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Stabilizing S.P.H. with Conservative Smoothing

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Stabilizing S.P.H. with Conservative Smoothing

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Abstract

There is an instability in certain S.P.H. (Smoothed Particle Hydrodynamics method) material dynamics computations. Evidence from analyses and experiments suggests that the instabilities in S.P.H. are not removable with artificial viscosities. However, the analysis shows that a type of conservative smoothing does remove the instability. Also, numerical experiments, on certain test problems, show that SPHCS, an S.P.H. code with conservative smoothing, compares well in accuracy with computations based on the von Neumann-Richtmyer method.

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1.0 INTRODUCTION

The S.P.H. instability was first documented in [12]. In [8] it was shown that conservative smoothing could stabilize S.P.H. This paper is an extension of the previous analyses. In the previous stability analyses the local equation of first variation (our mathematical model for the propagation of small perturbations in the S.P.H. equations) was developed as a perturbation from a uniform stress state; the following analysis develops the local equation of first variation as a perturbation from a **nonuniform** stress state.

The von Neumann - Fourier stability analyses involve a Fourier series analysis of the local equation of first variation. Numerous computational experiments with material dynamics codes have shown a close correlation between the predictions of von Neumann - Fourier stability analyses and the actual behavior of computations with respect to discrete - numerical stability. For details on von Neumann - Fourier stability analyses of computational material dynamics methods see [7] and [11].

Let $t^n = n\Delta t$, $\mu_j = j\Delta\mu$,

$$(1) \quad \mathbf{x}_j = (x_{j-\ell}, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_{j+\ell}) \in \mathbb{R}^{2\ell+1},$$

and consider the following equation:

$$(2) \quad \frac{\Delta^2 x^n}{\Delta t^2_j} = F_j^n(\mathbf{x}_j^{n-1}, \mathbf{x}_j^n, \mathbf{x}_j^{n+1}),$$

where F is composed of difference operators:

$$(3) \quad \frac{\Delta}{\Delta t}, \frac{\Delta}{\Delta\mu}, \dots, \frac{\Delta^{m+\tau}}{\Delta t^\tau \Delta\mu^m},$$

where $\tau = 0$ or 1 and m is some nonnegative integer. The equation of first variation of (2) is:

$$(4) \quad \frac{\Delta^2 \delta x^n}{\Delta t^2_j} = \delta F_j^n(\mathbf{x}_j^{n-1}, \mathbf{x}_j^n, \mathbf{x}_j^{n+1}),$$

where

$$(5) \quad \delta F_j^n(\mathbf{x}_j^{n-1}, \mathbf{x}_j^n, \mathbf{x}_j^{n+1}) = \sum_{\nu=-1}^1 \mathbf{A}^{n,\nu,j} \bullet \delta \mathbf{x}_j^{n+\nu},$$

with

$$(6) \quad \mathbf{A}^{n,\nu,j} = \text{grad}_{\mathbf{x}_j^{n+\nu}} F_j^n(\mathbf{x}_j^{n-1}, \mathbf{x}_j^n, \mathbf{x}_j^{n+1}).$$

Further, $\mathbf{A}^{n,\nu,j}$ is a vector with $2\ell + 1$ components and composed of difference operators as in (3), having coefficients which are functions of

$$(7) \quad n, j, \Delta t, \Delta\mu, x, \frac{\Delta x}{\Delta t}, \frac{\Delta x}{\Delta\mu}, \dots, \frac{\Delta^{m+\tau} x}{\Delta t^\tau \Delta\mu^m}$$

where $\tau = 0$ or 1 and m is some nonnegative integer.

Let $A_c^{n,\nu,j}$ be $A^{n,\nu,j}$ evaluated at the constant - coefficient data vector, C_c , i.e.,

$$(8) \quad C_c = \left(n_c, j_c, \Delta t_c, \Delta \mu_c, x_c, \frac{\Delta x}{\Delta t_c}, \frac{\Delta x}{\Delta \mu_c}, \dots, \frac{\Delta^{m+\tau} x}{\Delta t^\tau \Delta \mu^m c} \right)$$

where all the components of C_c are constants. Then the equation

$$(9) \quad \frac{\Delta^2 \delta x^n}{\Delta t^2 j} = \sum_{\nu=-1}^1 A^{n,\nu,j} \bullet \delta x_j^{n+\nu}$$

is called the local (at C_c) equation of first variation of (2).

For simplicity, assume the material interval is $[0, 2\pi]$, the total number of zones is J ,

$$(10) \quad J = 2\pi / \Delta \mu,$$

and J is an even integer. Then the shortest wavelength is

$$(11) \quad \ell_{\min} = 2\Delta \mu$$

and this is associated with the largest wavenumber

$$(12) \quad k_{\max} = \pi / \Delta \mu.$$

The set of integers, k , such that

$$(13) \quad -k_{\max} \leq k \leq +k_{\max}.$$

is the set of all possible wave numbers.

We call the following the von Neumann transformation:

$$(14) \quad F_{vN} \langle \delta x_j^n \rangle = (E_k)^j (\lambda_k)^n.$$

Apply the von Neumann transformation to the difference equation in δx , i.e., (9), to transform it to the following quadratic equation in λ_k :

$$(15) \quad \lambda_k^2 - 2b\lambda_k + c = 0,$$

with roots

$$(16) \quad \lambda_k^\pm = b \pm (b^2 - c)^{1/2}.$$

Recall the solutions to the three - level difference equation

$$(17) \quad f^{n+1} - 2bf^n + cf^{n-1} = 0,$$

are, in the equal roots case, i.e., $b^2 = c$ and

$$(18) \quad \lambda_k = \lambda_k^+ = \lambda_k^-$$

of the form

$$(19) \quad f^n = (\alpha + \beta n)(\lambda_k)^n;$$

and in the distinct roots case, i.e., $b^2 \neq c$ and

$$(20) \quad \lambda_k^+ \neq \lambda_k^-$$

of the form

$$(21) \quad f^n = \alpha(\lambda_k^+)^n + \beta(\lambda_k^-)^n.$$

The amplification factor for the k^{th} wavenumber, r_k , is defined by

$$(22) \quad r_k = \max |\lambda_k^\pm|.$$

Definition: Let k , Δt_c , $\Delta \mu_c$, N_c , be fixed finite numbers with further constraints being: wavenumber

$$(23) \quad k \in [-k_{\max}, +k_{\max}];$$

temporal increment

$$(24) \quad \Delta t_c > 0;$$

material increment

$$(25) \quad \Delta \mu_c > 0;$$

total number of timesteps

$$(26) \quad N_c \geq 1.$$

C_c is the constant - coefficient data vector whose components are used in the coefficients of the local equation of first variation, i.e., (9).

In the distinct roots case, i.e.,

$$(27) \quad \lambda_k^+ \neq \lambda_k^-,$$

if

$$(28) \quad (r_k)^n \leq 1 \quad \text{for} \quad 1 \leq n \leq N_c;$$

or in the equal roots case, i.e.,

$$(29) \quad \lambda_k^+ = \lambda_k^-,$$

if

$$(30) \quad n(r_k)^n \leq 1 \quad \text{for } 1 \leq n \leq N_c;$$

then we say that (2) is **absolutely D.-N. stable** at $(k, \Delta t_c, \Delta \mu_c, N_c, C_c)$.

Definition: If (2) is **not absolutely D.-N. stable** at $(k, \Delta t_c, \Delta \mu_c, N_c, C_c)$, then we say that it has an **absolutely D.-N. instability** or is **unstable in the absolute D.-N. sense** at $(k, \Delta t_c, \Delta \mu_c, N_c, C_c)$.

Remark: A necessary and sufficient condition for

$$(31) \quad n(r_k)^n \leq 1$$

all integer $n \geq 0$ is

$$(32) \quad 0 \leq r_k \leq \left(\frac{1}{3}\right)^{\frac{1}{3}}.$$

2.0 S.P.H.

We present the S.P.H. methodology in five steps. For background on S.P.H. see Barnes-Hernquist(1993).

Step 1. This step involves the Dirac delta distribution [a.k.a., generalized function, see Gel'fand and Shilov (1964, page 1)] centered at $\mathbf{x} \in \mathbb{R}^3$, i.e., $\delta(\mathbf{x}_i - \mathbf{x})$, where $\mathbf{x}_i$ is the variable of integration. The Dirac delta identity may be written:

$$(33) \quad f(\mathbf{x}) = \iiint_{\mathbb{R}^3} f(\mathbf{x}_i) \delta(\mathbf{x}_i - \mathbf{x}) d\mathbf{x}_i.$$

Step 2. This step replaces the generalized function, $\delta(\mathbf{x}_i - \mathbf{x})$, by a true function, $W(\cdot, \cdot)$, called the S.P.H. kernel; $W(\cdot, \cdot)$ is a function of two variables with $\mathbf{x}_i - \mathbf{x}$ being the first variable and h , called the smoothing length, being the second variable. Replace $\delta(\mathbf{x}_i - \mathbf{x})$ with $W(\mathbf{x}_i - \mathbf{x}, h)$ to get

$$(34) \quad f(\mathbf{x}) \approx \iiint_{\mathbb{R}^3} f(\mathbf{x}_i) W(\mathbf{x}_i - \mathbf{x}, h) d\mathbf{x}_i,$$

where the ' $\approx$ ' approximation is defined by the

S.P.H. Kernel Constraints:

SPH1 (δ approximator) $[h \rightarrow 0] \Rightarrow [W \rightarrow \delta \text{ in the generalized function sense}]$;

SPH2 (normalized) $\iiint_{\mathbb{R}^3} W(\mathbf{x}, h) d\mathbf{x} = 1$;

SPH3 (compact support) $\|\mathbf{x}\| \geq 2h \Rightarrow W(\mathbf{x}, h) = 0$;

SPH4 (flat top) $\text{grad}_{\mathbf{x}} W(\mathbf{0}, h) = \mathbf{0}$;

SPH5 (smooth) $W(\cdot, h) \in C^n, n \geq 0$;

SPH6 (nonnegative) $\forall \mathbf{x}, W(\mathbf{x}, h) \geq 0$;

SPH7 (radially - symmetric) $\|\mathbf{x}\| = \|\mathbf{y}\| \Rightarrow W(\mathbf{x}, h) = W(\mathbf{y}, h)$;

SPH8 (radially - decreasing) $\|\mathbf{x}\| \geq \|\mathbf{y}\| \Rightarrow W(\mathbf{x}, h) \leq W(\mathbf{y}, h)$.

