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Theoretical and Numerical Studies in Magnetic Fusion

This past year has been devoted to the development of analytic and numerical kinetic theory for the edge physics problem in Tokamaks. This work is being carried out collaboratively with the Plasma Fusion Laboratory at MIT, and with the Plasma Physics Group at LLNL. A paper is being prepared for publication on the work on the Lorentz gas.

For the charged particles the general formal problem involves the solution of the system of coupled drift-kinetic equations

$$\frac{\partial f_j}{\partial t} + (u\mathbf{b} + \mathbf{v}_d) \cdot \nabla f_j(\mathbf{r}, \mu, u, t) = \oint \frac{d\phi}{2\pi} C_j \quad (1)$$

for the distribution functions f_j , and the Maxwell equations for the electromagnetic fields. Near a diverter the set of equations must be supplemented by Boltzmann equations for the uncharged species, and the situation is complicated by wall chemistry and sheaths.

The full problem is clearly beyond the ability of the current best in available analytic and numerical methods. Thus we have begun by considering simplified model kinetic theory models in order to explore phenomena and develop analytic and numerical techniques. The first effort was to use the kinetic equation of discharge theory, which while not quantitative, provides the easiest problem. Next we have completely solved a kinetic theory problem using the Lorentz gas approximation for the collision term. This has provided analytic and computational tools for use in dealing with the kinetic equation including energy diffusion. In order to deal most simply with energy diffusion we have developed a simple model collision terms C_j which conserves number, and for self collisions also conserves energy and momentum. It lends itself to rapid accurate computation. The associated solutions of the kinetic equation should be semi-quantitative and provide a useful tool for the rapid exploration of edge phenomena and the development of first approximations required for the iterative solution of the full kinetic equation.

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Discharge Theory of the Scrapeoff Ribbon

The first problem, done with R. Cohen of LLNL, was chosen because of its simplicity. It is aimed at diverters in Tokamaks of the kind indicated schematically in Fig. 1 below.

Fig. 1 Schematic Figure of the Scrapeoff Region

It employs equations for the electron distribution function $f = f_0(\mathbf{r}, \mathbf{v}, t) + f_1(\mathbf{r}, \mathbf{v}, t)$, where $f_0 \gg |f_1|$, taken from discharge theory, namely

$$\begin{aligned}
 \frac{\partial f_0}{\partial t} = & \nabla \cdot \left[\frac{v^2 \tau}{3} \left(\nabla f_0 - \frac{e}{mv} \frac{\partial f_0}{\partial v} \mathbf{E} \right) \cdot \mathbf{M} \right] \\
 & + \frac{1}{v^2} \frac{\partial}{\partial v} \left[\frac{v^3 \tau}{3} \left(\nabla f_0 - \frac{e}{mv} \frac{\partial f_0}{\partial v} \mathbf{E} \right) \cdot \mathbf{M} \cdot \left(-\frac{e}{m} \mathbf{E} \right) \right] \\
 & + \frac{m}{m'} \frac{1}{v^2} \frac{\partial}{\partial v} \left[\frac{v^3}{\tau} \left(f_0 + \frac{T'}{mv} \frac{\partial f_0}{\partial v} \right) \right] + L\{f_0\}
 \end{aligned}
 \tag{2}$$

$$f_1 = - \left(\nabla f_0 - \frac{e}{mv} \frac{\partial f_0}{\partial v} \mathbf{E} \right) \cdot \mathbf{M} \cdot \mathbf{v} \tau \quad (3)$$

where $\mathbf{M} = \mathbf{b}\mathbf{b} - \frac{\Omega\tau\mathbf{b} \times \mathbf{I}}{1 + \Omega^2\tau^2} + \frac{\mathbf{I} - \mathbf{b}\dot{\mathbf{b}}}{1 + \Omega^2\tau^2}$, $\mathbf{b} = \mathbf{B}/B$, $\Omega = \frac{eB}{mc}$, and $\tau = \frac{m^2 v^3}{4\pi N Z_{\text{effective}}^2 e^4 \ln \Lambda}$. The

