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COMPOSITE FORMATION
UNDER A BINARY RELATION

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1. INTRODUCTION

In this paper we study the properties of a relation, H , between two sets, X and Y , $H \subseteq X \times Y$. We wish to determine when the existence of H between some elements of X and Y implies the existence of H between other composite elements of X and Y . The aim is to characterize a relation by its extension or restriction to composite elements.

Composite elements are formed by containment relationships among other elements. Thus the properties studied depend on the structure on the X and Y sets. We first consider the general case of lattices, M and N , on X and Y respectively, where the containment relationships provide the partial ordering on the sets. Here the meet and join operations are available. Later, Boolean algebras and the additional structure implied by the complementation operation are considered.

Systems composed of lattices M , N and relation H are first formally defined. Here we define a group of system transformations to simplify later proofs.

The properties of systems are then defined. All possible consistent combinations of property values are established by the use of reduction theorems and by the construction of example systems. Using thirteen independent properties there are exactly one hundred and forty seven possible combinations of property values. This set can be generated from a minimum set of forty values by the group of system transformations. These results are quite general and may have applications to areas requiring a classificatory or composite analysis approach.

Some of the thirteen independent properties defined in this paper are related to other concepts of structure mappings. The preservation of corresponding relations of the same rank on two sets, which includes three of the thirteen independent properties and two logical combinations

of other properties, may be represented by a generalized pair algebra as defined by Yeh [1]. The preservation properties are also generalizations of various types of functional homomorphisms and generalized congruence relations, which are explained in Yeh [2] and Liu [3].

Motivation

The current work is intended as a theoretical foundation for the study of graphical transformations on a set of objects interrelated by binary relations.

The properties defined here arose in attempting to characterize binary relations of image processing. In this case, elements of a directed graph typically correspond to "regions" of the image, and composites (unions of elements) correspond to figures of the scene. The properties specify ways in which relations extend to unions.

After characterizing a set of relations, it may be possible to determine which graph transformations are valid and desirable in order to form composites from elements, operations and relations.

A paper in progress pursues this possibility by enumerating and considering some specific graph transformations. Transformations which are implied by properties of a single relation as defined in this paper are exhibited. Relations of a transformation are next considered jointly, and the partitions induced on a class of relations by their joint behavior in the transformation are studied.

2. LATTICE PRELIMINARIES

In the following assume M and N are lattices on X and Y with the defining operations meet, $\cap$, and join, $\cup$, $M = (X, \cup, \cap)$, $N = (Y, \cup, \cap)$. The basic principles of lattices used in this paper may be found in Chapter one of Birkhoff [4] or in Chapters one and two of Szasz [5].

Lattice Duality

The dual of a lattice theoretical proposition involving the operational symbols $\cap$ and $\cup$ is obtained by interchanging $\cap$ and $\cup$ everywhere in the proposition. Denote the dual of a proposition P by $D(P)$. By the lattice duality principle P is true if and only if $D(P)$ is true, i.e., $P \Leftrightarrow D(P)$.

Given a lattice $M_1 = (X, \cup_1, \cap_1)$ we can define a new lattice

$$M_2 = (X, \cup_2, \cap_2) \text{ by}$$

$$a \cap_1 b = a \cup_2 b$$

$$a \cup_1 b = a \cap_2 b \text{ for all } a, b \in X.$$

The lattice duality principle shows that M_2 is a lattice. M_2 is called the dual of M_1 , $M_2 = D(M_1)$. The ordering of M_2 is reversed from that of M_1 . We use less than or equal, $\leq$, and greater than or equal, $\geq$, to represent the lattice ordering.

$$\text{Since } a \leq b \Leftrightarrow a \cup b = b$$

$$\text{and } a \geq b \Leftrightarrow a \cap b = b$$

$$D(a \leq b) = a \geq b.$$

Clearly $D(D(M)) = M$. If $D(M)$ is lattice isomorphic to M then we say M is self-dual. A lattice isomorphism is a one-to-one and onto mapping which preserves both lattice operations.

We will represent finite lattices by diagrams. The upward direction in the diagram corresponds to "greater than". If point a is higher than point b in the diagram, and if points a and b are connected by a branch, then $a > b$. The diagram of the dual lattice may be obtained by inverting the diagram of the lattice.

3. DEFINITION OF A SYSTEM

We consider a class of systems $\mathcal{L}$ where $S \in \mathcal{L}$ is a system in $\mathcal{L}$.

Let X, Y be arbitrary sets, H a binary relation from X to Y , $H \subseteq X \times Y$, and M and N be lattices on the sets X and Y respectively,

$$M = (X, \cup, \cap), N = (Y, \cup, \cap).$$

Then $S \in \mathcal{L}$ if and only if $S = (H, M, N)$ for some X, Y, H, M and N .

Group of Operators on a System

Given a system $S \in \mathcal{L}$, we can derive other systems $S' \in \mathcal{L}$ by taking dual lattices and inverse relations. This will simplify the study of properties of a relation H in a system S , and the enumeration of all possible relations H with a set of properties.

Thus we will define a group of operators, G , which may be applied to a system S , or to a logical proposition, A , about S . If $T \in G$ then T applied to S yields a well defined system S' , i.e. $T(S) = S'$ or $T(H, M, N) = (H', M', N')$. T applied to a logical proposition A about S yields a well defined logical proposition A' about S' . Furthermore, A is true if and only if A' is true.

The new systems will be obtained by all possibilities of dualizing the lattices M and N , or inverting the relation H . The group G of operators can be generated by composition of a dual operator and inverse operator.

The group consists of eight operators defined in Table 1 below.

Let $S = (H, M, N)$, $S \in \mathcal{L}$, where $H \subseteq X \times Y$, $M = (X, U, \cap)$, $N = (Y, U, \cap)$. $T(H, M, N) = (H', M', N')$, $T \in G$. $T(S) = S'$, $S' \in \mathcal{L}$.

	T	H'	M'	N'
Identity	E	H	M	N
Dual	D	H	D(M)	D(N)
Left Dual	D_L	H	D(M)	N
Right Dual	D_R	H	M	D(N)
Inverse	I	H^{-1}	N	M
Inverse Dual	ID	H^{-1}	D(N)	D(M)
Inverse Left Dual	ID_L	H^{-1}	N	D(M)
Inverse Right Dual	ID_R	H^{-1}	D(N)	M

TABLE 1

The group operation is composition of operators. Let $T_1, T_2, T_3 \in G$. We denote $T_1 (T_2 (S))$ as $T_1 T_2 (S)$, and the composition of operators T_2 and then T_1 as $T_1 T_2$.

Operation composition is associative, i.e. $T_1(T_2 T_3) = (T_1 T_2)T_3 = T_1 T_2 T_3$ which means apply T_3, T_2, T_1 in that order. The last three operators in the table are designated as compositions. For example, ID is the composition of D and I.

We display Table 2 of operator composition below to show that it is a group with identity element E.