Remark: Sometimes one or more of these constraints are relaxed.

Step 3. This step substitutes

$$(35) \quad f(\mathbf{x}) = F(\mathbf{x})$$

into (34) to get

$$(36) \quad F(\mathbf{x}) \approx \iiint_{\mathbb{R}^3} F(\mathbf{x}_i) W(\mathbf{x}_i - \mathbf{x}, h) d\mathbf{x}_i$$

and then takes the divergence with respect to $\mathbf{x}$ of both sides of (36) to get

$$(37) \quad \text{div}_{\mathbf{x}} \mathbf{F}(\mathbf{x}) \approx - \iiint_{\mathbb{R}^3} \mathbf{F}(\mathbf{x}_i) \text{grad}_{\mathbf{x}} W(\mathbf{x}_i - \mathbf{x}, h) d\mathbf{x}_i.$$

Remark: Alternatively, the standard, but longer, S.P.H. procedure gets (37) by substituting $\text{div}_{\mathbf{x}} \mathbf{F}(\mathbf{x})$ into (34), integrating by parts, applying the Gauss divergence theorem, and then using the compact support constraint, i.e., SPH3, to arrive at (37).

Step 4. This step transforms continuum integrals to discrete sums in preparation for digital computation,

$$(38) \quad \iiint_{\mathbb{R}^3} \mathbf{H}(\mathbf{x}) d\mathbf{x} \doteq \sum_{i=1}^{N_p} \mathbf{H}(\mathbf{x}_i) \text{Vol}_i$$

where N_p is the number of S.P.H. particles, $\mathbf{x}_i$ is the spatial coordinate of the i^{th} particle, Vol_i is the volume and

$$(39) \quad \text{Vol}_i = m_i V_i = m_i / \rho_i,$$

where m_i is the mass, V_i is the specific volume, and ρ_i is the mass density of the i^{th} particle.

We thus approximate (34) as follows

$$(40) \quad f(\mathbf{x}) \approx \sum_{i=1}^{N_p} f(\mathbf{x}_i) W(\mathbf{x}_i - \mathbf{x}, h) \text{Vol}_i,$$

where $\approx$ indicates the composition of the $\approx$ and $\doteq$ approximations. Similarly, (37) may be approximated as follows

$$(41) \quad \text{div}_{\mathbf{x}} \mathbf{F}(\mathbf{x}) \approx - \sum_{i=1}^{N_p} \mathbf{F}(\mathbf{x}_i) \text{grad}_{\mathbf{x}} W(\mathbf{x}_i - \mathbf{x}, h) \text{Vol}_i.$$

Step 5. This step applies the $\approx$ approximation to continuum material dynamics equations to produce discrete material dynamics equations. We illustrate on the following momentum evolution equation:

$$(42) \quad \rho \dot{\mathbf{u}} = -\text{div} \sigma$$

where σ is the Cauchy stress and with the overdot indicating the time derivative at a fixed point, $\mathbf{X}_j$, in the material, i.e.,

$$(43) \quad \dot{\mathbf{u}}_j \equiv \left. \frac{\partial \mathbf{u}}{\partial t} \right|_{\mathbf{X}_j}$$

and thus $\dot{\mathbf{u}}_j$ is the acceleration of particle j . A substitution of σ for $\mathbf{F}$ in (41) yields

$$(44) \quad m_j \dot{\mathbf{u}}_j \approx \sum_{i=1}^{N_p} \sigma(\mathbf{x}_i) \text{grad}_{\mathbf{x}} W(\mathbf{x}_i - \mathbf{x}_j, h) \text{Vol}_i \text{Vol}_j.$$

3.0 Stability Analysis of S.P.H. with Artificial Viscosity

The S.P.H. instability occurs in one -, two -, and three - dimensions and as the one - dimensional case is much the simplest we, therefore, present a simple set of one - dimensional S.P.H. equations to be analyzed for stability.

Let X be the material - length - coordinate (a.k.a., Lagrange - length - coordinate) and let μ be the material - mass - coordinate (a.k.a., Lagrange - mass - coordinate) where μ is defined in terms of X and the initial mass density at X , $\rho^0(X)$, as follows

$$(45) \quad \mu = \int_0^X \rho^0(X_i) dX_i.$$

Let x be the spatial - length - coordinate (a.k.a., Euler - length - coordinate) and let $\chi(\cdot, \cdot)$ be the motion function, i.e., if

$$(46) \quad x_j^n = \chi(\mu_j, t^n),$$

then at time t^n , x_j^n is the position in the spatial reference frame of the material point labeled μ_j . Similarly, the specific momentum, u_j^n , in the spatial reference frame at (μ_j, t^n) is given by

$$(47) \quad u_j^n = \partial \chi(\mu_j, t^n) / \partial t$$

and likewise, the specific volume, V_j^n , at (μ_j, t^n) is given by

$$(48) \quad V_j^n = \partial \chi(\mu_j, t^n) / \partial \mu.$$

Assume uniform material increments, i.e.,

$$(49) \quad \mu_{j+1} - \mu_j = \Delta\mu = m/Area$$

and assume unit cross sectional area, i.e.,

$$(50) \quad Area = 1.$$

Using

$$(51) \quad V_j \Delta\mu = Vol_j$$

a one - dimensional S.P.H. equation for the evolution of the specific momentum may be written as

$$(52) \quad \dot{u}_j = \Delta\mu \sum_{i=1}^{N_p} V_i V_j \sigma_{i,j} W'_{i,j} sign(x_i - x_j),$$

where

$$(53) \quad W'_{i,j} sign(x_i - x_j) = \partial W(|x_i - x_j|) / \partial x_i$$

with

$$(54) \quad \sigma_{i,j} = p_i + q_{i,j}$$

and $q_{i,j}$ is the viscous stress between particles i and j .

Single increment difference operators are denoted by an upper case delta with subscripts μ or t , when needed, e.g.,

$$(55) \quad \Delta_\mu x_{j+1/2} = x_{j+1} - x_j, \quad \Delta_t x^{n+1/2} = x^{n+1} - x^n,$$

but subscripts are suppressed when clear from context, e.g.,

$$(56) \quad \frac{\Delta x}{\Delta \mu} = \frac{\Delta_\mu x}{\Delta \mu} \quad \text{and} \quad \frac{\Delta x}{\Delta t} = \frac{\Delta_t x}{\Delta t}.$$

Similarly, double increment difference operators are denoted by:

$$(57) \quad D_\mu x_j = (x_{j+1} - x_{j-1})/2 \quad \text{and} \quad \frac{Dx}{\Delta \mu} = \frac{D_\mu x}{\Delta \mu}.$$

If

$$(58) \quad 0 < \Delta_\mu x \leq 2h \leq 2D_\mu x,$$

then the sum in (52) reduces to

$$(59) \quad \dot{u}_j = \Delta \mu \sum_{i=j-1}^{j+1} V_i V_j \sigma_{i,j} W'_{i,j} \text{sign}(x_i - x_j),$$

which may be written as

$$(60) \quad -\frac{\partial u_j}{\partial t} = \frac{DP}{\Delta \mu} \Big|_j \equiv \frac{P_{j,j+1} - P_{j,j-1}}{2\Delta \mu},$$

where

$$(61) \quad P_{j,j\pm 1} = \sigma_{j,j\pm 1} A_{j,j\pm 1},$$

with

$$(62) \quad A_{j,j\pm 1} = -2W'_{j,j\pm 1} V_{j\pm 1} V_j (\Delta \mu)^2.$$

Material law: For purposes of stability analysis it is convenient to introduce the *linear pV law*:

$$(63) \quad P_H(V) = p_{mc} - a_{mc}^2 (V - V_{mc}),$$

which is a special case of Hooke's Law with a_{mc} being the acoustic impedance; the mc - subscript on a symbol indicates a material constant. This determines the p - part of

our material law and for the q - part we take a simple, single increment version, of the von Neumann - Richtmyer - Landshoff form

$$(64) \quad q_{j,j\pm 1} = -\nu a_{mc} \Delta_\mu u_{j\pm 1/2},$$

where $\nu \geq 0$ is the dimensionless coefficient of artificial viscosity. Our t - discretization of (60) is

$$(65) \quad \Delta_t u_j^n \equiv u_j^{n+1/2} - u_j^{n-1/2} = -\Delta t \left. \frac{DP}{\Delta \mu} \right|_j^n,$$

where

$$(66) \quad u_j^{n+1/2} = \frac{\Delta x}{\Delta t} \left|_j^{n+1/2}, \right.$$

and

$$(67) \quad \begin{aligned} P_{j,j\pm 1}^n &= \sigma_{j,j\pm 1}^n A_{j,j\pm 1}^n, \quad \sigma_{j,j\pm 1}^n = p_{j\pm 1}^n + q_{j\pm 1/2}^{n-1/2}, \\ q_{j\pm 1/2}^{n-1/2} &= -\nu a \Delta_\mu u_{j\pm 1/2}^{n-1/2}, \quad p_{j\pm 1}^n = P_H(V_{j\pm 1}^n). \end{aligned}$$

