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On the sputtering of metals and insulators: A nonlinear evolution problem with nonlinear boundary condition*

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In this paper we consider the following nonlinear evolution problem:

$$(P) \quad \begin{cases} u_t = \Delta \varphi(u) & \text{in } \Omega \times \mathbb{R}^+ \\ \frac{\partial \varphi(u)}{\partial n} = -\beta(\varphi(u)) & \text{on } \partial\Omega \times \mathbb{R}^+ \\ u(x, 0) = u_0(x) & \text{in } \Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) with smooth boundary $\Gamma \equiv \partial\Omega$, n denotes the outward normal at $x \in \partial\Omega$. The function φ is smooth such that $\varphi(0) = 0$, $\varphi'(s) > 0$ for $s > 0$ and $\varphi'(0) = 0$. $\beta(s)$ is a nonnegative Lipschitz continuous function on $\mathbb{R}^+$ with $\beta(0) = 0$ and $\beta(s) > 0$ for $s > 0$. $u_0 \in L^\infty(\Omega)$ is a nonnegative function (for precise assumptions, we refer to Section 1).

Problem (P) arises in the study of thermal evaporation of atoms and molecules from locally heated surface regions (spikes) invoked as one of several mechanisms of ion-bombardment-induced particle emission (sputtering). Let Ω denote the locally heated region and let u be the temperature. Then in the case of particle-induced evaporation, the Stefan-Boltzman law of heat loss by radiation is replaced by some activation law describing the loss of heat by evaporation. In Sigmund and Claussen [12], Sigmund and Symonske [13] the following model, when the spike is approxi-

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mated as an ideal gas, is proposed,

$$\begin{cases} u_t = \Delta(u^{3/2}) & \text{in } \Omega \times \mathbb{R}^+ \\ \frac{\partial u}{\partial n} = -u^{-2}e^{-1/u} & \partial\Omega \times \mathbb{R}^+ \\ u(x, 0) = u_0(x) & \Omega . \end{cases}$$

Notice in this example that the function $\beta(\cdot)$ in (P) is Lipschitz but non-monotone.

The equation in (P) is the so-called degenerate diffusion problem, which has been extensively studied in recent years. See Aronson [1] and Peletier [11] for references. However, when dealing with the nonlinear flux boundary condition, $\beta(\cdot)$ is usually assumed to be monotone, see e.g. Brezis and Strauss [5] and Benilan, Crandall and Sacks [3]. The purpose of this paper is to provide a general theory for problem (P) under a different assumption on $\beta(\cdot)$, i.e., Lipschitz continuity instead of monotonicity. The main idea of the proof used here is to choose an appropriate test function from the corresponding linearized dual space of the solution. The similar idea has been used by many authors, e.g., Aronson, Crandall and Peletier [2], Bertsch and Hilhorst [4] and Friedman [8]. We shall follow the proof of Bertsch and Hilhorst [4].

The paper is organized as follows. We begin by stating the precise assumptions on the functions involved in (P) and by defining a weak solution. Then, in Section 2 we prove the existence of the solution by the method of parabolic regularization. The uniqueness is proved in Section 3. Finally, we study the large time behaviour of the solution in Section 4.

§1. Preliminaries.

We begin with the following hypothesis on φ , β , u_0 .

$$\text{H1: } \varphi \in C^3(\mathbb{R}^+) \cap C^1(\overline{\mathbb{R}^+}), \varphi(0) = \varphi'(0) = 0, \varphi'(s) > 0 \text{ for } s > 0$$

$$\text{H2: } \beta \in C^{0,1}(\overline{\mathbb{R}^+}), \beta(0) = 0 \text{ and } \beta(s) > 0 \text{ for } s > 0$$

$$\text{H3: } u_0 \in L^\infty(\Omega), u_0 \geq 0 \text{ a.e. in } \Omega .$$

Definition 1.1. A function $u : [0, \infty) \rightarrow L^1(\Omega)$ is a weak solution of (P) if it satisfies

- i) $u \in C([0, T]; L^1(\Omega)) \cap L^\infty(Q_T)$ for any $T > 0$ and $Q_T \equiv \Omega \times (0, T)$
- ii) $\varphi(u) \in V_2(Q_T)$ for any $T > 0$, where $V_2(Q_T)$ is the Banach space equipped with norm

$$|u|_{V_2(Q_T)}^2 = \text{ess sup}_{0 < t < T} \int_{\Omega} u^2(t) + \iint_{Q_T} (\text{grad } u)^2$$

$$\begin{aligned} \text{iii) } \int_{\Omega} u(T)\psi(T) &= \int_{\Omega} u_0\psi(0) + \int_0^T \int_{\Omega_t} \{u\psi_t - \text{grad } \varphi(u) \cdot \text{grad } \psi\} \\ &\quad - \int_0^T \int_{\Gamma} \beta(\varphi(u))\psi \\ &\text{for any } \psi \in C^1(\overline{Q}_T) \text{ and } T > 0. \end{aligned}$$

It is well known that any function in $V_2(Q_T)$ admits a trace in $L^2(S_T)$ where $S_T = \partial\Omega \times (0, T]$, see, e.g., Ladyzenskaja et. al [10, p.78]. Therefore the integrals in iii) are well defined.

A weak subsolution (supersolution, resp.) of (P) is defined to be a function $u : [0, \infty) \rightarrow L^1(\Omega)$ which satisfies i), ii) and iii) with equality replaced by $\leq$ ($\geq$, resp.).