		$T_1 \quad T_2$						
$T_1 \backslash T_2$	E	D	D_L	D_R	I	ID	ID_L	ID_R
E	E	D	D_L	D_R	I	ID	ID_L	ID_R
D	D	E	D_R	D_L	ID	I	ID_R	ID_L
D_L	D_L	D_R	E	D	ID_R	ID_L	ID	I
D_R	D_R	D_L	D	E	ID_L	ID_R	I	ID
I	I	ID	ID_L	ID_R	E	D	D_L	D_R
ID	ID	I	ID_R	ID_L	D	E	D_R	D_L
ID_L	ID_L	ID_R	I	ID	D_R	D_L	D	E
ID_R	ID_R	ID_L	ID	I	D_L	D_R	E	D

TABLE 2

Note that all elements have order 2 except ID_L and ID_R which have order 4 and the group is not commutative, since (e.g.)

$$ID_L = D_R I \neq D_L I .$$

The group G is isomorphic to the group of symmetries of a square. This can be seen by taking I to be reflection through the line $y = x$ and D_L to be reflection in the Y axis.

Given a system $S = (H, M, N)$, $S \in \mathcal{S}$, $T(S) = S'$ is a well defined element of $\mathcal{S}$ for all $T \in G$. This follows from the definition of T and the fact that $H^{-1} = \{(y, x) \mid (x, y) \in H\}$ is a well defined binary relation between Y and X , $D(M)$ is a well defined lattice on X , and $D(N)$ is a well defined lattice on Y .

Now assume we have a logical statement, A , about a system, S , involving the lattice operations in M and N , the relation H , variables, and expressions of logic. Then for all T in G we can form a statement $T(A) = A'$ about S' as shown in Table 3.

T	$T(A)$
E	A
D_L	A with lattice operations in M replaced by their dual operations.
D_R	A with lattice operations in N replaced by their dual operations.
I	A with H replaced by H^{-1}

TABLE 3

The other operators in G are compositions of the operators shown in Table 3.

For example if $A = ((x, y_1 \cup y_2) \in H)$,
then $I(A) = ((x, y_1 \cup y_2) \in H^{-1}) = ((y_1 \cup y_2, x) \in H)$,
 $D_R(A) = ((x, y_1 \cap y_2) \in H)$,
 $ID_R(A) = ((x, y_1 \cap y_2) \in H^{-1}) = ((y_1 \cap y_2, x) \in H)$,
 $D_R I(A) = ((y_1 \cup y_2, x) \in H)$.

By the definition of H^{-1} , the lattice theoretical duality principle, and the fact that S' always exists and is well defined, T changes true statements about $S = (H, M, N)$ to true statements about $S' = (H', M', N')$. Thus, a logical statement A concerning S is true if and only if the transformed statement $A' = T(A)$ concerning $S' = T(S)$ is true. That is, $A \Leftrightarrow T(A)$.

4. PROPERTIES OF H

We will now define properties of a relation H in a system $S \in \mathcal{L}$, $S = (H, M, N)$. The properties considered involve the preservation of the lattice operations: meet, join, less than or equal, and greater than or equal, in one or both lattices.

A property P is a logical statement about S and thus $T(P)$, $T \in G$, is a statement about $T(S)$. We will define twelve properties which are closed under operations in T . The properties are logically independent and imply all lattice operation preserving properties of a system S . The properties will be denoted: $P_1, P_1^{-1}, P_2, P_2^{-1}, P_3, P_3^{-1}, P_4, P_4^{-1}, P_5, P_6, P_7, P_6^{-1}$. In some cases the P will be omitted. $I(P)$ is denoted by P^{-1} unless $I(P) = P$ in which case P is a symmetric property. P_5 and P_7 are symmetric properties. The properties are defined below, where a 's and b 's are used as variables in M and N respectively.

Recall, according to the definition, that $I(P)$ is formed by replacing H by H^{-1} in P . For consistency in the definition of P^{-1} , we also interchange a and b variables. This is only a notational convenience so that a 's and b 's still refer to variables in M and N , respectively, since H goes between M and N in S . For example, the definition of P_3^{-1} is obtained from the definition of P_3 as follows:

$$P_3: aHb_1 \ \& \ aHb_2 \Rightarrow aH(b_1 \cup b_2)$$

$$I(P_3) = P_3^{-1}: aH^{-1}b_1 \ \& \ aH^{-1}b_2 \Rightarrow aH^{-1}(b_1 \cup b_2) \quad \Leftrightarrow$$

$$b_1Ha \ \& \ b_2Ha \Rightarrow (b_1 \cup b_2) H a$$

changing notation:

$$P_3^{-1}: a_1Hb \ \& \ a_2Hb \Rightarrow (a_1 \cup a_2)Hb.$$

Definition of Properties

P1: Propagates Images Up

$$a \text{ H } b_1 \ \& \ b_2 \geq b_1 \Rightarrow a \text{ H } b_2$$

$$a \rightarrow b$$

$$a \rightarrow \underline{\geq} b$$

P1⁻¹: Propagates Inverse Images Up

$$a_1 \text{ H } b \ \& \ a_2 \geq a_1 \Rightarrow a_2 \text{ H } b$$

$$a \rightarrow b$$

$$\underline{\geq} a \rightarrow b$$

P2: Propagates Images Down

$$a \text{ H } b_1 \ \& \ b_2 \leq b_1 \Rightarrow a \text{ H } b_2$$

$$a \rightarrow b$$

$$a \rightarrow \underline{\leq} b$$

P2⁻¹: Propagates Inverse Images Down

$$a_1 \text{ H } b \ \& \ a_2 \leq a_1 \Rightarrow a_2 \text{ H } b$$

$$a \rightarrow b$$

$$\underline{\leq} a \rightarrow b$$

P3: Joins Images

$$a \text{ H } b_1 \ \& \ a \text{ H } b_2 \Rightarrow a \text{ H } (b_1 \cup b_2)$$

$$a \rightarrow b_1$$

$$a \rightarrow b_2$$

$$a \rightarrow b_1 \cup b_2$$

P3⁻¹: Joins Inverse Images

$$a_1 \text{ H } b \ \& \ a_2 \text{ H } b \Rightarrow (a_1 \cup a_2) \text{ H } b$$

$$a_1 \rightarrow b$$

$$a_2 \rightarrow b$$

$$a_1 \cup a_2 \rightarrow b$$

P4: Meets Images

$$a \text{ H } b_1 \ \& \ a \text{ H } b_2 \Rightarrow a \text{ H } (b_1 \cap b_2)$$