Taking the volume of particle j to be

$$(68) \quad Vol_j = D_\mu x_j,$$

yields

$$(69) \quad V_j = \frac{Dx}{\Delta \mu} \Big|_j.$$

Substitute (66) into (65) to get

$$(70) \quad \frac{\Delta^2 x}{\Delta t^2} \Big|_j^n = - \frac{DP(x)}{\Delta \mu} \Big|_j^n,$$

where

$$(71) \quad \frac{\Delta^2 x}{\Delta t^2} \Big|_j^n \equiv \frac{x_j^{n+1} - 2x_j^n + x_j^{n-1}}{\Delta t^2},$$

and $P(x)$ means we are going to write P in terms of x , i.e., $P(x) = \sigma(x)A(x)$, where

$$(72) \quad \sigma(x)_{j,j\pm 1}^n = P_H(0) - a_{mc}^2 D_\mu x_{j\pm 1}^n / \Delta \mu - \nu a_{mc} \Delta_\mu \Delta_t x_{j\pm 1/2}^{n-1/2} / \Delta t,$$

and

$$(73) \quad A(x)_{j,j\pm 1}^n = -2(\Delta \mu)^2 W'(x)_{j,j\pm 1}^n \frac{Dx}{\Delta \mu} \Big|_j^n \frac{Dx}{\Delta \mu} \Big|_{j\pm 1}^n,$$

further

$$(74) \quad W'(x)_{j,j\pm 1}^n = W'(\Delta_\mu x_{j\pm 1/2}^n, h).$$

Now we compute the equation of first variation for (70):

$$(75) \quad \frac{\Delta^2 \delta x}{\Delta t^2} \Big|_j^n = - \frac{D\delta[\sigma(x)A(x)]}{\Delta \mu} \Big|_j^n,$$

where

$$(76) \quad \delta[\sigma(x)A(x)] = \delta[\sigma(x)]A(x) + \sigma(x)\delta[A(x)],$$

with

$$(77) \quad \delta[\sigma(x)]_{j,j\pm 1}^n = -a_{mc}^2 D_\mu \delta x_{j\pm 1}^n / \Delta \mu - \nu a_{mc} \Delta_\mu \Delta_t \delta x_{j\pm 1/2}^{n-1/2} / \Delta t,$$

and

$$(78) \quad \begin{aligned} \delta[A(x)]_{j,j\pm 1}^n / (-2(\Delta \mu)^2) &= \delta W'(x)_{j,j\pm 1}^n \frac{Dx}{\Delta \mu} \Big|_j^n \frac{Dx}{\Delta \mu} \Big|_{j\pm 1}^n \\ &+ W'(x)_{j,j\pm 1}^n \frac{D\delta x}{\Delta \mu} \Big|_j^n \frac{Dx}{\Delta \mu} \Big|_{j\pm 1}^n + W'(x)_{j,j\pm 1}^n \frac{Dx}{\Delta \mu} \Big|_j^n \frac{D\delta x}{\Delta \mu} \Big|_{j\pm 1}^n, \end{aligned}$$

further

$$(79) \quad \delta W'(x)_{j,j\pm 1}^n = W''(x)_{j,j\pm 1}^n \Delta_\mu \delta x_{j\pm 1/2}^n.$$

Notation note: Double increment material averaging is denoted by an overbar; e.g., the double increment material average of σ at μ_j is denoted $\bar{\sigma}_j$ and given by

$$(80) \quad \bar{\sigma}_j = (\sigma_{j,j+1} + \sigma_{j,j-1})/2.$$

Observe that, by the discrete product rule, i.e., $D(xy) = \bar{x}D(y) + D(x)\bar{y}$, we have

$$(81) \quad \frac{D\delta[\sigma A]}{\Delta \mu} = \bar{\delta\sigma} \frac{DA}{\Delta \mu} + \frac{D\delta\sigma}{\Delta \mu} \bar{A} + \bar{\sigma} \frac{D\delta A}{\Delta \mu} + \frac{D\sigma}{\Delta \mu} \bar{\delta A},$$

hence (75) expands to

$$(82) \quad - \frac{\Delta^2 \delta x}{\Delta t^2} = \bar{\delta\sigma} \frac{DA}{\Delta \mu} + \frac{D\delta\sigma}{\Delta \mu} \bar{A} + \bar{\sigma} \frac{D\delta A}{\Delta \mu} + \frac{D\sigma}{\Delta \mu} \bar{\delta A}.$$

For simplicity in earlier analyses we chose to perturb from a state of uniform stress, σ , and uniform Δx . Here we are going to do a more general stability analysis where we perturb from a nonuniform state of stress but still uniform Δx . If we perturb from a uniform- Δx -state, where $\Delta x = \Delta x_c$ is constant uniformly in μ , then also uniformly constant are $V = V_c$, $W' = W'_c$, $W'' = W''_c$. Thus

$$(83) \quad A = A_c \quad \text{and} \quad \frac{DA}{\Delta \mu} = 0, \quad A = A_c.$$

Therefore, (82) reduces to

$$(84) \quad -\frac{\Delta^2 \delta x}{\Delta t^2} = \frac{D\delta\sigma}{\Delta\mu} A_c + \bar{\sigma} \frac{D\delta A}{\Delta\mu} + \frac{D\sigma}{\Delta\mu} \bar{\delta A}.$$

From (77):

$$(85) \quad \frac{D\delta\sigma^n}{\Delta\mu_j} = -a_{mc}^2 D_\mu^2 \delta x_j^n / (\Delta\mu)^2 - \nu a_{mc} \Delta_\mu^2 \Delta_t \delta x_j^{n-1/2} / (2\Delta\mu \Delta t).$$

From (78) and (79):

$$(86) \quad \delta A_{j,j\pm 1}^n / (-2(\Delta\mu)^2) = V_c^2 W_c'' \Delta_\mu \delta x_{j\pm 1/2}^n + W_c' V_c \left. \frac{D\delta x}{\Delta\mu} \right|_j^n + W_c' V_c \left. \frac{D\delta x}{\Delta\mu} \right|_{j\pm 1}^n,$$

therefore,

$$(87) \quad \bar{\delta A}_j^n / (-2(\Delta\mu)^2) = V_c^2 W_c'' D_\mu \delta x_j^n + W_c' V_c \left. \frac{D\delta x}{\Delta\mu} \right|_j^n + W_c' V_c \left. \frac{D\bar{\delta x}}{\Delta\mu} \right|_j^n,$$

and

$$(88) \quad \frac{D\delta A^n}{\Delta\mu_j} / (-2(\Delta\mu)^2) = V_c^2 W_c'' \Delta_\mu^2 \delta x_j^n / (2\Delta\mu) + W_c' V_c \left. \frac{D^2 \delta x}{\Delta\mu^2} \right|_j^n.$$

Recall that ν , a_{mc} , $\Delta\mu$, Δt are constants. We emphasize this point: $\Delta\mu$ and Δt are fixed, finite constants, as they are in a computer calculation; they should not be thought of as 'arbitrarily small' nor as 'going to zero'. Henceforth, to reduce notational clutter, we suppress the 'c' subscripts, unless they seem needed for clarity.

Let:

$$(89) \quad E_{stuff} = (E_k)^j (\lambda_k)^n,$$

and apply the $F_{vN} \langle \cdot \rangle$ transformation to get:

$$(90) \quad F_{vN} \left\langle \left. \frac{D^2 \delta x}{\Delta\mu^2} \right|_j^n \right\rangle = E_{stuff} (E_k^2 - 2 + E_k^{-2}) / [4(\Delta\mu)^2],$$

and:

$$(91) \quad F_{vN} \left\langle \left. \frac{\Delta^2 \delta x}{\Delta\mu^2} \right|_j^n \right\rangle = E_{stuff} (E_k - 2 + E_k^{-1}) / (\Delta\mu)^2,$$

further:

$$(92) \quad F_{vN} \left\langle \left. \frac{\Delta^3 \delta x}{\Delta\mu^2 \Delta t} \right|_j^{n-1/2} \right\rangle = E_{stuff} (E_k - 2 + E_k^{-1}) (1 - \lambda_k^{-1}) / [\Delta t (\Delta\mu)^2],$$

furthermore:

$$(93) \quad F_{vN} \left\langle \frac{D\delta x}{\Delta\mu} \Big|_j^n \right\rangle = E_{stuff}(E_k - E_k^{-1})/(2\Delta\mu),$$

moreover:

$$(94) \quad F_{vN} \left\langle \frac{D\overline{\delta x}}{\Delta\mu} \Big|_j^n \right\rangle = E_{stuff}(E_k^2 - E_k^{-2})/(4\Delta\mu),$$

finally:

$$(95) \quad F_{vN} \left\langle \frac{\Delta^2 \delta x}{\Delta t^2} \Big|_j^n \right\rangle = E_{stuff}(\lambda_k - 2 + \lambda_k^{-1})/(\Delta t)^2$$

Cancelling out the E_{stuff} , the von Neumann-Fourier stability analysis of (84) reduces to

$$(96) \quad \begin{aligned} & (\lambda_k - 2 + \lambda_k^{-1})/(\Delta t)^2 = \\ & A\{a^2(E_k^2 - 2 + E_k^{-2})/(2\Delta\mu)^2 + \nu a(E_k - 2 + E_k^{-1})(1 - \lambda_k^{-1})/(2\Delta t\Delta\mu)\} \\ & + \overline{\sigma}\{\Delta\mu V^2 W''(E_k - 2 + E_k^{-1}) + W'V(E_k^2 - 2 + E_k^{-2})/2\} \\ & + \frac{D\sigma}{\Delta\mu}\{2\Delta\mu[(\Delta\mu V^2 W'' + W'V)(E_k - E_k^{-1}) + W'V(E_k^2 - E_k^{-2})/2]\}. \end{aligned}$$