formal term L_0 represents inelastic collisions. The model is quantitatively valid when collisions are predominantly with high Z ions. It provides qualitative insight into the scrapeoff electron distribution function even when not precise. For the case of steady state, strong magnetic field, negligible L_0 , and a spatially one dimensional electrostatic potential, a rough model of the scrapeoff flux tube, the equation for f_0 reduces to

$$\frac{\partial}{\partial s} \left[\frac{\tau^2}{3} \frac{\partial f_0}{\partial s} \right] + m \frac{\partial}{\partial H} \left(f_0 + T \frac{\partial f_0}{\partial H} \right) = 0 \quad (4)$$

where if Φ denotes the electrostatic potential, the energy $H = \frac{1}{2}mv^2 - e\Phi$. In order to model the effect of the electrostatic field without having to self consistently determine it, we have assumed a square well. The situation with boundary conditions is presented in the figure below.

Fig.2 Schematic Diagram of H, s space Showing Boundary Conditions

Representative solutions are given in Figs. 3, 4, and 5 below. They display the transition from the near Maxwellian distribution of the trapped particles and those near $s=0$ in the case of relatively short mean free path to the depleted untrapped electron distribution associated with the absorbing wall.

Fig. 3 Typical result for f_0

The Lorentz Gas Model

An instructive model kinetic equation which has been dealt with comprehensively and which both provides insight and leads to generalizable analytic and numerical methods is the Lorentz gas described by the collision term

$$C = \nabla_v \cdot \left[\frac{v^2 \mathbf{I} - \mathbf{v} \mathbf{v}}{\tau(v)} \cdot \nabla_v f \right] \quad (5)$$

where

$$\tau = \frac{m^2 v^3}{4\pi N Z_{\text{effective}}^2 e^4 \ln \Lambda} \quad (6)$$

We have considered in detail the case where the distribution function depends only on the cartesian space coordinate ξ such that in spherical polar coordinates $v, \arccos \mu, \phi$ in velocity space the kinetic equation

$$v\mu \frac{\partial f}{\partial \xi} = \frac{4\pi N(\xi) Z_{\text{effective}}^2 e^4 \ln \Lambda}{m^2 v^3} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial f}{\partial \mu} \right] \quad (7)$$

If we define the dimensionless quantities

$$\alpha(v) = \frac{4\pi Z_{\text{effective}}^2 e^4 \ln \Lambda \int_0^L d\xi N(\xi)}{m^2 v^4} \quad (8)$$

$$x = \frac{\int_0^\xi d\xi N(\xi)}{\int_0^L d\xi N(\xi)} \quad (9)$$

equation (7) reduces to

$$\mu \frac{\partial f}{\partial \xi} = \alpha \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial f}{\partial \mu} \right] \quad (10)$$

The domain in x, μ space and boundary conditions are shown in Fig. 4.

Fig. 4 The variable space and Boundary Conditions

The results of the research are as follows. The problem is most easily solved if $f(x,0)$ is known since then the equations in the domains $0 \leq \mu \leq 1$ and $-1 \leq \mu \leq 0$ are separately parabolic and can be dealt with numerically in a standard way. One can find $f(x,0)$ by solving an integral equation. To derive the integral equation one defines eigenfunctions via the Sturm-Liouville equation ($0 \leq x \leq 1, 0 \leq \mu \leq 1$)

$$\frac{d}{d\mu} \left[(1-\mu^2) \frac{dg_n(\mu)}{d\mu} \right] + \lambda_n \mu g_n(\mu) = 0 \quad (11)$$

subject to the conditions $g_n(1) < \infty$ and $g_n(0) = 0$. Then for one can demonstrate that $\lambda_1 = 2.6337$, $\lambda_2 = 18.5533$, $\lambda_n \sim 6.8751 \left(n + \frac{2}{3} \right)^2$ and that

$$f(\mu, x) = f(x, 0) + \int_0^1 K(\mu, v, x) [f(0, \mu) - f(0, 0)] v dv - \int_0^x f_x(t, 0) dt \int_0^1 K(\mu, v, x-t) v dv \quad (12)$$

where

$$K(\mu, v, x) = \sum_{n=0}^{\infty} g_n(\mu) g_n(v) e^{-\alpha \lambda_n x} \quad (12)$$