$$a \rightarrow b_1$$

$$a \rightarrow b_2$$

$$a \rightarrow b_1 \cap b_2$$

P4⁻¹: Meets Inverse Images

$$a_1 H b \ \& \ a_2 H b \Rightarrow (a_1 \cap a_2) H b$$

$$a_1 \rightarrow b$$

$$a_2 \rightarrow b$$

$$\Downarrow \\ a_1 \cap a_2 \rightarrow b$$

P5: Preserves Joins

$$a_1 H b_1 \ \& \ a_2 H b_2 \Rightarrow (a_1 \cup a_2) H (b_1 \cup b_2)$$

$$a_1 \rightarrow b_1$$

$$a_2 \rightarrow b_2$$

$$\Downarrow \\ a_1 \cup a_2 \rightarrow b_1 \cup b_2$$

P6: Dualizes Joins

$$a_1 H b_1 \ \& \ a_2 H b_2 \Rightarrow (a_1 \cup a_2) H (b_1 \cap b_2)$$

$$a_1 \rightarrow b_1$$

$$a_2 \rightarrow b_2$$

$$\Downarrow \\ a_1 \cup a_2 \rightarrow b_1 \cap b_2$$

P6⁻¹: Dualizes Meets

$$a_1 H b_1 \ \& \ a_2 H b_2 \Rightarrow (a_1 \cap a_2) H (b_1 \cup b_2)$$

$$a_1 \rightarrow b_1$$

$$a_2 \rightarrow b_2$$

$$\Downarrow \\ a_1 \cap a_2 \rightarrow b_1 \cup b_2$$

P7: Preserves Meets

$$a_1 H b_1 \ \& \ a_2 H b_2 \Rightarrow (a_1 \cap a_2) H (b_1 \cap b_2)$$

$$a_1 \rightarrow b_1$$

$$a_2 \rightarrow b_2$$

$$\Downarrow \\ a_1 \cap a_2 \rightarrow b_1 \cap b_2$$

Other properties can be defined which are logical combinations of the twelve independent properties. We show four such properties below.

$P2 \ \& \ P2^{-1}$: Preserves Less Than

$$\begin{array}{c} a \rightarrow b \\ \Downarrow \\ \leq a \rightarrow \leq b \end{array}$$

$P1 \ \& \ P1^{-1}$: Preserves Greater Than

$$\begin{array}{c} a \rightarrow b \\ \Downarrow \\ \geq a \rightarrow \geq b \end{array}$$

$P2^{-1} \ \& \ P1$: Dualizes Less Than

$$\begin{array}{c} a \rightarrow b \\ \Downarrow \\ \leq a \rightarrow \geq b \end{array}$$

$P1^{-1} \ \& \ P2$: Dualizes Greater Than

$$\begin{array}{c} a \rightarrow b \\ \Downarrow \\ \geq a \rightarrow \leq b \end{array}$$

Permutations of Property Values

Table 4 shows $T(P)$ for all twelve properties P and all T in G . Since each $T(P)$ is a permutation of the properties P , we can determine the true or false values of the properties $T(P)$ of the system $T(S)$, given the properties P of S .

Table 4 also shows that the twelve properties are partitioned into three sets of four properties such that each set is closed under all T in G . These three sets are separated by double lines in Table 4, and will be referred to in the following as the P_1 set, the P_3 set, and the P_5 set. Any combination of values of the twelve properties will be referred to by an index, (i, j, k) , where i, j, k are indices of the values of the properties in the P_1, P_3 , and P_5 set respectively. The values of the indices i, j, k are determined by partitioning the sixteen possible true or false values of four properties into the following six sets, with indices 0, 1, 2, ..., 5:

<u>Index</u>	<u>Set</u>
0	0000
1	0001, 0010, 0100, 1000
2	1010, 0101
3	0011, 0110, 1001, 1100
4	1110, 1101, 1011, 0111
5	1111

Each of these six sets of property values is closed under all T in G , and each element of each set is a generator of the set under T in G .

Thus the index, (i, j, k) of any value combination of the twelve properties does not change under T in G . That is, if values P are indexed by (i, j, k) , the values $T(P)$ are also indexed by (i, j, k) . These facts will allow us to easily derive a minimum set of generators for all consistent cases of properties. This derivation will be carried out after proving theorems interrelating the twelve properties in the next section.

T(P).

T \ P	1	1^{-1}	2	2^{-1}	3	3^{-1}	4	4^{-1}	5	6	7	6^{-1}
D	2	2^{-1}	1	1^{-1}	4	4^{-1}	3	3^{-1}	7	6^{-1}	5	6
D_L	1	2^{-1}	2	1^{-1}	3	4^{-1}	4	3^{-1}	6^{-1}	7	6	5
D_R	2	1^{-1}	1	2^{-1}	4	3^{-1}	3	4^{-1}	6	5	6^{-1}	7
I	1^{-1}	1	2^{-1}	2	3^{-1}	3	4^{-1}	4	5	6^{-1}	7	6
ID	2^{-1}	2	1^{-1}	1	4^{-1}	4	3^{-1}	3	7	6	5	6^{-1}
ID_L	1^{-1}	2	2^{-1}	1	3^{-1}	4	4^{-1}	3	6	7	6^{-1}	5
ID_R	2^{-1}	1	1^{-1}	2	4^{-1}	3	3^{-1}	4	6^{-1}	5	6	7

TABLE 4

5. ENUMERATION OF POSSIBLE COMBINATIONS OF PROPERTIES

We now wish to determine all logically consistent combinations of the twelve properties P of a relation H in a system $S = (H, M, N)$.

This section reduces the number of possibly consistent value combinations by proving and applying theorems which interrelate the twelve properties.

The next section proves that this set of theorems is complete by constructing examples of systems having all value combinations in the reduced set.

We will prove theorems involving properties P and expressions of logic. Corresponding to a theorem, A , are eight theorems $T(A)$, $T \in G$, which may or may not be independent of A , which are obtained by replacing P by $T(P)$ for all P involved in A . Since $T(P) \Leftrightarrow P$, and the properties involve arbitrary system S , it follows directly that $T(A) \Leftrightarrow A$. That is, a proof of A implies a proof of $T(A)$. This will considerably simplify the proofs to establish all consistent cases.

Theorems Relating Properties

In Table 5 we list twenty independent theorems, numbered one through twenty, and a diagrammatic representation of the theorems in a form which will make the application of the theorems in the reduction process readily apparent.

Theorem

Diagram

	1	1^{-1}	2	2^{-1}	3	3^{-1}	4	4^{-1}	5	6	7	6^{-1}
1. $1 \Rightarrow 3$	•				▷							
2. $1^{-1} \Rightarrow 3^{-1}$		•			▷							
3. $2 \Rightarrow 4$			•				▷					
4. $2^{-1} \Rightarrow 4^{-1}$				•			▷					
5. $5 \Rightarrow 3$					◁	◁			•			
6. $5 \Rightarrow 3^{-1}$												
7. $6 \Rightarrow 3^{-1}$						◁	◁			•		
8. $6 \Rightarrow 4$												
9. $7 \Rightarrow 4$												
10. $7 \Rightarrow 4^{-1}$							◁	◁			•	
11. $6^{-1} \Rightarrow 3$					◁			◁				•
12. $6^{-1} \Rightarrow 4^{-1}$												
13. $1 \text{ \& } 3^{-1} \Rightarrow 5$	•					•			▷			
14. $1 \text{ \& } 4^{-1} \Rightarrow 6^{-1}$	•							•				▷
15. $1^{-1} \text{ \& } 3 \Rightarrow 5$		•			•				▷			
16. $1^{-1} \text{ \& } 4 \Rightarrow 6$		•					•			▷		
17. $2 \text{ \& } 3^{-1} \Rightarrow 6$			•			•				▷		
18. $2 \text{ \& } 4^{-1} \Rightarrow 7$			•					•			▷	
19. $2^{-1} \text{ \& } 3 \Rightarrow 6^{-1}$				•	•							▷
20. $2^{-1} \text{ \& } 4 \Rightarrow 7$				•			•					▷

TABLE 5

Table 6 shows that transformations , T, applied to theorems one, five and thirteen generate all twenty theorems, and that these twenty theorems are closed under T in G.