Now let $s_k = \sin k\Delta\mu$ and $c_k = \cos k\Delta\mu$, then Euler's identity, $E_k = c_k + is_k$, yields the following trigonometric identities:

$$(97) \quad E_k + E_k^{-1} = 2c_k, \quad E_k - E_k^{-1} = 2is_k,$$

and:

$$(98) \quad E_k - 2 + E_k^{-1} = 2(c_k - 1),$$

further:

$$(99) \quad E_k^2 - 2 + E_k^{-2} = (E_k - E_k^{-1})^2 = -4s_k^2,$$

furthermore:

$$(100) \quad E_k^2 - E_k^{-2} = (E_k - E_k^{-1})(E_k + E_k^{-1}) = 4ic_k s_k.$$

Using the foregoing trigonometric identities reduces (96) to:

$$(101) \quad \begin{aligned} & \lambda_k - 2 + \lambda_k^{-1} = \\ & A\{-s_k^2 C_{FL}^2 + \nu C_{FL}(c_k - 1)(1 - \lambda_k^{-1})\} \\ & - 2(\Delta t)^2 \overline{\sigma}\{\Delta\mu V^2 W''(1 - c_k) + W'V s_k^2\} \\ & + 2is_k(\Delta t)^2 D_\mu \sigma\{\Delta\mu V^2 W'' + W'V(1 + c_k)\}, \end{aligned}$$

where $C_{FL} = a\Delta t/\Delta\mu$ is the Courant-Friedrichs-Lewy(1928) number.

Remark: Note that the $D_\mu\sigma$ -term in (101), i.e., the last term in (101):

$$(102) \quad 2is_k(\Delta t)^2 D_\mu\sigma\{\Delta\mu V^2 W'' + W'V(1 + c_k)\},$$

is the extra term introduced by going from a uniform to a nonuniform stress in the stability analysis. Further, note that this term contains the only imaginary numbers in (101), i.e., all the other terms contain real numbers. Note that in the $\bar{\sigma}$ -term in (101), i.e., the second term in (101):

$$(103) \quad -2(\Delta t)^2 \bar{\sigma}\{\Delta\mu V^2 W''(1 - c_k) + W'V s_k^2\}$$

there is a certain product: $\bar{\sigma}W''$. This term, i.e.,

$$(104) \quad \bar{\sigma}_j W_c'' = W_c''(\sigma_{j,j+1} + \sigma_{j,j-1})/2$$

turns out to be the crucial term in the SPH instability.

Thus the stability polynomial equation is

$$(105) \quad \lambda_k^2 - 2B\lambda_k + C = 0,$$

where

$$(106) \quad C = 1 - \alpha, \quad \alpha = \nu AC_{FL}(1 - c_k),$$

and

$$(107) \quad B = 1 - \Xi_{FL} - \alpha/2 + i\beta,$$

where

$$(108) \quad \Xi_{FL} = \{(1 + c_k)[AC_{FL}^2/2 + \bar{\sigma}W'V(\Delta t)^2] + \bar{\sigma}W''V^2\Delta\mu(\Delta t)^2\}(1 - c_k),$$

and

$$(109) \quad \beta = s_k(\Delta t)^2 D_\mu\sigma\{\Delta\mu V^2 W'' + W'V(1 + c_k)\}.$$

Remark: Note that β is the new term we get in the stability polynomial when we go from a uniform to a nonuniform stress in the stability analysis. Observe that $\beta = 0$ at $k = k_{\max}$ and therefore (105) becomes

$$(110) \quad \lambda_k^2 - 2(1 - \Xi_{FL} - \alpha/2)\lambda_k + C = 0,$$

which is exactly the same stability polynomial equation we get if we perturb from a uniform- σ - Δx -state. Therefore, the same theorems as before hold.

Theorem 3.1. *A sufficient condition for absolute D.-N. instability of (70) at $(k_{\max}, N, \Delta t, \Delta\mu, \Delta x, \bar{\sigma}, W'', V)$, where $V^2\Delta\mu(\Delta t)^2 > 0$, is*

$$(111) \quad \bar{\sigma}W'' < 0.$$

Corollary 3.2. *Artificial viscosities of the general von Neumann-Richtmyer-Landshoff form, i.e., $q = -\nu a \Delta_\mu u$, with $\nu a \geq 0$, cannot make (70) at $(k_{\max}, N, \Delta t, \Delta \mu, \Delta x, \bar{\sigma}, W'', V)$ absolutely D.-N. stable when $\overline{p+q}W'' < 0$.*

Remark: Computational experiments with typical hypervelocity impacts in solid materials show that the case $\overline{p+q} < 0$ and $W'' > 0$ is not unusual and thus Corollary 3.2. is not vacuous. We see that artificial viscosity cannot overcome the SPH instability. For proofs of Theorem 3.1. and Corollary 3.2. see [8].

4.0 CONSERVATIVE SMOOTHING

We illustrate the fundamental idea in conservative smoothing using the von Neumann - Richtmyer discrete representation of conservation of volume:

$$(112) \quad V_t = u_\mu,$$

where

$$(113) \quad (V_t)_{j+1/2}^{n+1/2} = \frac{V_{j+1/2}^{n+1} - V_{j+1/2}^n}{t^{n+1} - t^n}$$

and

$$(114) \quad (u_\mu)_{j+1/2}^{n+1/2} = \frac{u_{j+1}^{n+1/2} - u_j^{n+1/2}}{\mu_{j+1} - \mu_j}.$$

For simplicity, consider a uniform material mesh

$$(115) \quad \mu_j = j\Delta\mu, \quad 0 \leq j \leq J$$

with boundaries

$$(116) \quad \mu_{lb} = \mu_0, \quad \mu_{rb} = \mu_J.$$

Remark: Although we here discuss a one-dimensional problem, observe that if we had J mass chunks, $0 \leq j \leq J - 1$,

$$(117) \quad m_{j+1/2} = \text{Area}(\mu_{j+1} - \mu_j),$$

(*Area* is an area factor whose value we choose as unity for simplicity) with specific volumes, $V_{j+1/2}$, distributed throughout a two - or three - dimensional material, then this illustration of the fundamental idea in conservative smoothing would hold, mutatis mutandis, as well in three - or two - as it does in one - dimension(s). Caveat: Observe though, that implicit here is the assumption that a linear ordering has been introduced, (115), along with a tagging of the boundary points, (116). Further, to maintain local smoothing (i.e., to avoid 'action at a distance') suggests that there should be contiguity, in some sense, between $m_{j-1/2}$ and $m_{j+1/2}$. Furthermore, it seems that $m_{1/2}$ and $m_{J-1/2}$ should be boundary masses of the two - or three - dimensional problem.

Let's define the midpoint mesh by

$$(118) \quad \mu_{j+1/2} = (\mu_j + \mu_{j+1})/2, \quad \text{for } 0 \leq j \leq J - 1.$$

Now let's do a conservative redefining of the $V_\bullet$ -distribution on the original mesh to the $\hat{V}_\bullet$ -distribution on the midpoint mesh by

$$(119) \quad \hat{V}_j = (V_{j-1/2} + V_{j+1/2})/2, \quad \text{for } 1 \leq j \leq J - 1,$$

with boundary conditions

$$(120) \quad \hat{V}_0 = V_{1/2}, \quad \hat{V}_J = V_{J-1/2}$$

We say that this is a conservative redefining (a.k.a., conservative rezoning) because the total volume before and after is the same, i.e., the sum total volume before is given by

$$(121) \quad \Sigma_{V_{before}} = A \sum_{j=0}^{J-1} V_{j+1/2}(\mu_{j+1} - \mu_j)$$

and the sum total volume after is given by

$$(122) \quad \Sigma_{V_{after}} = A \left\{ \hat{V}_0(\mu_{1/2} - \mu_0) + \sum_{j=1}^{J-1} \hat{V}_j(\mu_{j+1/2} - \mu_{j-1/2}) + \hat{V}_J(\mu_J - \mu_{J-1/2}) \right\}$$

and it is straightforward to verify that

$$(123) \quad \Sigma_{V_{before}} = \Sigma_{V_{after}}.$$

Next, we conservatively rezone the $\hat{V}_\bullet$ -distribution on the midpoint mesh to the $\tilde{V}_\bullet$ -distribution on the original mesh via

$$(124) \quad \tilde{V}_{j+1/2} = (\hat{V}_{j+1} + \hat{V}_j)/2, \text{ for } 0 \leq j \leq J-1,$$

which in terms of the original $V_\bullet$ -distribution is

$$(125) \quad \tilde{V}_{j+1/2} = (V_{j+3/2} + 2V_{j+1/2} + V_{j-1/2})/4, \text{ for } 1 \leq j \leq J-2,$$

with boundary values

$$(126) \quad \tilde{V}_{1/2} = (V_{3/2} + 2V_{1/2} + V_{1/2})/4, \quad \tilde{V}_{J-1/2} = (V_{J-1/2} + 2V_{J-1/2} + V_{J-3/2})/4.$$