§2. Existence

Consider the following approximating problem (P_ε)

$$(P_\varepsilon) \quad \begin{cases} u_t = \Delta \varphi_\varepsilon(u) & \text{in } Q_T \\ \frac{\partial \varphi_\varepsilon(u)}{\partial n} = \beta_\varepsilon(\varphi_\varepsilon(u)) & \text{on } \partial\Omega \times (0, T) \\ u(x, 0) = u_{0\varepsilon}(x) & \text{in } \Omega \end{cases}$$

where $\varphi_\varepsilon \in C^\infty(\mathbb{R}^+)$, $\varphi_\varepsilon(0) = 0$, $\varphi'_\varepsilon(s) \geq C(\varepsilon) > 0$ for any $s \geq 0$, φ_ε and φ'_ε converge uniformly to φ and φ' respectively on compact subsets of $\mathbb{R}^+$ as $\varepsilon \rightarrow 0$;

where $\beta_\varepsilon \in C^\infty(\mathbb{R}^+) \cap C^{0,1}(\overline{\mathbb{R}}^+)$, $\beta_\varepsilon(0) = 0$, $\beta_\varepsilon(s) > 0$ for all $s \geq 0$, β_ε converges uniformly to β on $\overline{\mathbb{R}}^+$ and $|\beta'_\varepsilon| \leq C$ for some $C > 0$.

and where $u_{0\varepsilon} \in C^\infty(\bar{\Omega})$, $0 \leq u_{0\varepsilon} \leq \|u_0\|_{L^\infty(\Omega)}$ and $u_{0\varepsilon}$ satisfies the compatibility condition

$$\frac{\partial}{\partial n} \varphi_\varepsilon(u_{0\varepsilon}) = -\beta_\varepsilon(\varphi_\varepsilon(u_{0\varepsilon})) \quad \text{on } \partial\Omega.$$

and

$$\|u_{0\varepsilon} - u_0\|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } \varepsilon \downarrow 0.$$

The construction of φ_ε and β_ε is standard and it is omitted here. For the construction of $u_{0\varepsilon}$, we refer to [4].

Then, for each $\varepsilon > 0$, the classical theory of parabolic equation asserts that there exists a unique solution $u_\varepsilon \in C^{2,1}(\bar{Q}_T)$ of problem (P_ε) . Next we derive some a-priori estimates on u_ε . We begin with the following comparison principle.

Lemma 2.1. *Let u and $v \in C^{2,1}(\bar{Q}_T)$ be the two solutions of problem (P_ε) corresponding to the initial functions u_0 and v_0 respectively. Then $u(t) \leq v(t)$ if $u_0 \leq v_0$.*

Proof. Let $z = u - v$. Then z satisfies

$$\begin{cases} z_t = \Delta(a_\varepsilon z) & \text{in } Q_T \\ \frac{\partial(a_\varepsilon z)}{\partial n} + b_\varepsilon z = 0 & \text{on } \partial\Omega \times (0, T) \\ z(x, 0) = z_0(x) \equiv u_0(x) - v_0(x) & \text{in } \Omega \end{cases}$$

where

$$\begin{aligned} a_\varepsilon &= \int_0^1 \varphi'_\varepsilon(\theta u_\varepsilon + (1 - \theta)v_\varepsilon) d\theta \\ b_\varepsilon &= \int_0^1 \beta'_\varepsilon(\theta \varphi_\varepsilon(u_\varepsilon) + (1 - \theta)\varphi_\varepsilon(v_\varepsilon)) d\theta. \end{aligned}$$

The result of the lemma follows from the comparison principle for the linear problem. ■

Lemma 2.2. *Let $u_\varepsilon \in C^{2,1}(\bar{Q}_T)$ be a solution of P_ε . Then, $0 \leq u_\varepsilon \leq \|u_0\|_{L^\infty}$ in $\bar{Q}_T$.*

Proof. Since 0 is a solution of (P_ε) , therefore $u_\varepsilon \geq 0$ by Lemma 2.1.

By the strong maximum principle, u_ε cannot have a positive maximum on $\partial\Omega \times (0, T)$, because otherwise $\frac{\partial \varphi_\varepsilon(u_\varepsilon)}{\partial n} > 0$ at this point. But $\frac{\partial \varphi_\varepsilon(u_\varepsilon)}{\partial n} = -\beta_\varepsilon(\varphi_\varepsilon(u_\varepsilon)) \leq 0$, which is a contradiction. Therefore the maximum of u_ε must be achieved on $t = 0$, i.e.

$$u_\varepsilon \leq \|u_{0\varepsilon}\|_{L^\infty} \leq \|u_0\|_{L^\infty} . \quad \blacksquare$$

Lemma 2.3. The problem (P_ε) has a unique solution: $u_\varepsilon \in C^{2+\alpha}(\overline{Q}_T)$ for each $\alpha \in (0, 1)$.