A \ T	T(A)		
	1	5	13
D_L	1	11	14
I	2	6	15
ID_L	2	7	16
D_R	3	8	17
D	3	9	18
ID_R	4	12	19
ID	4	10	20

TABLE 6

Below we give proofs of theorems one, five and thirteen, thus implying proofs of all twenty theorems.

Thm. 1 $P1 \Rightarrow P3$

Proof: $a H b_1 \in aHb_2 \Rightarrow$
 $(\text{by } P1 \text{ and } b_1 \cup b_2 \geq b_1 \text{ or } b_1 \cup b_2 \geq b_2)$
 $a H (b_1 \cup b_2)$ Q.E.D.

Thm. 5 $P5 \Rightarrow P3$

Proof: $a H b_1 \in aHb_2 \Rightarrow (\text{by } P5)$
 $(aUa) H (b_1 \cup b_2) \Rightarrow$
 $a H (b_1 \cup b_2)$ Q.E.D.

Thm. 13 $P1 \& P3^{-1} \Rightarrow P5$

Proof: $a_1 H b_1 \& a_2 H b_2 \Rightarrow$

(by $(b_1 \cup b_2) \geq b_1$ and $(b_1 \cup b_2) \geq b_2$ and P1)

$a_1 H (b_1 \cup b_2) \& a_2 H (b_1 \cup b_2) \Rightarrow$ (by $P3^{-1}$)

$(a_1 \cup a_2) H (b_1 \cup b_2)$

Q.E.D.

Figure 1 represents all of the twenty theorems by an implication graph whose nodes represent the twelve properties. The following symmetries of the properties and implications are apparent from Figure 1.

The transformation D_R corresponds to reflection about a line through nodes 1^{-1} and 2^{-1} . D_L corresponds to reflection about a line through nodes 3 and 4. $D(P)$ corresponds to reflection through the center of the diagram on a line through node P, for any property P.

Reduction of Possible Consistent Cases

There are $2^{12} = 4096$ possible combinations of binary values of the twelve properties. By applying the twenty theorems, the number of possible consistent combinations is reduced to 136. Further the set of values, y , where $y = \{v\}$, is partitioned into subsets, called blocks, by the equivalence relation: v is equivalent to v' if and only if $v = T(v')$ for some T in G . Every element in each block is a generator of the block under T in G . We will show that there are 33 consistent blocks and, therefore, 33 generators can generate all consistent cases.

The reduction process and derivation of a minimum set of generators is illustrated in Table 7. Each combination is indexed by (i,j,k) , as explained in the previous section. We will refer to a value, v , by $v_1 v_2 v_3$, where v_1 is the 4 bit value of the P1 set of properties, v_2 of the P3 set and v_3 of the P5 set. In the derivation, for each $j = 0, 1, 2, 3, 4$ all possible consistent values of v_1 and of v_3 are listed for one possible

FIGURE 1

value of v_2 . For $j = 5$, since v_2 has only one value, all possible consistent values of v_3 are listed for one value of v_1 for each $i = 1, 2, 3, 4$. For $i = 0, j = 5$, one value of v_3 is listed for each $k = 0, 1, 2, 3, 4, 5$.

It is easily seen that the values in Table 7 are generators for all possible consistent values. For any consistent value $v_1 v_2 v_3$, with index (i,j,k) , we can always find a T such that $T(v_1 v_2 v_3) = v'_1, v'_2, v'_3$, and v'_1, v'_2, v'_3 is in Table 7. There are three cases:

- 1) $j = 0, 1, 2, 3, 4$ and T is the transformation which takes v_2 into the v'_2 used in the table.
- 2) $j = 5, i = 1, 2, 3, 4$ and T takes v_1 into the v'_1 used in the table.
- 3) $j = 5, i = 0$ and T takes v_3 into the v'_3 used in the table.

Since there are 32 different (i,j,k) blocks in Table 7, and any T does not alter the block of its operand, there must be at least 32 consistent blocks. By applying transformations to the values in Table 7, we see that there are at most 33 consistent blocks. The set indexed by $(1,4,3)$ requires two generators, while the other consistent (i,j,k) sets require only one generator. We index the 33 possibly consistent blocks by (i,j,k) if $(i,j,k) \neq (1,4,3)$, and by $(1,4,3) (1)$ and $(1,4,3) (2)$.

Table 8 lists a set of 33 generators from the values of Table 7. The values in Table 7 which are not used in Table 8 have an entry under column T specifying the transformation which generates them from the value in Table 8 in the same block. Table 8 also lists the number of elements in each block. All 136 values are listed in Appendix 1.

TABLE 7

Reduction of Set of Possible Consistent Values of Twelve Properties

1. All possible values in all i, k sets are listed for one representative value in the j set, for $j = 0, 1, 2, 3, 4$.

Index $i \ j \ k$	T	Properties											
		1	1 ⁻¹	2	2 ⁻¹	3	3 ⁻¹	4	4 ⁻¹	5	6	7	6 ⁻¹
000				0000		0000				0000			
010				0000		0001				0000			
110				0001						0000			
020				0000		1010				0000			
120				0010						0000			
120	D			1000						0000			
220				1010						0000			
030				0000		0011				0000			
031				0000						0010			
131				0001						0010			
131	I			0010						0010			
331				0011						0010			
040				0000		1110				0000			
041				0000						0100			
041	D _R			0000						1000			
043				0000						1100			
141				0010						0100			
143(1)				0010						1100			
143(2)				0100						1100			
343				0110						1100			
141	D _R			1000						1000			
143(1)	D _R			1000						1100			
243				1010						1100			
343	D _R			1100						1100			
443				1110						1100			

2. All possible values for all k sets are listed for $j = 5$ and for one representative value in the i set, for $i = 0, 1, 2, 3, 4, 5$.