Note that if we define the 'phantom zone' values $V_{-1/2}$ and $V_{J+1/2}$ by

$$(127) \quad V_{-1/2} = V_{1/2}, \quad V_{J+1/2} = V_{J-1/2},$$

then the single recipe

$$(128) \quad \tilde{V}_{j+1/2} = (V_{j+3/2} + 2V_{j+1/2} + V_{j-1/2})/4, \text{ for } 0 \leq j \leq J-1$$

includes both the interior points, (125), and the boundary points, (126), recipes. Also observe that if we wish to leave the boundary values, $V_{+1/2}$ and $V_{J-1/2}$, invariant in this form of conservative smoothing, then we may accomplish this with the following recipes. For the interior values of $\tilde{V}_{j+1/2}$, $2 \leq j \leq J-3$,

$$(129) \quad \tilde{V}_{j+1/2} = (V_{j+3/2} + 2V_{j+1/2} + V_{j-1/2})/4$$

and for the 'near-boundary' values of $\tilde{V}_{j+1/2}$, $j = 1, J-2$,

$$(130) \quad \tilde{V}_{3/2} = (V_{5/2} + 3V_{3/2})/4, \quad \tilde{V}_{J-3/2} = (3V_{J-3/2} + V_{J-5/2})/4,$$

and for the boundary values of $\tilde{V}_{j+1/2}$, $j = 0, J - 1$

$$(131) \quad \tilde{V}_{1/2} = V_{1/2}, \quad \tilde{V}_{J-1/2} = V_{J-1/2}.$$

Note that we may write (129) as

$$(132) \quad \tilde{V} = V + c_s \Delta_\mu^2 V$$

where $c_s = 1/4$ and

$$(133) \quad (\Delta_\mu^2 V)_{j+1/2} = V_{j+3/2} - 2V_{j+1/2} + V_{j-1/2}.$$

Thus the effect of a conservative smoothing of the type here discussed on the von Neumann - Richtmyer representation of the conservation of volume, $V_t = u_\mu$, would be to transform it to

$$(134) \quad V_t = [ac_s \Delta \mu / C_{FL}] V_{\mu\mu} + u_\mu,$$

where

$$(135) \quad V_{\mu\mu} = \Delta_\mu^2 V / (\Delta \mu)^2.$$

We call this **conservative smoothing of type d** because it introduces a term, $[ac_s \Delta \mu / C_{FL}] V_{\mu\mu}$, which looks like a *diffusion* term. Consistency with the standard continuum model,

$$(136) \quad \partial V / \partial t = \partial u / \partial \mu,$$

would require $[ac_s \Delta \mu / C_{FL}] V_{\mu\mu}$ to go to zero as the material and temporal increments vanished.

For further details on conservative smoothing, see [4].

5.0 Stability Analysis with Conservative Smoothing

Recall that the SPH scheme we are analyzing is

$$(137) \quad u_j^{n+1/2} = u_j^{n-1/2} - \Delta t \left. \frac{DP}{\Delta\mu} \right|_j^n, \quad x_j^{n+1} = x_j^n + \Delta t u_j^{n+1/2}, \quad V_j^{n+1} = \left. \frac{Dx}{\Delta\mu} \right|_j^{n+1}.$$

This is equivalent to

$$(138) \quad \begin{aligned} u_j^{n+1/2} &= u_j^{n-1/2} - \Delta t \left. \frac{DP}{\Delta\mu} \right|_j^n, \quad x_j^{n+1} = x_j^n + \Delta t u_j^{n+1/2}, \\ V_j^{n+1} &= V_j^n + \Delta t \left. \frac{Dx}{\Delta\mu} \right|_j^{n+1/2}. \end{aligned}$$

We turn off the artificial viscosity and turn on the conservative smoothing of type d with $c_s = 1/4$, i.e.,

$$(139) \quad \widetilde{\mathbf{U}}_j = \mathbf{U}_j + (\mathbf{U}_{j+1} - 2\mathbf{U}_j + \mathbf{U}_{j-1})/4, \quad \mathbf{U}_j = (V_j, u_j, x_j)^T.$$

Combining (138) and (139) yields the following scheme:

$$(140) \quad \begin{aligned} u_j^{n+1/2} &= \widetilde{u}_j^{n-1/2} - \Delta t \left. \frac{D[p(\widetilde{V})A(\widetilde{x})]}{\Delta\mu} \right|_j^n, \\ x_j^{n+1} &= \widetilde{x}_j^n + \Delta t \widetilde{u}_j^{n+1/2}, \quad V_j^{n+1} = \widetilde{V}_j^n + \Delta t \left. \frac{D\widetilde{u}}{\Delta\mu} \right|_j^{n+1/2} \end{aligned}$$

The stability polynomial equation becomes

$$(141) \quad \lambda_k^2 - 2\widetilde{B}\lambda_k + \widetilde{C} = 0,$$

where

$$(142) \quad \widetilde{B} = BA_{cs}(k), \quad \widetilde{C} = CA_{cs}^2(k);$$

recall

$$(143) \quad A_{cs}(k) = (1 + c_k)/2, \quad c_k = \cos k\Delta\mu.$$

Since the artificial viscosity is turned off, we have

$$(144) \quad C = 1, \quad B = 1 - \Xi_{FL} + i\beta,$$

with

$$(145) \quad \Xi_{FL} = \{(1 + c_k)[AC_{FL}^2/2 + \bar{\sigma}W'V(\Delta t)^2] + \bar{\sigma}W''V^2\Delta\mu(\Delta t)^2\}(1 - c_k),$$

and

$$(146) \quad \beta = s_k(\Delta t)^2 D_\mu \sigma \{\Delta\mu V^2 W'' + W'V(1 + c_k)\}.$$

Lemma 5.0. Given $\lambda^2 - 2B\lambda + C = 0$, $\lambda_{\pm} = B \pm (B^2 - C)^{1/2}$, $r = \max|\lambda_{\pm}|$, $B = a + ib$, $a, b \in \mathbb{R}$, $C = 0$, then

$$(147) \quad a^2 + b^2 \leq 1/4 \Leftrightarrow r \leq 1.$$

Proof: Straightforward because $r = 2|B|$.

Lemma 5.1. Given $\lambda^2 - 2B\lambda + C = 0$, $\lambda_{\pm} = B \pm (B^2 - C)^{1/2}$, $r = \max|\lambda_{\pm}|$, $B = a + ib$, $a, b \in \mathbb{R}$, $b \neq 0$, $C \geq 0$, then

$$(148) \quad a^2 + b^2 + C/2 \leq 1/4 \Rightarrow r \leq 1.$$

Proof: First, the case where $C = 0$ is taken care of by **Lemma 5.0.**, so assume henceforth $C > 0$. Note that

$$(149) \quad \lambda_{\pm} = a + ib \pm [(a + ib)^2 - C]^{1/2}.$$

Let u, v be defined by

$$(150) \quad [(a + ib)^2 - C]^{1/2} = u + iv.$$

Therefore,

$$(151) \quad |\lambda_{\pm}|^2 = (a \pm u)^2 + (b \pm v)^2 = a^2 \pm 2au + u^2 + b^2 \pm 2bv + v^2.$$

Square both sides of (163) and compare real and imaginary parts to get

$$(152) \quad a^2 - b^2 - C = u^2 - v^2,$$

and

$$(153) \quad ab = uv.$$

If $a \neq 0$, then (153) implies $b/v = u/a$ and thus we see that $\text{sign}(au) = \text{sign}(bv)$. Since $r^2 = \max|\lambda_{\pm}|^2$, it follows that

$$(154) \quad r^2 = a^2 + 2|au| + u^2 + b^2 + 2|bv| + v^2;$$

note that this also holds for $a = 0$. Next observe that

$$(155) \quad 2|au| \leq a^2 + u^2$$

and therefore

$$(156) \quad r^2 \leq 2(a^2 + u^2 + b^2 + v^2).$$

From (152) and (153):

$$(157) \quad (u^2 + v^2)^2 = (a^2 - b^2 - C)^2 + 4a^2b^2 = (a^2 + b^2 + C)^2 - 4Ca^2.$$

Then using the fact that $C > 0$ yields

$$(158) \quad u^2 + v^2 \leq a^2 + b^2 + C$$

and thus

$$(159) \quad r^2 \leq 4(a^2 + b^2 + C/2).$$

The remainder of the proof is straightforward.

Lemma 5.2. Given $\lambda^2 - 2B\lambda + C = 0$, $\lambda_{\pm} = B \pm (B^2 - C)^{1/2}$, $r = \max|\lambda_{\pm}|$, $B = a + ib$, a & b & $C \in \mathbb{R}$, $a \neq 0 \neq b$, then

$$(160) \quad r \leq 1 \Leftrightarrow |B|^2 - C + [(|B|^2 - C)^2 + 4b^2C]^{1/2} \leq (1 - C)^2 / 2.$$

Proof: Express the roots in polar form, let

$$(161) \quad \lambda_1 = \rho e^{i\theta}, \quad \lambda_2 = \frac{C}{\rho} e^{-i\theta}$$

and, without loss of generality, take $\rho \geq \frac{C}{\rho}$, and therefore $\rho = r$. Note that

$$(162) \quad \rho e^{i\theta} + \frac{C}{\rho} e^{-i\theta} = 2(a + ib).$$

Compare real, i.e.,

$$(163) \quad \left(\rho + \frac{C}{\rho} \right) \cos \theta = 2a,$$

and imaginary parts, i.e.,

$$(164) \quad \left(\rho - \frac{C}{\rho} \right) \sin \theta = 2b.$$

Observe that $a \neq 0 \Rightarrow \cos \theta \neq 0$ and $b \neq 0 \Rightarrow \sin \theta \neq 0$ and $\rho > \frac{C}{\rho}$. Therefore,

$$(165) \quad \rho + \frac{C}{\rho} = 2a / \cos \theta > 0$$

and

$$(166) \quad \rho - \frac{C}{\rho} = 2b / \sin \theta > 0.$$

Adding the prior pair of equations yields

$$(167) \quad \rho = a / \cos \theta + b / \sin \theta.$$

Multiply equations (152) and (153) by ρ to get

$$(168) \quad \rho^2 - 2a\rho/\cos\theta + C = 0$$

and

$$(169) \quad \rho^2 - 2b\rho/\sin\theta - C = 0.$$

Writing down the roots, since $\rho > \frac{C}{\rho}$, we have

$$(170) \quad \rho = a/\cos\theta + [(a/\cos\theta)^2 - C]^{1/2}, \quad \frac{C}{\rho} = a/\cos\theta - [(a/\cos\theta)^2 - C]^{1/2},$$

and

$$(171) \quad \rho = b/\sin\theta + [(b/\sin\theta)^2 + C]^{1/2}, \quad \frac{C}{\rho} = b/\sin\theta - [(b/\sin\theta)^2 + C]^{1/2}.$$