Proof. See Theorem 7.4 [14, LSU]. ■

Lemma 2.4. Let $0 \leq \tau < t \leq T$. Then there exists a constant $C(T) > 0$, independent of ε , such that

$$\int_\tau^t \int_\Omega (\text{grad } \varphi_\varepsilon(u_\varepsilon))^2 \leq C(T) .$$

Proof. Multiplying equation by $\varphi_\varepsilon(u_\varepsilon)$ and integrating by parts. Set

$$\Phi_\varepsilon(s) = \int_0^s \varphi_\varepsilon(r) dr$$

Then,

$$\begin{aligned} \int_\Omega \Phi_\varepsilon(u_\varepsilon)(t) - \int_\Omega \Phi_\varepsilon(u_\varepsilon)(\tau) &= \int_\tau^t \int_\Omega \varphi_\varepsilon(u_\varepsilon) \Delta \varphi_\varepsilon(u_\varepsilon) \\ &= \int_\tau^t \int_{\partial\Omega} \varphi_\varepsilon(u_\varepsilon) \beta_\varepsilon(\varphi_\varepsilon(u_\varepsilon)) - \int_\tau^t \int_\Omega (\text{grad } \varphi_\varepsilon(u_\varepsilon))^2 \end{aligned}$$

Therefore, by the boundedness of u_ε ,

$$\int_\tau^t \int_\Omega (\text{grad } \varphi_\varepsilon(u_\varepsilon))^2 \leq C(T) . \quad \blacksquare$$

We shall use the following compactness result, which is due to DiBenedetto [7].

Lemma 2.5. For every $\tau > 0$ there exists $\omega_\tau(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\omega_\tau(0) = 0$, continuous and non-decreasing such that

$$|u_\varepsilon(x_1, t_1) - u_\varepsilon(x_2, t_2)| \leq \omega_\tau(|x_1 - x_2| + |t_1 - t_2|^{1/2}) \quad \forall (x_i, t_i) \in \overline{\Omega} \times [\tau, T], \quad i = 1, 2 .$$

In addition, if $u_0(x)$ is continuous in $\overline{\Omega}$, then $(x, t) \rightarrow u_\varepsilon(x, t)$ is continuous with modulus of continuity $\omega_0(\cdot)$.

Now we use the compactness result to prove existence theorem.

Theorem 2.6. *There exists a weak solution u of problem (P).*

Proof. From above estimates and compactness result, we conclude that there exists a $u \in L^\infty(Q_T) \cap C(\bar{\Omega} \times (0, T))$ and a subsequence of $\{u_\varepsilon\}$, which is denoted again by u_ε , such that

- i) $u_\varepsilon \rightarrow u$ uniformly on all sets of the form $\bar{\Omega} \times [\tau, T]$ with $\tau > 0$ (by Lemma 2.5)
- ii) $u_\varepsilon \rightarrow u$ strongly in $L^2(Q_T)$ and in $L^2(S_T)$ (This follows from i) and uniform boundedness of u_ε in $L^\infty(\Omega)$.)
- iii) $\varphi_\varepsilon(u_\varepsilon) \rightarrow \varphi(u)$ weakly in $L^2(0, T; H^1(\Omega))$ (This is the consequence of Lemma 2.4. One identifies limit to be $\varphi(u)$ by the fact that $\varphi_\varepsilon(u_\varepsilon) \rightarrow \varphi(u)$ strongly in $L^2(Q_T)$ and the dominated convergence theorem.)
- iv) $\beta_\varepsilon(\varphi_\varepsilon(u_\varepsilon)) \rightarrow \beta(\varphi(u))$ strongly in $L^2(S_T)$ where $S_T = \partial\Omega \times (0, T]$.

One easily checks that $u \in C((0, T); L^1(\Omega))$ and u satisfies ii) and the integral identity iii) in the definition of weak solution since u_ε satisfies a similar identity. It remains to show that $\|u(t)\|_{L^1(\Omega)}$ is continuous at zero. For this we consider a smooth approximation v_0 of u_0 , i.e., for any $\delta > 0$, choose v_0 smooth to be such that $\|v_0 - u_0\| \leq \delta/3$. Let v be a solution of problem (P) with initial function v_0 obtained as a limit of solution of problem (P_ε) . Then $\|v(t)\|_{L^1(\Omega)}$ is continuous at zero. (This is the consequence of the second part of Lemma 2.5.) Therefore, there exists a $t_0 > 0$ such that for all $t \leq t_0$,

$$\begin{aligned} \|u(t) - u_0\|_{L^1(\Omega)} &\leq \|u(t) - v(t)\|_{L^1(\Omega)} + \|v(t) - v_0\|_{L^1(\Omega)} + \|v_0 - u_0\|_{L^1(\Omega)} \\ &\leq \|u(t) - v(t)\|_{L^1(\Omega)} + \frac{2\delta}{3}. \end{aligned}$$

For the estimation of $\|u(t) - v(t)\|_{L^1(\Omega)}$, we have the following L^1 -contraction results:

Lemma 2.7. $\forall t > 0$,

$$\|u(t) - v(t)\|_{L^1(\Omega)} \leq \|u_0 - v_0\|_{L^1(\Omega)}. \quad (2.1)$$

Applying Lemma 2.7, we conclude that $\forall \delta > 0, \exists t_0 > 0$ such that for all $t \leq t_0$

$$\|u(t) - u_0\|_{L^1(\Omega)} \leq \|u_0 - v_0\|_{L^1(\Omega)} + \frac{2\delta}{3} \leq \delta.$$

Therefore $u(t) \in C([0, T]; L^1(\Omega))$. This concludes the proof of Theorem 2.6.