Index			Properties											
i	j	k	$1 \ 1^{-1} \ 2 \ 2^{-1}$				$3 \ 3^{-1} \ 4 \ 4^{-1}$				$5 \ 6 \ 7 \ 6^{-1}$			
050			0000				1111				0000			
051											0001			
052											1010			
053											0011			
054											1110			
055											1111			
153			0001				1111				0011			
154											0111			
154		D_R									1011			
155											1111			
255			1010				1111				1111			
354			0011				1111				0111			
355											1111			
455			1110				1111				1111			
555			1111				1111				1111			

TABLE 8

Minimum Set of 33 Generators of 136 Cases

Block Index			$1 \ 1^{-1} \ 2 \ 2^{-1}$				$3 \ 3^{-1} \ 4 \ 4^{-1}$				$5 \ 6 \ 7 \ 6^{-1}$				Order of Block
i	j	k													
000			0000				0000				0000				1
010			0000				0001				0000				4
020			0000				1010				0000				2
030			0000				0011				0000				4
031			0000				0011				0010				4
040			0000				1110				0000				4
041			0000				1110				0100				8
043			0000				1110				1100				4
050			0000				1111				0000				1
051			0000				1111				0001				4
052			0000				1111				1010				2
053			0000				1111				0011				4
054			0000				1111				1110				4
055			0000				1111				1111				1
110			0001				0001				0000				4
120			0010				1010				0000				4
131			0001				0011				0010				8
141			0010				1110				0100				8
143(1)			0010				1110				1100				8
143(2)			0100				1110				1100				4
153			0001				1111				0011				4
154			0001				1111				0111				8
155			0001				1111				1111				4
220			1010				1010				0000				2
243			1010				1110				1100				4
255			1010				1111				1111				2
331			0011				0011				0010				4
343			0110				1110				1100				8
354			0011				1111				0111				4
355			0011				1111				1111				4
443			1110				1110				1100				4
455			1110				1111				1111				4
555			1111				1111				1111				1

6. CONSTRUCTION OF EXAMPLE SYSTEMS

In this section an example system is constructed for one representative in each of the 33 possible consistent blocks, thus proving that there are exactly 136 possible consistent combinations of values of the twelve properties.

Factoring of Property Set

In order to determine a procedure for explicitly constructing examples of systems with given properties, we wish to determine the largest sets of properties which are completely independent of each other. Thus, we wish to factor the properties into sets whose values may be assigned independently, and which have a maximum of internal constraints so as to give a minimum number of values within each set.

Figure 1 shows that at least two nodes must be removed to obtain independent sets. By removing 6 and 6^{-1} , the ten remaining properties factor into two sets of five properties, 1, 1^{-1} , 3, 3^{-1} , 5 and 2, 2^{-1} , 4, 4^{-1} , 7, such that the two sets are congruent under duals. That is, the properties and theorems of one set are duals of the other set. Each set is also closed under inverses. Thus, each set has the same possible value assignments, and the two sets combined have all possible products of assignments of each set. This is an optimal factoring and the one that will be considered below.

An exactly analogous situation occurs if 5 and 7 are removed. The next best factorings are obtained by removing 7 and 6^{-1} , or 5 and 6, or 5 and 6^{-1} , or 7 and 6, which result in independent sets of 2 and 8 elements symmetric under D_R or D_L .

Now we will construct examples for each dual set of five properties, then determine a procedure to construct examples for ten properties from examples of each set of five, and finally alter the application of the procedure to obtain examples when permissible values of the other two properties, 6 and 6^{-1} , are also considered.

Simplest Examples for Factored Sets

The theorems eliminate all but ten combinations of the five properties 1, 1^{-1} , 3, 3^{-1} , 5 and the same ten combinations of 2, 2^{-1} , 4, 4^{-1} , 7. These ten possibilities are given in Table 9 below. To prove that all ten cases are consistent, diagrams of systems satisfying each case are shown in Figure 2. The example corresponding to each case is designated at the right of entry in Table 9. The Dual examples correspond to the dual properties. S_1 is the null example, i.e. $S_1 = D(S_1) = (\emptyset, \emptyset, \emptyset)$.

1	1^{-1}	3	3^{-1}	5	<u>Example</u>
1	1	1	1	1	S_1
1	0	1	1	1	S_2
1	0	1	0	0	S_3
0	0	1	1	1	S_4
0	0	1	1	0	S_5
0	0	1	0	0	S_6
0	0	0	0	0	S_7
0	1	1	1	1	$I(S_2)$
0	1	0	1	0	$I(S_3)$
0	0	0	1	0	$I(S_6)$

TABLE 9

The examples given use the simplest lattices and relations possible. That is, if we define a system $S = (H, M, N)$ of a set $\mathcal{L}'$ to be simplest if

- 1) both M and N have the fewest points and the most complete ordering of any S in $\mathcal{L}'$, and
- 2) H has the fewest elements of any S in $\mathcal{L}'$,

then we can show that each of our examples is simplest in the set $\mathcal{L}'$ of S in $\mathcal{L}$ satisfying the given properties.

$D(S_3) = S_3$

$D(S_5) = S_5$

$D(S_7) = S_7$

FIGURE 2

To show that these examples are simplest we need only show that $S_1, S_2, \dots, S_7$ are simplest.

Below are diagrams of all lattices of less than or equal to four points, ordered according to strictly decreasing simplicity. Let $L_i > L_j$ mean L_j is simpler than L_i .

$L_5 > L_4 > L_3 > L_2 > L_1 > L_0$

Let $|H|$ be the number of elements in H . Table 10 shows M , $|H|$, and N for the examples.

S	M	,	$ H $	,	N
S_1	L_0	,	0	,	L_0
S_2	L_2	,	1	,	L_1
S_3	L_5	,	2	,	L_1
S_4	L_2	,	1	,	L_2
S_5	L_2	,	2	,	L_2
S_6	L_5	,	2	,	L_2
S_7	L_5	,	3	,	L_5

TABLE 10

The following facts may be easily established by enumerating all simpler cases:

1. $\neg (P1^{-1}) \Rightarrow M > L_1$
2. $\neg (P1) \Rightarrow N > L_1$
3. $\neg (P3^{-1}) \Rightarrow M > L_4$
4. $\neg (P3) \Rightarrow N > L_4$
5. $\neg (P1 \text{ or } P1^{-1}) \Rightarrow |H| \geq 1$
6. $\neg (P3 \text{ or } P5 \text{ or } P3^{-1}) \Rightarrow |H| \geq 2$

These facts and Table 10 show directly that $S_1, S_2, \dots, S_6$ are simplest. Additionally we need only show that $|H| = 2$ is not possible for S_7 . Again this can be seen by enumeration of cases. Thus, $S_1, S_2, \dots, S_7$ are simplest examples.

Combinations of Ten Properties

Next consider the ten properties $1, 1^{-1}, 3, 3^{-1}, 5$ and $2, 2^{-1}, 4, 4^{-1}, 7$ together. There are one hundred possible consistent combinations of these ten properties, obtained by taking all combinations of the ten consistent cases for each set of five properties. We can prove that all one hundred cases are consistent by giving a procedure to construct an example satisfying the ten properties from two simplest examples satisfying the two sets of five properties. This procedure is given below.

Let $S_1 = (H_1, M_1, N_1)$ be the example satisfying a combination of properties $1, 1^{-1}, 3, 3^{-1}, 5$ and $S_2 = (H_2, M_2, N_2)$ be the example for $2, 2^{-1}, 4, 4^{-1}, 7$. Construct a new example $S = (H, M, N)$ as follows:

1. Form $M(N)$ by joining the minimum element of $M_1(N_1)$ to the maximum element of $M_2(N_2)$.
2. Form H'_1 in $X_1 \times Y_1$ by adding elements to H_1 implied by the 2, 2^{-1} , 4, 4^{-1} , 7 properties of S_2 . Form H'_2 in $X_2 \times Y_2$ by adding elements to H_2 implied by the 1, 1^{-1} , 3, 3^{-1} , 5 properties of S_1 .
3. Take $H = H'_1 \cup H'_2$.