From (154) and (158) we get

$$(172) \quad a/\cos\theta + b/\sin\theta = a/\cos\theta + [(a/\cos\theta)^2 - C]^{1/2}$$

and therefore

$$(173) \quad (b/\sin\theta)^2 = (a/\cos\theta)^2 - C.$$

Let $y \equiv (\sin\theta)^2$ to get

$$(174) \quad Cy^2 + (a^2 + b^2 - C)y - b^2 = 0.$$

Note that if $C = 0$, then $y = b^2/(a^2 + b^2)$ and this picks the correct root of the quadratic, i.e.,

$$(175) \quad (\sin\theta)^2 = \frac{2b^2}{a^2 + b^2 - C + [(a^2 + b^2 - C)^2 + 4b^2C]^{1/2}}.$$

From (159)

$$(176) \quad \rho \leq 1 \Leftrightarrow b/\sin\theta + [(b/\sin\theta)^2 + C]^{1/2} \leq 1$$

and therefore

$$(177) \quad \rho \leq 1 \Leftrightarrow (b/\sin\theta)^2 + C \leq (1 - b/\sin\theta)^2 \Leftrightarrow 2b/\sin\theta \leq 1 - C$$

and thus

$$(178) \quad \rho \leq 1 \Leftrightarrow 4b^2 \leq (1 - C)^2 (\sin\theta)^2.$$

Now put (164) with this to get

$$(179) \quad \rho \leq 1 \Leftrightarrow 2\{a^2 + b^2 - C + [(a^2 + b^2 - C)^2 + 4b^2C]^{1/2}\} \leq (1 - C)^2.$$

The remainder of the proof is straightforward.

Remark: Before stating the next theorem, some explanation of the theorem's form appears to be appropriate. To allow for the proper modelling of physical instabilities, it seems reasonable to allow bounded growth for the amplitudes of the Fourier components of the lesser wave numbers, i.e., $|k| < \theta k_{\max}$, where $0 < \theta < 1$, and only require absolute D.-N. stability for the larger wave numbers, i.e., $\theta k_{\max} \leq |k| \leq k_{\max}$. If $\beta = 0$ in (146), then the stability polynomial is the same as if we perturb from a uniform- σ - Δx -state and thus we have the following proposition from that prior analysis.

Theorem 5.1. When $\beta = 0$ in (146), a sufficient condition for absolute D.-N. stability of (140) at $(k, \Delta t, \Delta \mu, N, C^u)$ for $\frac{1}{2}k_{\max} \leq |k| \leq k_{\max}$ is as follows:

$$(180) \quad 2|\xi_{FL}| + 4|W''_{stuff}| \leq 1,$$

with

$$(181) \quad \xi_{FL} = -2(a^2 - \rho\bar{\sigma})(\Delta t/\Delta \mu)^2 W' V^2 \Delta \mu^2$$

and

$$(182) \quad W''_{stuff} = \bar{\sigma} W'' V^2 \Delta t^2 \Delta \mu.$$

Proof: See [8].

Theorem 5.2. When $\beta \neq 0$ in (146), a sufficient condition for absolute D.-N. stability of (140) at $(k, \Delta t, \Delta \mu, N, C^u)$ for $\frac{2}{3}k_{\max} \leq |k| \leq k_{\max}$ is as follows:

$$(183) \quad |\xi_{FL}| + 4|W''_{stuff}| \leq \sqrt{7} - 2$$

and

$$(184) \quad |\eta_{D\sigma}| + 4|W''_{new}| \leq 4\sqrt{7/3}$$

where

$$(185) \quad \xi_{FL} = -2(a^2 - \rho\bar{\sigma})(\Delta t/\Delta \mu)^2 W' V^2 \Delta \mu^2$$

and also

$$(186) \quad W''_{stuff} = \bar{\sigma} W'' V^2 \Delta t^2 \Delta \mu$$

further

$$(187) \quad \eta_{D\sigma} = 2W' V \Delta t^2 D_\mu \sigma$$

and finally

$$(188) \quad W''_{new} = W'' V^2 \Delta t^2 \Delta \mu D_\mu \sigma.$$

Proof: Note that $\frac{2}{3}k_{\max} \leq |k| \leq k_{\max}$ implies

$$(189) \quad 0 \leq A_{cs}(k) \leq 1/4.$$

Since $\beta \neq 0$, the stability polynomial can only have distinct roots, therefore the sufficient condition for absolute D.-N. stability is $r \leq 1$. Recall that the stability polynomial equation is

$$(190) \quad \lambda_k^2 - 2\tilde{B}\lambda_k + \tilde{C} = 0,$$

where

$$(191) \quad \tilde{B} = (1 - \Xi_{FL} + i\beta)A_{cs}, \quad \tilde{C} = A_{cs}^2,$$

with

$$(192) \quad \begin{aligned} \Xi_{FL} &= (1 - c_k)[\xi_{FL}A_{cs} + W''_{stuff}], \quad A = -2W'V^2\Delta\mu^2, \\ \xi_{FL} &= AC_{FL}^2 + 2\bar{\sigma}W'V\Delta t^2, \quad W''_{stuff} = \bar{\sigma}W''V^2\Delta\mu\Delta t^2 \end{aligned}$$

and

$$(193) \quad \beta = s_k(\eta_{D\sigma}A_{cs} + W''_{new}), \quad \eta_{D\sigma} = 2\Delta t^2 D_\mu \sigma W'V, \quad W''_{new} = \Delta t^2 D_\mu \sigma \Delta\mu V^2 W''$$

Observe that

$$(194) \quad |\tilde{B}|^2 = [(1 - \Xi_{FL})^2 + \beta^2]A_{cs}^2.$$

If

$$(195) \quad (1 - \Xi_{FL})^2 \leq 7/4$$

and

$$(196) \quad \beta^2 \leq 7/4,$$

then

$$(197) \quad |\tilde{B}|^2 + \tilde{C}/2 \leq 1/4$$

which, by Lemma 5.1, is a sufficient condition for absolute D.-N. stability. Thus we see that (195) and (196) are sufficient for stability. Now we seek sufficient conditions for (195) and (196). Note that

$$(198) \quad (1 - \Xi_{FL})^2 \leq 7/4 \Leftrightarrow 1 - \sqrt{7}/2 \leq \Xi_{FL} \leq 1 + \sqrt{7}/2$$

and then, by the definition of Ξ_{FL} , (198) is equivalent to

$$(199) \quad (2 - \sqrt{7})/2 \leq (1 - c_k)[\xi_{FL}A_{cs} + W''_{stuff}] \leq (2 + \sqrt{7})/2.$$

If $\frac{2}{3}k_{\max} \leq |k| \leq k_{\max}$, then

$$(200) \quad 3/2 \leq 1 - c_k \leq 2$$

and therefore we have

$$(201) \quad (2 - \sqrt{7})/4 \leq \xi_{FL}A_{cs} + W''_{stuff} \leq (2 + \sqrt{7})/4.$$

First, consider the righthand inequality in (201), recall that $A_{cs} \in [0, 1/4]$, and note that there are two cases, i.e.,

$$Case(+): \xi_{FL} > 0 \quad \text{and} \quad Case(-): \xi_{FL} \leq 0.$$

In $Case(+): \xi_{FL} > 0$:

$$(202) \quad \xi_{FL}/4 + W''_{stuff} \leq (2 + \sqrt{7})/4 \Rightarrow \xi_{FL}A_{cs} + W''_{stuff} \leq (2 + \sqrt{7})/4.$$

In $Case(-): \xi_{FL} \leq 0$:

$$(203) \quad W''_{stuff} \leq (2 + \sqrt{7})/4 \Rightarrow \xi_{FL}A_{cs} + W''_{stuff} \leq (2 + \sqrt{7})/4.$$

Using the fact that $\max(a, b) = (a + b)/2 + |a - b|/2$, we get

$$(204) \quad \xi_{FL}/8 + W''_{stuff} + |\xi_{FL}|/8 \leq (2 + \sqrt{7})/4 \Rightarrow \xi_{FL}A_{cs} + W''_{stuff} \leq (2 + \sqrt{7})/4$$

for all $A_{cs} \in [0, 1/4]$.

Second, consider the lefthand inequality in (201) and again consider the two cases.

In $Case(+): \xi_{FL} > 0$:

$$(205) \quad (2 - \sqrt{7})/4 \leq W''_{stuff} \Rightarrow (2 - \sqrt{7})/4 \leq \xi_{FL}A_{cs} + W''_{stuff}.$$

In $Case(-): \xi_{FL} \leq 0$:

$$(206) \quad (2 - \sqrt{7})/4 \leq \xi_{FL}/4 + W''_{stuff} \Rightarrow (2 - \sqrt{7})/4 \leq \xi_{FL}A_{cs} + W''_{stuff}.$$

Using the fact that $\min(a, b) = (a + b)/2 - |a - b|/2$, we get

$$(207) \quad (2 - \sqrt{7})/4 \leq \xi_{FL}/8 + W''_{stuff} - |\xi_{FL}|/8 \Rightarrow (2 - \sqrt{7})/4 \leq \xi_{FL}A_{cs} + W''_{stuff}$$

for all $A_{cs} \in [0, 1/4]$.