Proof of Lemma 2.7. Let $u_\varepsilon, v_\varepsilon$ be the solutions of approximating problem (P_ε) corresponding to the smooth initial data $u_{0\varepsilon}$ and $v_{0\varepsilon}$ such that

$$\|u_{0\varepsilon} - u_0\|_{L^1(\Omega)} \rightarrow 0 \text{ and } \|v_{0\varepsilon} - v_0\|_{L^1(\Omega)} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

since both u_ε and v_ε satisfy the integral identity in iii) of the definition of weak solution. Integrating by parts and subtracting on both sides, we obtain, for $T > 0$,

$$\begin{aligned} \int_{\Omega} (u_\varepsilon - v_\varepsilon) \psi(T) - \int_{\Omega} (u_{0\varepsilon} - v_{0\varepsilon}) \psi(0) &= \int_0^T \int_{\Omega} \left\{ (u_\varepsilon - v_\varepsilon) \psi_t + (\varphi_\varepsilon(u_\varepsilon) - \varphi_\varepsilon(v_\varepsilon)) \Delta \psi \right\} \\ &\quad - \int_0^T \int_{\Gamma} (\varphi_\varepsilon(u_\varepsilon) - \varphi_\varepsilon(v_\varepsilon)) \frac{\partial \psi}{\partial n} \\ &\quad - \int_0^T \int_{\Gamma} (\beta_\varepsilon(\varphi_\varepsilon(u_\varepsilon)) - \beta_\varepsilon(\varphi_\varepsilon(v_\varepsilon))) \psi \\ &= \int_0^T \int_{\Omega} (u_\varepsilon - v_\varepsilon) \{ \psi_t + a(u_\varepsilon, v_\varepsilon) \Delta \psi \} \\ &\quad - \int_0^T \int_{\Gamma} (\varphi_\varepsilon(u_\varepsilon) - \varphi_\varepsilon(v_\varepsilon)) \left(\frac{\partial \psi}{\partial n} + b(u_\varepsilon, v_\varepsilon) \psi \right) \end{aligned}$$

where

$$\begin{aligned} a(u_\varepsilon, v_\varepsilon) &= \begin{cases} \frac{\varphi_\varepsilon(u_\varepsilon) - \varphi_\varepsilon(v_\varepsilon)}{u_\varepsilon - v_\varepsilon} & \text{if } u_\varepsilon \neq v_\varepsilon \\ \varphi'_\varepsilon(u_\varepsilon) & \text{if } u_\varepsilon = v_\varepsilon \end{cases} \\ b(u_\varepsilon, v_\varepsilon) &= \begin{cases} \frac{\beta_\varepsilon(\varphi_\varepsilon(u_\varepsilon)) - \beta_\varepsilon(\varphi_\varepsilon(v_\varepsilon))}{\varphi_\varepsilon(u_\varepsilon) - \varphi_\varepsilon(v_\varepsilon)} & \text{if } u_\varepsilon \neq v_\varepsilon \\ \beta'_\varepsilon(\varphi_\varepsilon(u_\varepsilon)) & \text{if } u_\varepsilon = v_\varepsilon. \end{cases} \end{aligned}$$

Both $a(u_\varepsilon, v_\varepsilon)$ and $b(u_\varepsilon, v_\varepsilon)$ are smooth, $a(u_\varepsilon, v_\varepsilon) \geq c(\varepsilon)$ and $|b(u_\varepsilon, v_\varepsilon)| \leq C(\varepsilon)$ for some positive constants $c(\varepsilon), C(\varepsilon)$.

Let ψ be the smooth solution of the following problem

$$\begin{cases} \psi_t + a(u_\varepsilon, v_\varepsilon)\Delta\psi = 0 & \text{in } \Omega \times (0, T) \\ \frac{\partial\psi}{\partial n} + b(u_\varepsilon, v_\varepsilon)\psi = 0 & \text{on } \partial\Omega \times (0, T) \\ \psi(x, T) = \chi(x) & \text{in } \Omega. \end{cases}$$

where $\chi \in C_0^\infty(\Omega)$ is such that $0 \leq \chi \leq 1$.

Then $\psi(x, t) \in C^{2,1}(\overline{Q}_T)$, $\psi(x, t) \geq 0$ and

$$\int_{\Omega} (u_\varepsilon - v_\varepsilon)(T)\chi \leq \int_{\Omega} (u_{0\varepsilon} - v_{0\varepsilon})\psi(0) \leq \int_{\Omega} (u_{0\varepsilon} - v_{0\varepsilon})^+ \quad (2.2)$$

where $(\cdot)^+ = \max(\cdot, 0)$.

Let $\bar{\chi}$ be

$$\bar{\chi}(x) = \begin{cases} 1 & \text{if } u_\varepsilon(x, T) > v_\varepsilon(x, T) \\ 0 & \text{otherwise} \end{cases}$$

and choose $\chi = \chi_n \rightarrow \bar{\chi}$ in $L^2(\Omega)$ as $n \rightarrow \infty$. Then, letting $n \rightarrow \infty$ in (2.2), we obtain

$$\int_{\Omega} (u_\varepsilon - v_\varepsilon)^+(T) \leq \int_{\Omega} (u_{0\varepsilon} - v_{0\varepsilon})^+$$

Identically we can prove that

$$\int_{\Omega} (v_\varepsilon - u_\varepsilon)^+(T) \leq \int_{\Omega} (v_{0\varepsilon} - u_{0\varepsilon})^+.$$

Therefore,

$$\|u_\varepsilon(T) - v_\varepsilon(T)\|_{L^1(\Omega)} \leq \|u_{0\varepsilon} - v_{0\varepsilon}\|_{L^1(\Omega)}$$

(2.1) is obtained by letting $\varepsilon \rightarrow 0$.