There is a counterpart solution for $-1 \leq \mu \leq 0$. Since the solution must be analytic at all points interior to the domain, $f_\mu(x, 0)$ must be continuous which yields the integral equation

$$\int_0^x dt F(x-t) f_x(t, 0) - \int_x^1 d\tau F(\tau-x) f_x(t, 0) = s(x), \quad 0 < x < 1 \quad (13)$$

where

$$F(x) = \int_0^1 K_\mu(0, \nu, x) \nu d\nu = \sum_{n=0}^{\infty} \frac{[g'_n(0)]^2}{\lambda_n} e^{-\alpha \lambda_n x}, \quad x > 0 \quad (14)$$

$$s(x) = \int_0^1 \{ K_\mu(0, \nu, x) [f(0, \nu) - f(0, 0)] + K_\mu(0, \nu, 1-x) [f(1, -\nu) - f(1, 0)] \} \nu d\nu \quad (15)$$

In the short mean free path limit corresponding to $\alpha \gg 1$ one can show that near the boundaries at $x = 0$ and 1 there are boundary layers of thickness λ_1/α , away from which the solution in this transport limit is given by

$$f_a(\mu, x) = A + B \left(x - \frac{\mu}{2\alpha} \right) \quad (16)$$

where to good approximation

$$B(\alpha) = - \left(\frac{1}{2} + \frac{1}{3\alpha} \right)^{-1} \left[\int_0^1 \mu \psi(\mu) d\mu + \int_{-1}^0 \mu \phi(\mu) d\mu \right] \quad (17)$$

$$A(\alpha) = \frac{B(\alpha)}{3\alpha} + 2 \int_0^1 \mu \psi(\mu) d\mu$$

In the long mean free path regime $\alpha \ll 1$ away from $\mu = 0$ one has approximately $f(x, \mu) \underset{\mu > 0}{\sim} f(0, \mu)$ and $f(x, \mu) \underset{\mu < 0}{\sim} f(1, \mu)$. Straddling the x axis there is a boundary layer of thickness $\alpha^{1/3}$ in μ which makes the distribution function pass smoothly from one limiting situation to the other. One can solve the boundary layer problem exactly. The result leads to the approximate solution for the whole x, μ space

$$f(\mu, x) = \begin{cases} \psi(\mu) + [f(0, \mu) - f(0, 0)] L(x, \alpha^{-1/3} \mu) & 0 \leq x \leq 1, 0 \leq \mu \leq 1 \\ \phi(\mu) - [f(1, \mu) - f(1, 0)] L(1-x, -\alpha^{-1/3} \mu) & 0 \leq x \leq 1, -1 \leq \mu \leq 0 \end{cases} \quad (18)$$

where

$$L(x, y) = \frac{1}{\Gamma(1/3)} \int_0^{9x/y^3} U(x - y^3\tau/9) \frac{e^{-1/\tau}}{\tau^{4/3}} d\tau \quad (19)$$

and as determined by solving an integral equation

$$U(x) = \frac{2^{5/3}\pi}{\Gamma^3(1/3)} x^{1/6} \int_0^1 (1 - xz^6)^{-5/6} dz, \quad 0 \leq x \leq 1 \quad (20)$$

It is readily seen from this result that $f_x(x, 0)$ is singular at $x = 0, 1$. This indicates that great care must be given in any numerical solution of any such kinetic problem to the neighborhood of the counterpart boundary points in order to obtain the desired accuracy and rapid convergence of the iterative or relaxation procedures customarily employed.

Case of $\psi(\mu) = 1$ and $\phi(\mu) = 0$.