Next note that

$$(208) \quad \begin{aligned} (2 - \sqrt{7})/4 &\leq \xi_{FL}/8 + W''_{stuff} - |\xi_{FL}|/8 \\ &\Leftrightarrow \\ \xi_{FL}/8 - W''_{stuff} - |\xi_{FL}|/8 &\leq (-2 + \sqrt{7})/4. \end{aligned}$$

Thus both inequalities in (201) are satisfied provided

$$(209) \quad \xi_{FL}/8 - W''_{stuff} - |\xi_{FL}|/8 \leq (-2 + \sqrt{7})/4$$

and

$$(210) \quad \xi_{FL}/8 + W''_{stuff} + |\xi_{FL}|/8 \leq (2 + \sqrt{7})/4.$$

Observe that (183) suffices for both (209) and (210).

Next recall that we need a sufficient condition for

$$(211) \quad \beta^2 \leq 7/4$$

and this leads us to consider

$$(212) \quad -\sqrt{7}/2 \leq \beta \leq \sqrt{7}/2.$$

Recall that

$$(213) \quad \beta = s_k(\eta_{D\sigma} A_{cs} + W''_{new})$$

and therefore we want a sufficient condition for

$$(214) \quad -\sqrt{7}/2 \leq s_k(\eta_{D\sigma} A_{cs} + W''_{new}) \leq \sqrt{7}/2$$

for $\frac{2}{3}k_{\max} \leq |k| \leq k_{\max}$ where

$$(215) \quad 0 \leq s_k \leq \sqrt{3}/2.$$

Thus we see that we want a sufficient condition for

$$(216) \quad -\sqrt{7/3} \leq \eta_{D\sigma} A_{cs} + W''_{new} \leq \sqrt{7/3}.$$

As before, we first consider the righthand inequality in (216), recall that $A_{cs} \in [0, 1/4]$, and the two cases are

$$Case(+): \eta_{D\sigma} > 0 \quad \text{and} \quad Case(-): \eta_{D\sigma} \leq 0.$$

In $Case(+): \eta_{D\sigma} > 0$:

$$(217) \quad \eta_{D\sigma}/4 + W''_{new} \leq \sqrt{7/3} \Rightarrow \eta_{D\sigma} A_{cs} + W''_{new} \leq \sqrt{7/3}.$$

In $Case(-): \eta_{D\sigma} \leq 0$

$$(218) \quad W''_{new} \leq \sqrt{7/3} \Rightarrow \eta_{D\sigma} A_{cs} + W''_{new} \leq \sqrt{7/3}.$$

Again as before, we use $\max(a, b) = (a + b)/2 + |a - b|/2$ to get

$$(219) \quad \eta_{D\sigma}/8 + W''_{new} + |\eta_{D\sigma}|/8 \leq \sqrt{7/3} \Rightarrow \eta_{D\sigma} A_{cs} + W''_{new} \leq \sqrt{7/3}.$$

Second, proceeding in parallel to the previous analysis, we consider the lefthand inequality in (216). In $Case(+): \eta_{D\sigma} > 0$:

$$(220) \quad -\sqrt{7/3} \leq W''_{new} \Rightarrow -\sqrt{7/3} \leq \eta_{D\sigma} A_{cs} + W''_{new}.$$

In *Case*(-): $\eta_{D\sigma} \leq 0$

$$(221) \quad -\sqrt{7/3} \leq \eta_{D\sigma}/4 + W''_{new} \Rightarrow -\sqrt{7/3} \leq \eta_{D\sigma}A_{cs} + W''_{new}.$$

Yet again as before, using $\min(a, b) = (a + b)/2 - |a - b|/2$, we get

$$(222) \quad -\sqrt{7/3} \leq \eta_{D\sigma}/8 + W''_{new} - |\eta_{D\sigma}|/8 \Rightarrow -\sqrt{7/3} \leq \eta_{D\sigma}A_{cs} + W''_{new}$$

for all $A_{cs} \in [0, 1/4]$. Note now that

$$(223) \quad \eta_{D\sigma}/8 - W''_{new} - |\eta_{D\sigma}|/8 \leq \sqrt{7/3}$$

and

$$(224) \quad \eta_{D\sigma}/8 + W''_{new} + |\eta_{D\sigma}|/8 \leq \sqrt{7/3}$$

suffice for both inequalities in (216) and, in turn,

$$(225) \quad |\eta_{D\sigma}| + 4|W''_{new}| \leq 4\sqrt{7/3}$$

suffices for both (223) and (224). Q.E.D.

We now apply Lemma 5.2 to obtain a necessary and sufficient condition for stability when $\beta \neq 0$.

Theorem 5.3. When $\beta \neq 0$ in (146), a necessary and sufficient condition for absolute D.-N. stability of (140) at $(k, \Delta t, \Delta\mu, N, C^u)$ for $0 \leq |k| \leq k_{\max}$ is as follows:

$$(226) \quad |\tilde{B}|^2 - A_{cs}^2 + [(|\tilde{B}|^2 - A_{cs}^2)^2 + 4\beta^2 A_{cs}^4]^{1/2} \leq (1 - A_{cs}^2)^2 / 2,$$

where

$$(227) \quad |\tilde{B}|^2 = \{(1 - \Xi_{FL})^2 + \beta^2\} A_{cs}^2, \quad A_{cs} = (1 + c_k)/2,$$

with

$$(228) \quad \Xi_{FL} = (1 - c_k)[\xi_{FL}A_{cs} + W''_{stuff}], \quad \xi_{FL} = AC_{FL}^2 + 2\bar{\sigma}W'V\Delta t^2,$$

$$A = -2W'V^2\Delta\mu^2, \quad W''_{stuff} = \bar{\sigma}W''V^2\Delta\mu\Delta t^2$$

further

$$(229) \quad \beta = s_k(\eta_{D\sigma}A_{cs} + W''_{new}), \quad \eta_{D\sigma} = 2\Delta t^2 D_\mu \sigma W'V, \quad W''_{new} = \Delta t^2 D_\mu \sigma \Delta\mu V^2 W''$$

and finally

$$(230) \quad c_k = \cos \pi k / k_{\max}, \quad s_k = \sin \pi k / k_{\max}.$$

Proof: Recall that Lemma 5.2. says: Given $\lambda^2 - 2B\lambda + C = 0$, $\lambda_{\pm} = B \pm (B^2 - C)^{1/2}$, $r = \max |\lambda_{\pm}|$, $B = a + ib$, $a, b, C \in \mathbb{R}$, $a \neq 0 \neq b$, then

$$(231) \quad r \leq 1 \Leftrightarrow |B|^2 - C + [(|B|^2 - C)^2 + 4b^2C]^{1/2} \leq (1 - C)^2 / 2.$$

Then substitute $\tilde{B}$ for B , where $\tilde{B} = \tilde{a} + i\tilde{b}$ and

$$(232) \quad \tilde{a} = (1 - \Xi_{Fl})A_{cs}, \quad \tilde{b} = \beta A_{cs}, \quad C = A_{cs}^2$$

to get (226). Further, recall that the stability polynomial can only have distinct roots for $\beta \neq 0$; therefore, $r \leq 1$ is necessary and sufficient. Q.E.D.

Corollary 5.3A. When $\beta \neq 0$ in (146), a necessary and sufficient condition for absolute D.-N. stability of (140) at $(k, \Delta t, \Delta \mu, N, C^u)$ for $0 \leq |k| \leq k_{\max}$ is as follows:

$$(233) \quad A_{cs}^2[(1 - |B|^2)^2 + 4\beta^2]^{1/2} \leq A_{cs}^2(1 - |B|^2) + (1 - A_{cs}^2)^2 / 2,$$

where

$$(234) \quad |B|^2 = \{(1 - \Xi_{FL})^2 + \beta^2\}.$$

Proof: Straightforward from Theorem 5.3.

Corollary 5.3B. When $\beta \neq 0$ in (146), a necessary and sufficient condition for absolute D.-N. stability of (140) at $(k, \Delta t, \Delta \mu, N, C^u)$ for $0 \leq |k| \leq k_{\max}$ is as follows:

$$(235) \quad (-2\Xi_{FL} + \Xi_{FL}^2)A_{cs}^2(1 - A_{cs}^2)^2 + \beta^2[A_{cs}^2(1 - A_{cs}^2)^2 + 4A_{cs}^4] \leq (1 - A_{cs}^2)^4 / 4.$$

Proof: Square both sides of (233), cancel like terms, and expand $|B|^2$.

Corollary 5.3C. When $\beta \neq 0$ in (146), a sufficient condition for absolute D.-N. stability of (140) at $(k, \Delta t, \Delta \mu, N, C^u)$ for $k_{\max}/2 \leq |k| \leq k_{\max}$ is as follows:

$$(236) \quad \{||\xi_{FL}| + 2|W_{stuff}''| + (|\xi_{FL}|/2 + |W_{stuff}''|)^2\} \left(\frac{3}{4}\right)^2 + (|\eta_{D\sigma}|/2 + |W_{new}''|)^2 \left[\left(\frac{3}{4}\right)^2 + 1\right] \leq \left(\frac{3}{4}\right)^4.$$

Proof: $k_{\max}/2 \leq |k| \leq k_{\max} \Rightarrow 0 \leq A_{cs}(k) \leq 1/2$ & $0 \leq s_k \leq 1$ & $0 \leq c_k \leq 1$. The rest is straightforward from Corollary 5.3B.