Then we have a lattice M on X , N on Y , and $H \subseteq X \times Y$.

X is a set with distinct points corresponding to $X_1 - \min(M_1)$ and $X_2 - \max(M_2)$ and one point corresponding to the two points $\min(M_1), \max(M_2)$. Y is obtained similarly.

For example if M_1, M_2, N_1, N_2 all have the structure:

then $S = (H, M, N)$ has the structure:

Thm.

If S_1 and S_2 are simplest examples with properties
 $1, 1^{-1}, 3, 3^{-1}, 5$ and $2, 2^{-1}, 4, 4^{-1}, 7$
 respectively, then S as constructed above has the same
 $1, 1^{-1}, 3, 3^{-1}, 5, 2, 2^{-1}, 4, 4^{-1}, 7$ properties.

Proof: i)

$H'_2 \subseteq X_2 \times Y_2$; since $H_2 \subseteq X_2 \times Y_2$ and if H'_2 has an element
 (x', y') with x' in $(X-X_2)$, then $|X_1| > 1$ and S_1 must
 satisfy Pl^{-1} , since Pl^{-1} is the only property that can move
 an element of H_2 above the $\max(M_2)$. But, all simplest
 examples with $Pl^{-1} = 1$ have $|X_1| = 1$. Similarly H'_2 can
 have no element (x', y') with y' in $(Y-Y_2)$. Thus $H'_2 \subseteq X_2 \times Y_2$,
 or the properties of S_1 can move no elements of H_2 upward
 into $(X-X_2) \times (Y-Y_2)$.

ii) In forming H'_1 , the properties of S_2 can only move elements
 of H_1 downward.

iii) Since no higher elements are added in $(X-X_2) \times (Y-Y_2)$, S
 fails to satisfy the same properties as S_1 in a simplest
 example.

iv) Adding lower elements in a simplest example cannot change
 any of the S_1 properties.

Thus S has the same properties as S_1 .

The dual argument shows S has the same properties as S_2 . Thus
 S has the same $1, 1^{-1}, 3, 3^{-1}, 5$ properties as S_1 and the same
 $2, 2^{-1}, 4, 4^{-1}, 7$ properties as S_2 .

Q.E.D.

This procedure will not necessarily give the simplest example in
 each case. It does establish that all one hundred values are consistent.

Consistent Values of Twelve Properties

We will now construct 33 examples, one for each block, of systems considering all twelve properties. The procedure given for constructing example systems having a desired value of ten properties is used and then modified if possible to give the proper value of properties 6 and 6^{-1} .

First we notice that we can reduce the set of 100 combinations of ten properties to a smaller generator set by using the transformations I, D, and ID, under which the 10 properties are closed. Denote the 100 possible values by (x,y) where x and y gives the index of the value of 5 properties in Table 9. Then, the set is reduced to 34 values, indicated by 0's or 1's in the matrix of Table 11, by using the four transformations below:

- 1) $D(x,y) = (y,x)$, so only the lower triangular of the ten by ten matrix is necessary.
- 2) $I(I(x), I(y)) = (x,y)$ so columns $I(2)$, $I(3)$, $I(6)$ are not needed.
- 3) $I(I(x),y) = (x,y)$ for $y = 1, 4, 5, 7$ so that these four y columns are not needed with the last 3 rows.
- 4) $ID(I(x),y) = (I(y), x)$ so only the lower triangular of the submatrix consisting of rows $I(2)$, $I(3)$, $I(6)$ and columns 2,3,6 is necessary.

Next we consider the possible values of $P6$ and $P6^{-1}$ consistent with the theorems for each of the 34 cases. This adds only 16 values. We then pick one combination for each of the 33 blocks. Table 12 shows the 33 representatives. To the left of the value are the x and y indices. Only the values of x and y marked by 1's in Table 11 are used. To the right of the entry in Table 12 is the block index, and columns P and S .

A modified procedure is used to construct example systems for each of the 33 values. These examples are displayed in Figure 3 by diagrams, where M is represented by the left lattice, N by the right lattice, and H by directed branches from left to right.

The modified procedure used first constructs the example for the ten properties as explained above, then modifies it intuitively, either to satisfy the values of P_6 and P_6^{-1} , or to simplify the example, or both. Properties of the final example arrived at in Figure 3 are indicated by columns P and S of Table 12. Column P is marked if the example is exactly the same as the example produced by the original procedure, and column S is marked if the example is simpler.

Example systems for all 136 possible values can be obtained by applying a proper transformation, such as the one listed in Appendix 1, to the representative in Figure 3.

$\begin{array}{c} y \\ \backslash \\ x \end{array}$	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	1	1				
4	1	1	0	1			
5	1	1	0	1	1		
6	1	1	1	1	1	1	
7	1	1	1	1	1	1	1
$I(2)$		0					
$I(3)$		0	0				
$I(6)$		1	0			0	

TABLE 11

#	x	y	1 1 ⁻¹ 2 2 ⁻¹				3 3 ⁻¹ 4 4 ⁻¹				5 6 7 6 ⁻¹				Block Index i j k			P	S
			1	1 ⁻¹	2	2 ⁻¹	3	3 ⁻¹	4	4 ⁻¹	5	6	7	6 ⁻¹	i	j	k		
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	5	5	5	1	
2	2	1	1	0	1	1	1	1	1	1	1	1	1	1	4	5	5	1	
3	3	1	1	0	1	1	1	1	0	1	1	0	0	1	4	4	3	1	
4	4	1	0	0	1	1	1	1	1	1	1	1	1	1	3	5	5	1	
5	5	1	0	0	1	1	1	1	1	1	0	1	1	1	3	5	4	1	
6	6	1	0	0	1	1	1	1	0	1	1	0	0	1	3	4	3	1	
7	7	1	0	0	1	1	0	0	1	1	0	0	0	1	3	3	1		1
8	2	2	1	0	1	0	1	1	1	1	1	1	1	1	2	5	5	1	
9	3	2	1	0	1	0	1	1	0	1	1	0	0	1	2	4	3	1	
10	4	2	0	0	1	0	1	1	1	1	1	1	1	1	1	5	5	1	
11	4	2	0	0	1	0	1	1	1	1	1	1	1	0	1	5	4		
12	5	2	0	0	1	0	1	1	1	1	0	1	1	0	1	5	3		
13	6	2	0	0	1	0	1	1	0	1	1	0	0	1	1	4	3(1)	1	
14	6	2	0	0	1	0	1	1	0	1	1	0	0	1	1	4	1		
15	I(6)	2	0	0	1	0	0	1	1	1	0	1	1	0	1	4	3(2)	1	
16	7	2	0	0	1	0	0	1	1	1	0	0	1	0	1	3	1		1
17	3	3	1	0	1	0	1	0	1	0	0	0	0	0	2	2	0	1	
18	6	3	0	0	1	0	1	0	1	0	0	0	0	0	1	2	0		1
19	7	3	0	0	1	0	0	1	0	1	0	0	0	0	1	1	0		1
20	4	4	0	0	0	0	1	1	1	1	1	0	1	0	0	5	2		
21	4	4	0	0	0	0	1	1	1	1	1	0	1	1	0	5	4		
22	4	4	0	0	0	0	1	1	1	1	1	1	1	1	0	5	5	1	
23	5	4	0	0	0	0	1	1	1	1	0	0	1	0	0	5	1		
24	5	4	0	0	0	0	1	1	1	1	0	0	1	1	0	5	3		
25	6	4	0	0	0	0	1	0	1	1	0	0	1	0	0	4	1		
26	6	4	0	0	0	0	1	0	1	1	0	0	1	1	0	4	3	1	
27	7	4	0	0	0	0	0	1	1	1	0	0	1	0	0	3	1		1
28	5	5	0	0	0	0	1	1	1	1	0	0	0	0	0	5	0		
29	6	5	0	0	0	0	1	0	1	1	0	0	0	0	0	4	0		
30	7	5	0	0	0	0	0	1	1	1	0	0	0	0	0	3	0		1
31	6	6	0	0	0	0	1	0	1	0	0	0	0	0	0	2	0		1
32	7	6	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0		1
33	7	7	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0		1

