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AMENDMENT

Technical Report for DOE Grant DE-FG02-92ER25123
"Singularities and Symmetries of Nonlinear Ordinary and
Partial Differential Equations"

Extension period 5/15/93 - 5/14/94

The above grant supported research for the one year period 5/15/92-5/14/93. A one year extension covering the period 5/15/93 - 5/14/94 was granted to use up the small portion of unused funds. During this period the research activities described in the original report (attached) were completed or extended. Two of the manuscripts resulting from the original grant were published and completed and accepted for publication. These were, respectively:

I. Klapper and M. Tabor, "Current Sheet Formation in Two Dimensional MHD Turbulence from a Lagrangian Point of View" Geophys. Astrophys. Fluid Dynamics, 73, 109-122 (1993).

A. Goriely and M. Tabor, "The Singularity Analysis for nearly Integrable Systems: Homoclinic Intersections and Local Multivaluedness" accepted for publication in Physica D (1994).

The completion of this second paper was one of the main achievements of the extension period.

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DOE/ER/25123-F

Technical Report for DOE Grant DE-FG02-92ER25123 "Singularities and Symmetries of Nonlinear Ordinary and Partial Differential Equations"

One year period 5/15/92 - 5/14/93

At the beginning of 1992 the PI moved from Columbia University to the University of Arizona to become Head of the Applied Mathematics Program there. The remnants of the second year of funds from the three year grant DE-FG02-84ER13190 "Singularities and Symmetries of Nonlinear Ordinary and Partial Differential Equations" were transferred to the University of Arizona in the form of a subcontract from Columbia University. The third year funds were re-issued in the form of a special one-year grant, DE-FG02-92ER25123 "Singularities and Symmetries of Nonlinear Ordinary and Partial Differential Equations" directly to the University of Arizona. (The technical report given below is essentially the same as that given in the final technical report for DE-FG02-84ER13190 which was submitted via Columbia University).

In this project we have studied singularities as both fundamental mathematical objects in their own right determining, for example, integrable and nonintegrable properties of nonlinear differential equations; and the determining mechanism of crucial physical processes such as self-focusing singularities and current sheet formation. This approach defines broad based, interdisciplinary research program relevant to the DOE/AMS mission. During the past year progress has been made on the following fronts:

1. Painlevé Property and the Geometry of Solvable Groups

For systems of ordinary differential equations the "Painlevé test" is now recognized as one of the few direct (but by no means fool proof) tests of "integrability" [1]. In its simplest form, it consists of demonstrating that all solutions of the equation in question have the "Laurent property", namely that they can be expanded about the movable complex time singularities (t_0) in Laurent series of the form

$$x(t) = \sum_j a_j (t-t_0)^{j-\alpha}$$

where α is the (integer) "leading order" and a_j the set of expansion coefficients. For the general solution there must be as many arbitrary coefficients, termed "resonances" as the order of the system and for any other, singular, solution "branches" (or "balances") less. All solution branches must have the Laurent property in order for the system to be deemed as having the Painlevé property. Despite the empirical success of the Painlevé methods their workings have proved to be somewhat mysterious and in need of deeper geometric insight. Here we have explored the connections between the Painlevé analysis and solvable groups. The possible connection between these, ostensibly different, techniques was first investigated in the thesis of F. Cariello (supported under this project) in which it was shown, for a limited class of examples, that the symmetry generators could be determined from the first few terms in the Laurent expansions. The position and role of the resonances in the Laurent expansions is still, however, proving to be baffling and in need of further investigation. Collaboration at the University of Arizona with N. Ercolani and graduate student Y. Liu has now made it possible to continue this study. The older results appeared to be restricted to a very small class of problems with a rather modest group structure. More recently we have been able to make a more comprehensive classification of this class

- which now seems larger than originally anticipated. Our preliminary results suggest that equations of the form

$$y'' = y^3 \phi(v) + 2(y')^2/y - y/\sqrt{2}$$

where $v = y'/y^2 - 1/\sqrt{2}y$, and the function ϕ is subject to only quite modest constraints; can all be proved to have both the Painlevé property and a solvable group. All of these equations can be integrated in terms of elliptic functions. These results open the way for a deeper understanding of the geometry of these integrable systems and their symmetries and new insights into Painlevé's original classification of second order equations with movable poles.