6.0 Results of Computational Experiments

The test problem here involves the collision of two mirror-symmetric shock waves; it is a special case of the Riemann problem and is also a variation on a test problem introduced in [5]. Proceeding from left to right, the initial values are the lefthand state,

$$(237) \quad \begin{aligned} p_\ell &= 1.35 \times 10^4 \text{ dynes/cm}^2, \quad V_\ell = 7.5 \times 10^5 \text{ cm}^3/\text{gm}, \\ u_\ell &= 2.958 \times 10^4 \text{ cm/sec}, \end{aligned}$$

connected by a rightward shock to the initial middle state,

$$(238) \quad p_0 = 10^4 \text{ dynes/cm}^2, \quad V_0 = 10^6 \text{ cm}^3/\text{gm}, \quad u_0 = 0,$$

which in turn is connected by a leftward shock to the righthand state,

$$(239) \quad \begin{aligned} p_r &= 1.35 \times 10^4 \text{ dynes/cm}^2, \quad V_r = 7.5 \times 10^5 \text{ cm}^3/\text{gm}, \\ u_r &= -2.958 \times 10^4 \text{ cm/sec}. \end{aligned}$$

After collision, the lefthand state is connected by a leftward shock to the final middle state,

$$(240) \quad p_m = 1.7 \times 10^4 \text{ dynes/cm}^2, \quad V_m = 5 \times 10^5 \text{ cm}^3/\text{gm}, \quad u_m = 0,$$

which in turn is connected by a rightward shock to the righthand state. In this test problem we have a uniform material increment, i.e.,

$$(241) \quad \Delta\mu = 10^{-4} \text{ gm/cm}^2.$$

The material law is Hooke's law. The CFL number is

$$(242) \quad C_{FL} = .2$$

The program (see [14] for a program listing) computes the exact solution, vNR solution, and S.P.H. solution at collision and after collision and compares the nondimensional, normalized ℓ_1 -norm of the difference between the exact solution and the vNR solution; it also computes this error measure for the S.P.H. solution.

ERROR MEASURES: Let U_{excon} be the exact continuum solution and U_{dcomp} be the digital computer solution, where U may be specific volume, specific momentum, specific energy, et cetera. Let $U_{\max}$ be the maximum magnitude of U_{excon} and

$$(243) \quad \Delta_{err} U \equiv \sum_{j=1}^{nzones} |U_{excon}(j) - U_{dcomp}(j)| / (U_{\max} * nzones).$$

For these test problems

$$(244) \quad n_{\text{zones}} = 200.$$

ERROR COMPARISONS: The $\Delta_{\text{err}}U$ numbers in the following table are for: the $vNR_{0,0}$ code (a von Neumann-Richtmyer code using artificial viscosity coefficients $c_1 = 0, c_2 = 0$); the $vNR_{0,02}$ code (using $c_1 = 0, c_2 = .02$); the *SPHCS* hydrocode is the S.P.H. code with conservative smoothing as follows. First, full ($c_s = 1/4$) conservative smoothing of type d was done three times on the initial data. Thereafter, the conservative smoothing coefficient at $\mu_{j+1/2}$ was given by

$$(245) \quad c_{sj+1/2} = \min\{1/4, 5 \times 10^{-4} \max[1, \frac{h^2}{10^{-12} + (x_{j+1} - x_j)^2}]\},$$

where h is the smoothing length and, for these calculations,

$$(246) \quad h = 100\text{cm}.$$

By the way, these calculations used the B-Spline Kernel, i.e.,

$$(247) \quad W'(x) = \frac{1}{h^2} \begin{cases} 2 \left[-\frac{x}{h} + \frac{3}{4} \left(\frac{x}{h} \right)^2 \right], & \text{if } 0 \leq x \leq h \\ -\frac{1}{2} \left[2 - \frac{x}{h} \right]^2, & \text{if } h \leq x \leq 2h \\ 0, & \text{otherwise} \end{cases}.$$

With respect to the conservative smoothing: the smoothing recipe is

$$(248) \quad U_j = U_j + c_{sj+1/2}(U_{j+1} - U_j) - c_{sj-1/2}(U_{j-1} - U_j)$$

with $c_{sj+1/2}$ as defined in (245). Further details and a listing of the $vNR_{0,0}$, $vNR_{0,02}$ and *SPHCS* hydrocodes may be found in [14].

Comparison of $\Delta_{\text{err}}U$ Values At Collision :

$t_{\text{col}} = 2.789 \times 10^{-2}$ seconds; $n = 1680$ cycles :

Quantity	$vNR_{0,0}$	$vNR_{0,02}$	<i>SPHCS</i>	worst/best
u	1.585×10^{-4}	1.588×10^{-4}	1.284×10^{-4}	1.24
V	4.834×10^{-5}	4.840×10^{-5}	2.616×10^{-5}	1.85
p	3.759×10^{-5}	3.764×10^{-5}	2.035×10^{-5}	1.85

Observe that in all cases *SPHCS*, i.e., the S.P.H. code with conservative smoothing, gives the least error. The last column in the table above is the ratio of the worst (i.e., the largest) to the best (i.e., the least) error; this provides a quantitative measure of how much improvement conservative smoothing can make. For example, in the V row we have the worst error occurring in the $vNR_{0,02}$ code with $\Delta_{\text{err}}V \doteq 4.840 \times 10^{-5}$ and the best error occurring in the *SPHCS* code with $\Delta_{\text{err}}V \doteq 2.616 \times 10^{-5}$; the

ratio of worst to best here is approximately 1.85. Thus, by this measure, the *SPHCS* code, an *S.P.H.* code with conservative smoothing, did almost two times better, with respect to *V*-accuracy, than the *vNR_{0,.02}* code, a von Neumann-Richtmyer code with artificial viscosity coefficients $c_1 = 0$, $c_2 = .02$, $C_{FL} = .2$ and on a Hooke's law test problem. Recall that the von Neumann-Richtmyer-Landshoff (see Landshoff (1955)) artificial viscosity is of the form

$$q = -c_1 a \Delta_\mu u - c_2 \rho |\Delta_\mu u| \Delta_\mu u.$$

Our computational experiments indicate that, on this test problem (with Hooke's law and $C_{FL} = .2$) the *vNR* code gets smaller errors with the artificial viscosity turned off. We also know that the *vNR* codes get smaller errors as $C_{FL} \rightarrow 1$ but we wanted to compare the *SPH* code with the *vNR* code at the same C_{FL} number, hence these comparisons. Someone might also protest that this is a rather 'wimpy' test problem and we confess that we agree; we intend eventually to do more extensive testing and include problems with stronger shocks and more difficult material laws such as ideal gas, Mie-Grüneisen, et cetera.

Comparison of $\Delta_{err}U$ Values After Collision :

$t_{aftcol} = 4.225 \times 10^{-2}$ seconds; $n = 2530$ cycles :

<i>Quantity</i>	<i>vNR_{0,0}</i>	<i>vNR_{0,.02}</i>	<i>SPHCS</i>	<i>worst/best</i>
<i>u</i>	2.296×10^{-4}	2.300×10^{-4}	1.738×10^{-4}	1.32
<i>V</i>	6.960×10^{-5}	6.975×10^{-5}	5.692×10^{-5}	1.23
<i>p</i>	4.889×10^{-5}	4.899×10^{-5}	4.002×10^{-5}	1.22

From the 'worst/best' column, we see that the *SPHCS* code is doing from about 22% to about 32% better than the *vNR_{0,.02}* code by these error measures on this test problem.

GRAPHICAL PRESENTATIONS: Now we present the results in graphical form.

The next table provides a 'table of contents' for the graphs which follow.

<i>FIGURE</i>	<i>DESCRIPTION</i>
6.1 top left	u_{excon} at $t = 0$
6.1 top right	V_{excon} at $t = 0$
6.1 bottom	p_{excon} at $t = 0$
6.2	vNR error versus c_2
6.3	$vNR_{0,0} u$ at $t = t_{col}$
6.4	$SPHCS u$ at $t = t_{col}$
6.5	$vNR_{0,0} V$ at $t = t_{col}$
6.6	$SPHCS V$ at $t = t_{col}$
6.7	$vNR_{0,0} p$ at $t = t_{col}$
6.8	$SPHCS p$ at $t = t_{col}$
6.9	$vNR_{0,0} u$ at $t = t_{aftcol}$
6.10	$SPHCS u$ at $t = t_{aftcol}$
6.11	$vNR_{0,0} V$ at $t = t_{aftcol}$
6.12	$SPHCS V$ at $t = t_{aftcol}$
6.13	$vNR_{0,0} p$ at $t = t_{aftcol}$
6.14	$SPHCS p$ at $t = t_{aftcol}$

INITIAL DATA OF TEST PROBLEM

Figure 6.1

x-coordinate: C2 in artificial viscosity

y-coordinate: l_1 norm of error of specific momentum at collision

Conclusion: vNR gives us better results with artificial viscosity turned off when we use Hooke's Law as our material law and with $C_{FL} = 0.2$.

Figure 6.2

vNR SOLUTION of specific momentum at collision

Figure 6.3

S.P.H. SOLUTION of specific momentum at collision

Figure 6.4

vNR SOLUTION of specific volume at collision

Figure 6.5

S.P.H. SOLUTION of specific volume at collision

Figure 6.6

vNR SOLUTION of pressure at collision

Figure 6.7

S.P.H. SOLUTION of pressure at collision

Figure 6.8

vNR SOLUTION of specific momentum after collision

Figure 6.9

S.P.H. SOLUTION of specific momentum after collision

Figure 6.10

vNR SOLUTION of specific volume after collision

Figure 6.11

S.P.H. SOLUTION of specific volume after collision

Figure 6.12

vNR SOLUTION of pressure after collision

Figure 6.13

S.P.H. SOLUTION of pressure after collision

Figure 6.14

7.0 Summary

The analysis in this report shows that the SPH method has an instability; this stability analysis analyzes the growth of a perturbation from a nonuniform state of stress and a uniform state of Δx . This instability is not stabilizeable with artificial viscosities. However, the analysis shows that it is stabilizeable with conservative smoothing.

The computational results in this report show that *SPHCS*, an SPH code with conservative smoothing, can produce more accurate solutions than *vNR*, a code based on the von Neumann - Richtmyer method, on a test problem with $C_{FL} = .2$. This is illustrated using the *vNR* scheme on a test problem with known exact solution involving a shock collision in a Hooke's law material.

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