PWR.

M.P. TEMP./BEDDED SALT/SF 75 KW/ACRE

(RSI-0064)

Figure A.1.6. Repository Mid-Plane Temperatures for the Bedded Salt Repository Model
Considering a Gross Thermal Loading of 75 kW/Acre of Spent Fuel 10 Years
Out of a Fresh CO_2 Fueled PWR.

Figure 6.1. Comparison of Analytical and Finite-Element Vertical Thermoelastic Displacements at the Surface (Example Problem 1).

(RSI-0064)

Figure A.4.9. Dome Salt Repository Model Centerline Temperatures Considering a Gross Thermal Loading of 150 kW/Acre of High Level Waste from a Fresh UO_2 Fueled PWR 10 Years After Reprocessing.

M.P. TEMP. /DOME SALT/HLW150 KW/ACRE

(RSI-0064)

Figure A.4.10. Repository Mid-Plane Temperatures for the Dome Salt Repository Model Considering a Gross Thermal Loading of 150 kW/Acre of High Level Waste from a Fresh UO_2 Fueled PWR 10 Years After Reprocessing.

Figure 6.2. Comparison of Analytical and Finite-Element Vertical Thermoelastic Displacements at the Surface (Example Problem 2).

LIST OF REFERENCES

- (1) Callahan, G. D. and Ratigan, J. L.; "Thermoelastic Analyses of Spent Fuel Repositories in Bedded and Dome Salt", Technical Memorandum Report RSI-0054, Prepared for the Office of Waste Isolation under Subcontract No. 89Y-22303C with Union Carbide Corporation, Nuclear Division, May 1977, Y/OWI/SUB-77/22303/4.
- (2) Boley, B. A. and Weiner, J. J.: Theory of Thermal Stresses, John Wiley and Sons, 1960.
- (3) Timoshenko, S. P. and Goodier, J. N.; Theory of Elasticity, McGraw-Hill, Inc., 1970.