$\alpha = 0.1$

Fig. 5 $f(x, \mu)$ for the case $\alpha = 0.1$, $\psi(\mu) = 1$, and $\phi(\mu) = 0$

Fig. 6 $f(x, \mu)$ for the case $\alpha = 1$, $\psi(\mu) = 1$, and $\phi(\mu) = 0$

Fig. 5 $f(x, \mu)$ for the case $\alpha = 10$, $\psi(\mu) = 1$, and $\phi(\mu) = 0$

A Model Collision Operator

The Fokker-Planck collision terms are non-linear and integro-differential. The associated numerical problems are very big. It would be desirable to have a model collision term with most of the features of the full expression in order to develop numerical methods and obtain rapid semi-quantitative results. To this end we have derived the expression for electrons

$$C_{e-e} = \nabla_{\mathbf{v}} \cdot \left\{ \frac{1}{\tau_1} \left[(\mathbf{v} - \mathbf{u})^2 \mathbf{I} - (\mathbf{v} - \mathbf{u})(\mathbf{v} - \mathbf{u}) \right] \cdot (\nabla_{\mathbf{v}} f) + \frac{(\mathbf{v} - \mathbf{u})}{\tau_2} \left[\frac{T}{m} \frac{(\mathbf{v} - \mathbf{u})}{|\mathbf{v} - \mathbf{u}|^2} \cdot (\nabla_{\mathbf{v}} f) + f \right] \right\} \\ + \nabla_{\mathbf{v}} \cdot \left\{ \frac{1}{\tau_L} \left[(\mathbf{v} - \mathbf{u})^2 \mathbf{I} - (\mathbf{v} - \mathbf{u})(\mathbf{v} - \mathbf{u}) \right] \cdot (\nabla_{\mathbf{v}} f) \right\} \quad (21)$$

where the last term on the right in (21) represents electron-proton collisions and the first electron-electron collisions. The various collision times are given by

$$\frac{1}{\tau_L} = \frac{2\pi N e^4 \ln \Lambda}{m^2 |\mathbf{v} - \mathbf{u}|^3} \quad (22)$$

$$\frac{1}{\tau_B} = \frac{2\pi N e^4 \ln \Lambda}{m^2} \frac{8\pi}{3} \left(\frac{m}{2\pi T} \right)^{3/2} \quad (23)$$

$$\tau_1 = \tau_L + \frac{m(\mathbf{v} - \mathbf{u})^2}{T} \tau_B \quad (24)$$

$$\tau_2 = \frac{\tau_L}{2} + \tau_B \quad (25)$$

The parameters T and $\mathbf{u}$ can be determined in order to yield conservation of energy and momentum under self collisions. When they are interpreted as the temperature and mean velocity the limiting forms of (21) for large and small $\mathbf{v}$ agree with the asymptotic results obtained from the full integro-differential expressions. A parallel result can be gotten for ions. For example one can adopt as a natural next step after the Lorentz gas the model problem obtained by choosing $\mathbf{u} = 0$ in (21)

$$\begin{aligned}
\frac{\partial}{\partial \xi} [\mu v^3 f(\xi, v, \mu)] = & \frac{\partial}{\partial v} \left\{ \frac{1}{\bar{\tau}} \frac{v^3}{\frac{\tau_L}{\bar{\tau}} + 1} \left[\frac{T}{mv} \frac{\partial f(\xi, v, \mu)}{\partial v} + f(z, v, x) \right] + v^2 \frac{1}{m} \frac{dV(\xi)}{d\xi} x f(\xi, v, \mu) \right\} \\
& + \frac{\partial}{\partial \mu} \left[\frac{1}{\bar{\tau}} \frac{1 - \mu^2}{\frac{\tau_L}{\bar{\tau}} + \frac{mv^2}{2T}} \frac{\partial f(\xi, v, \mu)}{\partial \mu} + \frac{1}{m} \frac{dV(\xi)}{d\xi} (1 - \mu^2) v f(\xi, v, \mu) \right]
\end{aligned}
\tag{26}$$

which incorporates energy diffusion.

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