2. Singularity Clustering and Psi-Lambda Series

Earlier work investigated and developed techniques to study the complicated analytic structure of systems with logarithmic (or other multi-sheeted) singularities. Thus in the case of the Duffing equation

($y'' + \lambda y' - \mu y + 2y^3 = 0$), excluding the special λ/μ^2 ratios for which the system has the Painlevé property, the system is found to have logarithmic branch points and a local psi-series expansion of the form

$$y(x) = \sum_j \sum_k a_{jk} x^{j-1} (x^4 \log x)^k$$

where $x = (t - t_0)$. Such series were shown to result in recursive singularity clustering in the form of patterns of four-armed stars of singularities. Series of this form were studied using a novel resummation technique leading to expansions of the form

$$y(x) = \sum_j \Theta_j(z) x^{j-1}$$

where the $\Theta_j(z)$ are a set of functions in the variable $z = x^4 \log x$.

The Θ_j ($j > 1$) were found to satisfy a hierarchy of inhomogeneous Lamé equations and there was some hope that the resummed series might have a non-zero radius of convergence. More careful examination subsequently showed that the Θ_j exhibited their own pathologies and the convergence of the expansion was called into doubt.

These difficulties have suggested that we should, instead, ask more geometric questions about the *onset* of logarithmic branch points as represented by the original psi-series. The original psi-series analysis was not perturbative in nature and a natural question to ask is, in fact, how does the underlying geometry change as the system is perturbed away from the Painlevé cases? The Riemann surfaces associated with these cases correspond to nice varieties (typically a finite genus torus). Once the meromorphic structure is lost these surfaces must "break up" in some way. In order to investigate this we have recently devised a perturbative approach to the psi-series which we have dubbed psi-lambda series. For example, the Duffing equation has the Painlevé property for $\lambda = 0$ and, as already stated, its solutions can be locally expressed as a simple Laurent series. For $\lambda \neq 0$ the local representation should be in the form of the above psi-series. Our idea has been to work in the neighborhood of $\lambda = 0$ (or the other Painlevé case) and write the psi-series in a "perturbative" form

$$y(x) = \sum_j \sum_k a_j^{(k)}(z) \lambda^k x^{j-1}$$

where the $a_j^{(k)}(z)$ are certain functions (different from the $\Theta_j(z)$ in our previous resummation efforts) of the variable $z = x^4 \log x$. Before, the expansions were substituted into the differential equations and ordered in powers of x ; now the ordering is initially made in orders of λ . At each order of λ we find much more tractable equations and it now seems possible to make some precise statements about the convergence of this type of expansion.

This type of perturbative singularity expansion appears to be a rather powerful new tool with application to a wide range of problems. Through its use we now plan to tackle some long standing questions such as the relation between singularity clustering and partial integrability and connections between our approach to (clustered) branch points and other techniques such as Melnikov's method, Ziglin's method and the poly-Painlevé method. In particular, striking new results have been obtained connecting these expansions with Melnikov's method. We have demonstrated, rigorously, precise connections between the logarithmic coefficients in the new expansion and the Melnikov function. These results now make it possible, subject to certain conditions, to evaluate the Melnikov function directly from the local expansion. This evaluation is almost trivially generalized to the evaluation of the Melnikov function for multi-dimensional systems. In addition, these results demonstrate, for the first time, rigorous connections between multivaluedness in complex time (the logarithmic branch points) and multivaluedness in phase space (homoclinic tangles resulting from transverse intersections).