§3. Uniqueness

To show uniqueness we use an idea due to Kalashnikov [9], see also Bertsch and Hilhorst [4], to compare an arbitrary generalized solution of problem (P) with a solution obtained as the limit of a sequence of classical solutions of the approximating problem (P_ε) . Following Bertsch and Hilhorst [4], we approximate solutions in two steps. First by

$$(P_n) \quad \begin{cases} u_t = \Delta\varphi(u) & \text{in } Q_T \\ \frac{\partial}{\partial n}\varphi(u) + \beta(\varphi(u)) = \frac{A}{n}e^{-\lambda t} & \text{on } \partial\Omega \times (0, T) \\ u(x, 0) = u_0(x) + \frac{1}{n} & \text{in } \Omega \end{cases}$$

which in turn is approximated by

$$(P_{\varepsilon n}) \quad \begin{cases} u_t = \Delta \tilde{\varphi}(u) & \text{in } Q_T \\ \frac{\partial}{\partial n} \tilde{\varphi}(u) + \beta_\varepsilon(\tilde{\varphi}(u)) = \frac{A}{n} e^{-\lambda t} & \text{on } \partial\Omega \times (0, T) \\ u(x, 0) = u_{0\varepsilon}(x) + \frac{1}{n} & \text{in } \Omega \end{cases}$$

where β_ε is smooth, $\|\beta'_\varepsilon \cdot \tilde{\varphi}'\|_{L^\infty} \leq A$ for some constant $A > 0$ and converges to β uniformly on $\overline{\mathbb{R}^+}$ as $\varepsilon \rightarrow 0$.

Where $u_{0\varepsilon}$ is such that

$$u_{0\varepsilon} \in C^\infty(\Omega), \quad \|u_{0\varepsilon} - u_0\|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

and $u_{0\varepsilon}$ satisfies the compatibility condition

$$\tilde{\varphi}^{-1}\left(u_{0\varepsilon} + \frac{1}{n}\right) \frac{\partial u_{0\varepsilon}}{\partial n} + \beta_\varepsilon\left(\tilde{\varphi}\left(u_{0\varepsilon} + \frac{1}{n}\right)\right) = \frac{A}{n} e^{-\lambda t} \quad \text{on } \partial\Omega$$

where λ is determined from Lemma 3.1 below and $\tilde{\varphi}(s)$ is smooth and is defined as

$$\tilde{\varphi}(s) = \begin{cases} \varphi(s) & \text{if } s \leq \tilde{C} \\ \text{smooth} & \text{if } \tilde{C} < s < \tilde{C} + 1 \\ \varphi(\tilde{C} + 1) & \text{if } s \geq \tilde{C} + 1 \end{cases}$$

with $\tilde{C} = \varphi^{-1}(2\varphi(2\|u_0\|_{L^\infty}))$ from lemma 3.1 below.

Lemma 3.1. *The solution $u_{n\varepsilon} \in C^{2,1}(\overline{Q})$ of problem $(P_{\varepsilon n})$ satisfies, for n large enough, that*

$$\frac{1}{n} e^{-\lambda t} \leq u_{n\varepsilon}(x, t) \leq \tilde{C} \equiv \varphi^{-1}(2\varphi(2\|u_0\|_{L^\infty})) \quad \text{for any } (x, t) \in \overline{Q}_T$$

where the constant $\tilde{C}$ does not depend on n and t .

Proof. Let $L(u) = u_t - \Delta \tilde{\varphi}(u)$ and choose $h(x, t) = \frac{1}{n} e^{-\lambda t}$. Then

$$\begin{cases} L(h) = -\frac{\lambda}{n} e^{-\lambda t} \leq 0 = L(u_{n\varepsilon}) & \text{in } \Omega \times (0, T) \\ \frac{\partial \tilde{\varphi}(h)}{\partial n} + \beta_\varepsilon(\tilde{\varphi}(h)) = \beta_\varepsilon(\tilde{\varphi}(h)) \leq \frac{A}{n} e^{-\lambda t} = \frac{\partial \tilde{\varphi}(u)}{\partial n} + \beta_\varepsilon(\tilde{\varphi}(u_{n\varepsilon})) & \text{on } \partial\Omega \times (0, T) \\ h(x, 0) = \frac{1}{n} \leq u_{0\varepsilon} & \text{in } \partial\Omega \end{cases}$$

By comparison principle

$$u_{n\varepsilon}(x, t) \geq \frac{1}{n}e^{-\lambda t} \text{ for all } (x, t) \in Q_T.$$

In order to find an upper bound we introduce the following function $w(x, t) = \frac{1}{3-e^{-\lambda t}\xi(x)}\tilde{\varphi}(u_{n\varepsilon})$ where $\xi(x) \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that

$$1 \leq \xi(x) \leq 2, \quad -\frac{\partial \xi}{\partial n} \Big|_{\Gamma} \geq \delta \text{ for some } \delta > 0.$$