TABLE 12

1.

 $\emptyset$

2.

3.

4.

5.

6.

7.

8.

9.

10.

11.

12.

FIGURE 3 SHEET 1

13.

14.

15.

16.

17.

18.

19.

20.

21.

22.

23.

24.

FIGURE 3 SHEET 2

25.

27.

29.

31.

33.

26.

28.

30.

32.

FIGURE 3 SHEET 3

7. COMPLEMENTATION

We now introduce one additional property for the case when M and N are Boolean algebras. Then, a unique complement, $\bar{x}$, exists for each element x in M or N. We will use DeMorgan's laws:

$$\overline{(a \cup b)} = \bar{a} \cap \bar{b}$$

$$\overline{(a \cap b)} = \bar{a} \cup \bar{b}$$

and the identity, $a \leq b \Leftrightarrow \bar{a} \geq \bar{b}$, which hold in a Boolean algebra.

The new property is

P8: Preserves Complements

$$a \text{ H } b \Rightarrow \bar{a} \text{ H } \bar{b}$$

P8 is invariant under T, i.e. $T(P8) = P8$ for all T in G. Thus, T still permutes the set of all thirteen properties. The following theorems relate P8 to the first twelve properties.

Thm. 21 $P \text{ \& P8 } \Rightarrow D(P)$

Proof: Because of DeMorgan's laws, $D(P)$ can be obtained, if P8. is true, by complementing variables on the left hand side of the implication sign in P, applying P to the complemented variables, and then complementing the elements of H in the resulting statement.

Q.E.D.

For example we prove:

$P1 \text{ \& P8 } \Rightarrow P2.$

Proof: $a \text{ H } b_1 \text{ \& } b_2 \leq b_1 \Rightarrow (\text{by P8})$
 $\bar{a} \text{ H } \bar{b}_1 \text{ \& } \bar{b}_2 \geq \bar{b}_1 \Rightarrow (\text{by P1})$
 $\bar{a} \text{ H } \bar{b}_2 \Rightarrow (\text{by P8})$
 $\bar{\bar{a}} \text{ H } \bar{\bar{b}}_2 \Rightarrow$
 $a \text{ H } b_2$

Q.E.D.

Thm. 22 $P1 \& P1^{-1} \& P2 \& P2^{-1} \Rightarrow P8$

Proof: Let 0 represent the minimum element of the Boolean algebra.

$$a \vee b \Rightarrow (\text{by } P2 \& P2^{-1})$$

$$0 \vee 0 \Rightarrow (\text{by } P1 \& P1^{-1})$$

$$\overline{a} \vee \overline{b}$$

Q.E.D.

Now we wish to determine all possible consistent cases of all thirteen properties when M and N are Boolean algebras.

Applying Theorem 21 we get:

$$P8 \Rightarrow (P1 = P2) \& (P1^{-1} = P2^{-1}) \& (P3 = P4) \& (P3^{-1} = P4^{-1}) \& (P5 = P7) \& (P6 = P6^{-1}).$$

Thus P8 cannot be true unless all properties imply their dual properties. This could also be proven by showing that Boolean algebras are self-dual lattices with complementation as the isomorphism between L and D(L).

The theorems determine P8 uniquely in all but eleven of the one hundred thirty six cases of the first twelve properties, or seven of the thirty three generator cases, giving a possible one hundred forty seven total cases or forty generator cases.

We show that all of these cases are possible by giving examples of the fourteen generators where P8 is not uniquely determined by the first twelve properties. The diagrams of Boolean algebras in Figure 4 are n-cubes where diagonal points are complements. Table 13 gives the index of the example in Figure 4 or Figure 3 which corresponds to each of the fourteen cases.

Thus, examples of all one hundred and forty seven cases of the thirteen properties can be generated from the examples in Figures 3 and 4.

Index of Basic Case of First 12 Properties	P8	Illustration Number
8	0	C1
8	1	C2
17	0	C3
17	1	17
20	0	C4
20	1	20
22	0	C5
22	1	C6
28	0	C7
28	1	28
31	0	C8
31	1	C9
33	0	33
33	1	C10

TABLE 13

FIGURE 4

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APPENDIX 1

All 136 Consistent Cases are listed by block. The first entry in each block is the generator in Table 8. A transformation which generates each of the other elements from the first is specified under Column T.

Block Index

i j k	1 1 ⁻¹ 2 2 ⁻¹	3 3 ⁻¹ 4 4 ⁻¹	5 6 7 6 ⁻¹	T
000	0000	0000	0000	
010	0000	0001	0000	
	0000	0010	0000	I
	0000	0100	0000	D
	0000	1000	0000	ID
020	0000	1010	0000	
	0000	0101	0000	I
030	0000	0011	0000	
	0000	0110	0000	D _L
	0000	1001	0000	D _R
	0000	1100	0000	D
031	0000	0011	0010	
	0000	0110	0100	D _L
	0000	1001	0001	D _R
	0000	1100	1000	D
040	0000	1110	0000	
	0000	1101	0000	I
	0000	1011	0000	D
	0000	0111	0000	ID