3. Current Sheet Formation in Magneto-Hydrodynamics

The equations of magneto-hydrodynamics (mhd) are notoriously difficult to solve. Numerical simulations stretch current computing resources to their limit and are difficult to interpret. The nonlinear coupling between the magnetic and velocity fields (the Lorentz force) also makes analytical treatment very difficult unless major approximations are made. A well known example of such an approximation is the so called kinematic dynamo approximation in which the Lorentz term is dropped. In this way one can study the effect of the fluid velocity on the magnetic field - without, however, feedback or saturation effects - and follow field amplification mechanisms due to the stretching and folding of magnetic field lines. Here, we have been studying, both analytically and numerical, reasonable models which keep the essential coupling between field and fluid. Our main emphasis is on the stretching and alignment of both the magnetic field and the vorticity. We have developed a Lagrangian description for the evolution of gradient quantities - this is a natural framework for studying alignment questions and draws on some of the ideas used in our work on stretching and alignment in fluid mechanical turbulence. Some useful preliminary results for two dimensional incompressible mhd turbulence have been obtained. Although dynamo action (field amplification) is not of interest in two dimensions, other significant effects, such as current singularities, can still occur.

Our approach is to start from the ideal incompressible mhd equations

$$\frac{du}{dt} = -\nabla p^* + B \cdot \nabla B$$

$$\frac{dB}{dt} = -B \cdot \nabla u$$

where p^* is the modified pressure, $p^* = p + B^2/2$, $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the usual advective derivative and we assume the incompressibilities $\nabla \cdot \mathbf{u} = 0$ and $\nabla \cdot \mathbf{B} = 0$. In order to study stretching and alignment we have developed the equations of motion for the various symmetric and anti symmetric gradients, namely

$$S = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T)$$

$$W = \frac{1}{2} (\nabla \mathbf{u} - \nabla \mathbf{u}^T)$$

$$C = \frac{1}{2} (\nabla \mathbf{B} + \nabla \mathbf{B}^T)$$

$$J = \frac{1}{2} (\nabla \mathbf{B} - \nabla \mathbf{B}^T)$$

So far we have studied our gradient evolution equations in two dimensions with radially isotropic models of the pressure field. Around magnetic null points ($B = 0$), current sheet formation can be predicted and analyzed rather easily. For the $B \neq 0$ case, non-local dynamics are introduced, but use of new variables (the Elsasser variables) formally remove the non-local terms and the Lagrangian particles can be thought of as being advected by the two velocity fields $\mathbf{u} \pm \mathbf{B}$. By working with closed field-lines, a Lagrangian particle can be thought of as moving around the field line in the + or - direction with Alfvén velocity $|\mathbf{B}|$, while the field line itself moves through the fluid with velocity $\mathbf{u}$. Once this is recognized a straightforward dynamics can be developed for closed loops using a Fourier decomposition. In the ensuing analysis a key role is played by the velocity "balance of strain", $\sigma = S^2 - W^2$, and the "magnetic field balance of strain", $\mu = C^2 - J^2$. Our preliminary results indicate the possibility of a finite time singularity occurring in a time scale proportional to σ^{-1} .

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Papers resulting from project during the 1992/93 period

"Non generic connections corresponding to front solutions", J. Phys. A. 25, 3773 - 3796 (1992), (with J. A. Powell).

"Singularities and symmetries of nonlinear evolution equations", in "Painlevé transcendents, their asymptotics and physical applications", NATO ASI series B vol. 278 (editors: D. Levi and P. Winternitz; Plenum Press, 1992), (with F. Cariello).

"Investigation of femtosecond solitary waves in nonlinear optical fibres", submitted to J. Nonlinear Optics (1993), (with M.J. Potasek and J.M. Fang).

"Psi-series and their Summability for Nonintegrable Dynamical Systems" in "Large Scale Structure in Nonlinear Physics", eds: J. D. Fournier and P. L. Sulem. Springer-Verlag, New York (1992), (with J.D. Fournier).

"Current Sheet Formation in Two Dimensional Incompressible MHD Turbulence from a Lagrangian Point of View", Geophys. and Astrophys. Fluid Dynamics (in press), (with I. Klapper).

Local Meromorphicity and Homoclinic structure, in preparation (with A. Goriely).