Then $w(x, t)$ satisfies, for $(x, t) \in Q_T$

$$\begin{aligned} w_t = & \tilde{\varphi}'(u)\Delta w - \frac{\tilde{\varphi}''\tilde{\varphi}'}{\tilde{\varphi}}|\text{grad } u|^2 w - \frac{2e^{-\lambda t}\tilde{\varphi}'^2 \text{grad } u_{n\varepsilon} \text{grad } \xi}{(3-e^{-\lambda t}\xi)^2} \\ & - \frac{e^{-\lambda t}\tilde{\varphi}'}{3-\xi(x)e^{-\lambda t}}w\Delta \xi - \frac{2e^{-\lambda t}(\text{grad } \xi)^2\tilde{\varphi}'}{(3-e^{-\lambda t}\xi)^2}w \\ & + \frac{\tilde{\varphi}''\tilde{\varphi}'^2}{\tilde{\varphi}}(\text{grad } u_{n\varepsilon})^2 w + \frac{\lambda e^{-\lambda t}\xi(x)}{3-e^{-\lambda t}\xi(x)}w \end{aligned} \quad (3.1)$$

At the maximum point $(x_0, t_0) \in \Omega \times (0, T)$ of w^2 , we have

$$ww_t \geq 0, \quad w \text{grad } w = 0$$

$$\tilde{\varphi}'w\Delta w \leq 0$$

Substitute into equation (3.1), we obtain

$$-\frac{e^{-\lambda t}w^2}{3-\xi e^{-\lambda t}}(\lambda\xi + \tilde{\varphi}'\Delta\xi) \geq 0$$

i.e.,

$$\lambda + \tilde{\varphi}'\frac{\Delta\xi}{\xi} \leq 0.$$

Since $\xi \geq 1$ and $\Delta\xi$ is bounded, this is impossible if we choose λ large enough. Therefore, w^2 cannot have maximum in $\Omega \times (0, T)$. On the lateral boundary point where maximum of w^2 occurs, we have

$$w\frac{\partial w}{\partial n} \geq 0$$

i.e.,

$$w^2\left(\frac{\varphi'}{\varphi}\frac{\partial u_{n\varepsilon}}{\partial n} + \frac{e^{-\lambda t}\frac{\partial \xi}{\partial n}}{3-e^{-\lambda t}\xi}\right) \geq 0$$

i.e.,

$$-\frac{\partial \xi}{\partial n} w \leq \frac{A}{n} - e^{-\lambda t} \beta(\tilde{\varphi}(u_{n\varepsilon})) \leq \frac{A}{n}$$

By the choice of ξ ,

$$w \leq \frac{A}{n\delta}$$

Therefore, for $(x, t) \in \overline{Q}_T$

$$\begin{aligned} |w(x, t)| &\leq \max \left\{ \|w(x, 0)\|_{L^\infty}, \frac{A}{n\delta} \right\} \\ &\leq \max \left\{ \left\| \frac{\varphi(u_{0\varepsilon})}{3 - \xi(x)} \right\|_{L^\infty}, \frac{A}{n\delta} \right\} \\ &\leq \|\varphi(u_{0\varepsilon})\|_{L^\infty} \quad (\text{if } n \text{ is large enough}). \end{aligned}$$

Consequently

$$\begin{aligned} |\tilde{\varphi}(u_{n\varepsilon}(x, t))| &\leq (3 - e^{-\lambda t} \xi(x)) \|\varphi(u_0)\|_{L^\infty} \\ &\leq 2 \|\varphi(u_{0\varepsilon})\|_{L^\infty} \leq 2\varphi(2\|u_0\|_{L^\infty}) \end{aligned}$$

i.e.,

$$|u_{n\varepsilon}(x, t)| \leq \tilde{C} \quad \blacksquare$$

Now by the method of Section 2, one can obtain further a-priori estimates on $u_{n\varepsilon}$ of problem $(P_{n\varepsilon})$ to show that there exists a subsequence of $\{u_{n\varepsilon}\}$ which converges to the solution u_n of (P_n) and then that a subsequence of u_n , which is denoted by u_n , converges to the solution of problem (P) as $n \rightarrow \infty$. We will use this approximation to prove the following uniqueness theorem:

Theorem 3.2. *Let u be a solution of (P) obtained as the limit of the solution u_n of (P_n) and let $\bar{u}$ be any subsolution of (P) . Then*

$$\int_{\Omega} (\underline{u} - u)^+(t) \leq \int_{\Omega} (\underline{u}_0 - u_0)^+ \quad \forall t > 0 \quad (3.1)$$

Similarly for any supersolution $\bar{u}$

$$\int_{\Omega} (u - \bar{u})^+(t) \leq \int_{\Omega} (u_0 - \bar{u}_0)^+ \quad \forall t > 0 \quad (3.2)$$

Proof. Let $u_n \rightarrow u$ be approximating sequence as above. Then for any $\psi \in C^{2,1}(\overline{Q}_T)$

$$\begin{aligned} \int_{\Omega} u_n(T)\psi(T) &= \int_{\Omega} u_{0n}\psi(0) + \int_0^T \int_{\Omega} \{u_{0n}\psi_t + \varphi(u_n)\Delta\psi\} \\ &\quad - \int_0^t \int_{\Gamma} \left(\varphi(u_n) \frac{\partial\psi}{\partial n} + \beta(\varphi(u_n))\psi \right) + \int_0^t \int_{\Gamma} \frac{A}{n} e^{-\lambda t} \psi \\ \int_{\Omega} \underline{u}(T)\psi(T) &\leq \int_{\Omega} \underline{u}_0\psi(0) + \int_0^T \int_{\Omega} \{\underline{u}_0\psi_t + \varphi(\underline{u})\Delta\psi\} \\ &\quad - \int_0^T \int_{\Gamma} \left(\varphi(\underline{u}) \frac{\partial\psi}{\partial n} + \beta(\varphi(\underline{u}))\psi \right) \end{aligned}$$