i j k	1 1 ⁻¹ 2 2 ⁻¹	3 3 ⁻¹ 4 4 ⁻¹	5 6 7 6 ⁻¹	T
041	0000	1110	0100	
	0000	1110	1000	D _R
	0000	1101	0001	I
	0000	1101	1000	ID _R
	0000	1011	0001	D
	0000	1011	0010	D _L
	0000	0111	0100	ID
	0000	0111	0010	ID _L
043	0000	1110	1100	
	0000	1101	1001	I
	0000	1011	0011	D
	0000	0111	0110	ID
050	0000	1111	0000	
051	0000	1111	0001	
	0000	1111	0010	D _R
	0000	1111	0100	D
	0000	1111	1000	D _L
052	0000	1111	1010	
	0000	1111	0101	D _R
053	0000	1111	0011	
	0000	1111	0110	I
	0000	1111	1001	ID
	0000	1111	1100	D
054	0000	1111	1110	
	0000	1111	1101	D _R
	0000	1111	1011	D
	0000	1111	0111	D _L
055	0000	1111	1111	

i j k	1 1 ⁻¹ 2 2 ⁻¹	3 3 ⁻¹ 4 4 ⁻¹	5 6 7 6 ⁻¹	T
110	0001	0001	0000	
	0010	0010	0000	I
	0100	0100	0000	D
	1000	1000	0000	ID
120	0010	1010	0000	
	0001	0101	0000	I
	1000	1010	0000	D
	0100	0101	0000	ID
131	0001	0011	0010	
	0001	1001	0001	D _R
	0010	0011	0010	I
	0010	0110	0100	ID _R
	0100	1100	1000	D
	0100	0110	0100	D _L
	1000	1100	1000	ID
	1000	1001	0001	ID _L
141	0010	1110	0100	
	1000	1011	0001	D
	0010	1011	0010	D _L
	1000	1110	1000	D _R
	0001	1101	0001	I
	0100	0111	0100	ID
	0001	0111	0010	ID _L
	0100	1101	1000	ID _R
143(1)	0010	1110	1100	
	1000	1011	0011	D
	0010	1011	0011	D _L
	1000	1110	1100	D _R
	0001	1101	1001	I
	0100	0111	0110	ID
	0001	0111	0110	ID _L
	0100	1101	1001	ID _R

i j k	1 1 ⁻¹ 2 2 ⁻¹	3 3 ⁻¹ 4 4 ⁻¹	5 6 7 6 ⁻¹	T
143(2)	0100	1110	1100	
	1000	1101	1001	I
	0001	1011	0011	D
	0010	0111	0110	ID
153	0001	1111	0011	
	0010	1111	0110	I
	0100	1111	1100	D
	1000	1111	1001	ID
154	0001	1111	0111	
	0001	1111	1011	D _R
	0010	1111	0111	I
	0010	1111	1110	ID _R
	0100	1111	1101	D
	0100	1111	1110	D _L
	1000	1111	1101	ID
	1000	1111	1011	ID _L
155	0001	1111	1111	
	0010	1111	1111	I
	0100	1111	1111	D
	1000	1111	1111	ID
220	1010	1010	0000	
	0101	0101	0000	I
243	1010	1110	1100	
	1010	1011	0011	D
	0101	1101	1001	I
	0101	0111	0110	ID
255	1010	1111	1111	
	0101	1111	1111	I

i j k	1 1 ⁻¹ 2 2 ⁻¹	3 3 ⁻¹ 4 4 ⁻¹	5 6 7 6 ⁻¹	T
331	0011 1100 0110 1001	0011 1100 0110 1001	0010 1000 0100 0001	 D D _L D _R
343	0110 1001 0011 1100 1001 0110 0011 1100	1110 1011 1011 1110 1101 0111 0111 1101	1100 0011 0011 1100 1001 0110 0110 1001	 D D _L D _R I ID ID _L ID _R
354	0011 1100 0110 1001	1111 1111 1111 1111	0111 1101 1110 1011	 D D _L ID _L
355	0011 0110 1001 1100	1111 1111 1111 1111	1111 1111 1111 1111	 D _L D _R D
443	1110 1101 1011 0111	1110 1101 1011 0111	1100 1001 0011 0110	 I D ID
455	1110 1101 1011 0111	1111 1111 1111 1111	1111 1111 1111 1111	 I D ID
555	1111	1111	1111	

U. S. ATOMIC ENERGY COMMISSION
UNIVERSITY-TYPE CONTRACTOR'S RECOMMENDATION FOR
DISPOSITION OF SCIENTIFIC AND TECHNICAL DOCUMENT

(See Instructions on Reverse Side)

1. AEC REPORT NO. COO-1018-1189 Report No. 348	2. TITLE CONSISTENT PROPERTIES OF COMPOSITE FORMATION UNDER A BINARY RELATION
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3. TYPE OF DOCUMENT (Check one):

- ☒ a. Scientific and technical report
- ☐ b. Conference paper not to be published in a journal:
- Title of conference _____
- Date of conference _____
- Exact location of conference _____
- Sponsoring organization _____
- ☐ c. Other (Specify) _____

4. RECOMMENDED ANNOUNCEMENT AND DISTRIBUTION (Check one):

- ☐ a. AEC's normal announcement and distribution procedures may be followed.
- ☐ b. Make available only within AEC and to AEC contractors and other U.S. Government agencies and their contractors.
- ☐ c. Make no announcement or distribution.

5. REASON FOR RECOMMENDED RESTRICTIONS:

6. SUBMITTED BY: NAME AND POSITION (Please print or type)

John C. Schwebel, Research Assistant

Organization

Dept. of Computer Science, Univ. of Ill., Urbana, Ill. 61801

Signature

John C. Schwebel

Date

August 12, 1969

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7. AEC CONTRACT ADMINISTRATOR'S COMMENTS, IF ANY, ON ABOVE ANNOUNCEMENT AND DISTRIBUTION RECOMMENDATION:

8. PATENT CLEARANCE:

- ☐ a. AEC patent clearance has been granted by responsible AEC patent group.
- ☐ b. Report has been sent to responsible AEC patent group for clearance.
- ☐ c. Patent clearance not required.

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Item 1. The first element in the number shall be an AEC-approved code. This may be a code which is unique to the contractor, e.g., MIT, or it may be the code of the AEC Operations Office, i.e., NYO, COO, ORO, IDO, SRO, SAN, ALO, RLO, NVO. The contract administrator will specify the code which is to be used.

The code shall be followed by a sequential number, or by a contract number plus a sequential number, as follows: (a) Contractors with unique codes may complete the report number by adding a sequential number to the code, e.g., MIT-101, MIT-102, etc.; or they

may add the identifying portion of the contract number and a sequential number, e.g., ABC-2105-1, ABC-2105-2, etc.; (b) Contractors using the operations office code shall complete the report number by adding the identifying portion of the contract number and a sequential number, e.g., NYO-2200-1, NYO-2200-2, etc.

Item 2. Give title exactly as on the document itself.

Item 3. If box c is checked, indicate type of item being sent, e.g., thesis, translation, computer program, etc.

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Item 5. If 4b or 4c is checked, give reason for recommending announcement or distribution restrictions, e.g., "preliminary information", "prepared primarily for internal use", etc.

Item 6. Enter name of person to whom inquiries concerning the recommendations on this form may be addressed.

Item 7. AEC contract administrators may use this space to show concurrence or nonconcurrence with the recommendation in item 4 and to make other recommendations.

Item 8. AEC contract administrator or patent group representative should check a, b, or c, and forward this form and the document to:

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