Subtracting on both sides yields

$$\begin{aligned} \int(\underline{u} - u_n)\psi(T) - \int(\underline{u}_0 - u_{0n})\psi(0) &\leq \int_0^T \left\{ (\underline{u} - u_n)\psi_t + (\varphi(\underline{u}) - \varphi(u_n))\Delta\psi \right\} \\ &\quad - \int_0^T \int_{\Gamma} \left\{ (\varphi(\underline{u}) - \varphi(u_n)) \frac{\partial\psi}{\partial n} + (\beta(\varphi(\underline{u})) - \beta(\varphi(u_n)))\psi \right\} \\ &\quad - \int_0^T \int_{\Gamma} \frac{A}{n} e^{-\lambda t} \psi(t) \\ &= \int_0^T \int_{\Omega} (\underline{u} - u_n) \{ \psi_t + A_n \Delta\psi \} \\ &\quad - \int_0^T \int_{\Gamma} (\underline{u} - u_n) \left\{ \frac{\partial\psi}{\partial n} + B_n \psi \right\} \\ &\quad - \int_0^T \int_{\Gamma} \frac{A}{n} e^{-\lambda t} \psi(t) \end{aligned}$$

where

$$A_n = \frac{\varphi(\underline{u}) - \varphi(u_n)}{\underline{u} - u_n} \geq \varepsilon(n) > 0$$

and

$$B_n = \frac{\beta(\varphi(\underline{u})) - \beta(\varphi(u_n))}{\varphi(\underline{u}) - \varphi(u_n)},$$

which is bounded by a positive constant L independent of n . Now define two smooth sequences A_{nj} and B_{nk} such that

$$A_{nj} \geq \varepsilon(n) > 0 \quad \text{and} \quad \|A_{nj} - A_n\|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } j \rightarrow \infty$$

and

$$|B_{nk}| \leq L \quad \text{and} \quad \|B_{nk} - B_n\|_{L^2(\Gamma)} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Let ψ_{njk} be the solution of the following problem

$$\begin{cases} \psi_t + A_{nj}\Delta\psi = 0 & \text{in } Q_T \\ \frac{\partial\psi}{\partial n} + B_{nk}\psi = 0 & \text{on } \Gamma \times (0, T) \\ \psi(x, T) = \chi(x) & \text{in } \Omega \end{cases}$$

where $\chi(x) \in C_0^\infty(\Omega)$ is such that $0 \leq \chi(x) \leq 1$. Then ψ_{njk} satisfies the following estimates.

Lemma 3.3.

i) $0 \leq \psi_{njk}(x, t) \leq C(T)$ for $(x, t) \in \Omega \times (0, T)$ and C is independent of n, j, k .

ii) $\iint_{Q_T} (\Delta\psi_{njk})^2 \leq C(n, k, T)$.

Assuming lemma 3.3, we obtain

$$\begin{aligned} \int_{\Omega} (\underline{u} - u_n)(T)\chi - \int_{\Omega} (\underline{u}_0 - u_{0n})^+ &\leq \int_0^T \int_{\Omega} (\underline{u} - u_n)(A_n - A_{n,j})\Delta\psi_{njk} \\ &\quad - \int_0^T \int_{\Gamma} (B_n - B_{n,k})\psi_{njk}(\underline{u} - u_n) \\ &\quad - \int_0^T \int_{\Gamma} \frac{A}{n} e^{-\lambda t} \psi_{njk} \end{aligned}$$

If we let first $j \rightarrow \infty$, using ii) of lemma 3.3, then let $k \rightarrow \infty$, and then let $n \rightarrow \infty$, we arrive at

$$\int_{\Omega} (\underline{u} - u)(T)\chi \leq \int_{\Omega} (\underline{u}_0 - u_{0n})^+$$

As in the proof of lemma 2.7, we conclude that

$$\int_{\Omega} (\underline{u} - u)^+(T) \leq \int_{\Omega} (\underline{u}_0 - u_0)^+.$$

The proof of (3.2) is similar.

Proof of lemma 3.3.

i) $\psi_{njk} \geq 0$ is the consequence of maximum principle. For the upper bound, as in the proof of lemma 3.1 let $w(x, t) = e^{-M(T-t)}\xi(x)\psi(x, t)$. Then $w(x, t)$ satisfies

$$w_t + A_{nj}\Delta w - 2A_{nj}\frac{\text{grad } \xi}{\xi} \cdot \text{grad } w + \left(M - A_{nj}\frac{\Delta \xi(x)}{\xi(x)} + 2A_{nj}\frac{|\text{grad } \xi|^2}{\xi^2} \right) w = 0$$

Choose $\xi(x) = 1 + \alpha h(x)$ where

$$\begin{cases} \Delta h(x) = -1 & \text{in } \Omega, \\ h(x) = 0 & \text{in } \partial\Omega \end{cases}$$

and α is large enough such that $[-(\partial\xi/\partial n)/\xi]|_{\Gamma} \geq L$. (This is possible since $[-(\partial h(x)/\partial n)]|_{\Gamma} \geq \delta > 0$). Next we choose

$$M \geq A_{nj}\frac{\Delta \xi(x)}{\xi(x)} - 2A_{nj}\frac{|\text{grad } \xi|^2}{\xi^2}.$$

Then, similarly as in the proof of lemma 3.1, we conclude that the maximum of w^2 can only be attained at $t = T$, i.e.,

$$|w(x, t)| \leq \sup_{\Omega} |\xi(x)\chi(x)| \leq C$$

i.e.,

$$\psi(x, t) \leq C(T).$$

For the proof of ii), multiplying equation by $\Delta\psi$ and integrating by parts we obtain

$$\begin{aligned} \int_t^T \int_{\Omega} A_{nj}(\Delta\psi)^2 &= \int_t^T \int_{\Omega} \psi_t \Delta\psi \\ &= - \int_t^T \int_{\Gamma} \frac{\partial\psi}{\partial n} \psi_t + \int_0^T \int_{\Omega} (\text{grad } \psi)_t^2 \\ &= \int_{\Gamma} \int_t^T B_{nk} \psi \psi_t + \int_{\Omega} (\text{grad } \psi(T))^2 - \int_{\Omega} (\text{grad } \psi(t))^2 \\ &= \int_{\Gamma} \frac{B_{nk}}{2} \psi^2(T) - \int_{\Gamma} \frac{B_{nk}}{2} \psi^2(t) - \int_t^T \int_{\Gamma} B_{nkt} \frac{\psi^2}{2} \\ &\quad + \int_{\Omega} (\text{grad } \chi)^2 - \int_{\Omega} (\text{grad } \psi(t))^2 \end{aligned}$$

from which we conclude that

$$\int_t^T \int_{\Omega} A_{nj}(\Delta\psi)^2 + \int_{\Omega} (\text{grad } \psi(t))^2 \leq C(n, k, T)$$

ii) follows from $A_{nj} \geq \varepsilon(n)$. ■

§4. Large time behavior

We define the w -limit set

$$w = \{q \in L^1(\Omega) : \text{there exists a sequence } t_n \rightarrow \infty \text{ such that } u(t_n) \rightarrow q \text{ in } L^1(\Omega) \text{ as } t_n \rightarrow \infty\}$$

Then by lemma 2.5: the set $\{u(t; u_0) : t \geq 1\}$ is precompact in $C(\overline{\Omega})$ and therefore w is nonempty and $w \in C(\overline{\Omega})$.

Claim. $w = \{0\}$.

Therefore we have the following result by Proposition 2.1 of Dafermos [6].

Theorem 4.1. *Under the hypothesis H1-H3 the solution of problem (P) satisfies $u(t; u_0) \rightarrow 0$ in $C(\overline{\Omega})$.*

Proof of the claim. Let $E = \{q \in C(\overline{\Omega}) : q \geq 0 \text{ and } u(t; q) = q \text{ for } t \geq 0\}$. Then for any $q \in E$, we have

$$\begin{aligned} \int_{\Omega} u(t, q) \psi(t) &= \int_{\Omega} q \psi(0) + \int_0^t \int_{\Omega} (u \psi_t + \varphi(u) \Delta \psi) \\ &\quad - \int_0^t \int_{\Gamma} \left(\varphi(u) \frac{\partial \psi}{\partial n} + \beta(\varphi(u)) \psi \right) \end{aligned} \tag{4.1}$$

for any $\psi \in C^{2,1}(\overline{Q})$.

Now set $\psi = 1$, (4.1) becomes

$$\int_0^t \int_{\Gamma} \beta(\varphi(q)) \psi = 0 \tag{4.2}$$

i.e.,

$$q|_{\Gamma \times (0, t)} = 0.$$

Substituting this into (4.1), we obtain

$$\int_{\Omega} q(\psi(t) - \psi(0)) = \int_0^t \int_{\Omega} q \psi_t - \int_0^t \int_{\Omega} \varphi(q) \Delta \psi$$

i.e.,

$$\int_0^t \int_{\Omega} \varphi(q) \Delta \psi = 0 \text{ for any } \psi \in C^{2,1}(\overline{Q}).$$

Therefore $q = 0$ and $E = \{0\}$.

Next we define the functional $V : L^1(\Omega) \rightarrow [0, \infty)$

$$V(u) = \int_{\Omega} u, \quad u \in L^1(\Omega)$$

Since zero is a solution of problem (P) it follows from Theorem 3.2 that

$$V(u(t_2)) \leq V(u(t_1)) \text{ for an } t_2 \geq t_1 \geq 0.$$

Therefore V is a Lyapunov functional for problem (P) and V is continuous since $u \in C([0, \infty); L^1(\Omega))$. An application of Proposition 2.2 from Dafermos [6], that V is constant on w , i.e., for any $q \in w$

$$V(u(t; q)) = V(q) \text{ for an } t \geq 0$$

i.e.,

$$\int_{\Omega} u(t; q) = \int_{\Omega} q \tag{4.3}$$

If we choose $\psi = 1$ in (4.1), using (4.3); then

$$\int_0^t \int_{\Gamma} \beta(\varphi(q)) = 0$$

from which we conclude, as above, that

$$q \equiv 0 \text{ in } \Omega \times (0, t)$$

Thus $w = E = \{0\}$.

The proof of claim is completed and thus theorem 4.1 is proved. ■

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