10/10-18-94 JSD

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SAND94-1853 • UC-705

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Printed August 1994

Conservative Smoothing versus Artificial Viscosity

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Prepared by
Sandia National Laboratories
Albuquerque, New Mexico 87185 and Livermore, California 94550
for the United States Department of Energy
under Contract DE-AC04-94AL85000

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Conservative Smoothing versus Artificial Viscosity

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Abstract

This report was stimulated by some recent investigations of S.P.H. (Smoothed Particle Hydrodynamics method). Solid dynamics computations with S.P.H. show symptoms of instabilities which are not eliminated by artificial viscosities. Both analysis and experiment indicate that conservative smoothing eliminates the instabilities in S.P.H. computations which artificial viscosities cannot. Questions were raised as to whether conservative smoothing might smear solutions more than artificial viscosity. Conservative smoothing, properly used, can produce more accurate solutions than the von Neumann - Richtmyer - Landshoff artificial viscosity which has been the standard for many years. We illustrate this using the vNR scheme on a test problem with known exact solution involving a shock collision in an ideal gas. We show that the norms of the errors with conservative smoothing are significantly smaller than the norms of the errors with artificial viscosity.

† This work was conducted at Michigan Technological University.

†† This work, conducted at Sandia National Laboratories, was supported by the U. S. Department of Energy under Contract No. DE-AC04-94AL85000.

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Contents

1.0 Introduction.....	4
2.0 Artificial Viscosity.....	7
3.0 Conservative Smoothing.....	15
4.0 Artificial Viscosity as a Special Case of Conservative Smoothing	22
5.0 Analysis.....	24
6.0 Results of Numerical Experiments	32
7.0 Summary.....	43
References	44

1.0 INTRODUCTION

Almost all of the methods (e.g., finite cell methods, finite difference methods, finite element methods, et cetera) for computing multidimensional material hyperdynamics have experienced instabilities of some sort. For example, the material dynamics codes (a.k.a., hydrocodes, wave - codes) with grids fixed in the material (a.k.a., Lagrange - coordinate codes) often experience the hourglass - keystone instabilities. Although S.P.H. (the Smoothed Particle Hydrodynamics Method, see Barnes - Hernquist(1993)) does not appear to have hourglass - keystone - like instabilities, it does experience some instabilities which can be stabilized, as can the hourglass - keystone instabilities, by conservative smoothing. For an interesting discussion of the numerical instabilities encountered in finite element formulations see Tezduyar - Aliabadi - Behr - Johnson - Mittal (1993).

The S.P.H. instability was first documented in [15]. In [10] it is shown that conservative smoothing could stabilize the special case of S.P.H. where only nearest neighbors are included. Both analysis and experiment have indicated that conservative smoothing eliminates the instabilities in S.P.H. computations which artificial viscosities can not.

At the conference where we first presented these results some well known S.P.H. researchers were fearful that conservative smoothing might smear the solution much more than artificial viscosity. To address these fears, we show that conservative smoothing, properly used, can produce more accurate solutions than the von Neumann - Richtmyer - Landshoff artificial viscosity. The von Neumann - Richtmyer - Landshoff artificial viscosity has been the standard dissipative mechanism in material dynamics codes for many years. We illustrate this accuracy improvement using the vNR (von Neumann and Richtmyer) scheme on a test problem involving a shock collision in an ideal gas. In future reports we intend to illustrate this using an S.P.H. scheme and using the vNR results as a benchmark against which the S.P.H. scheme may be compared.

We present comparisons between exact solutions to the standard continuum model and the solutions produced by the vNR scheme with artificial viscosity and with conservative smoothing. The test problem used here was first used in comparing AFWL's PUFF code (vNR based) against LAXWEN (a code based on ideas of Lax, Wendroff, and Richtmyer) [9]; using this test problem and the previous results from [9] allows us to make comparisons among four different hydrocodes.

We use discrete representations of the conservation laws to simulate material dynamics on digital computers. For these discrete representations artificial viscosity was introduced by von Neumann - Richtmyer (1950) to produce physically acceptable approximations of shock waves without using explicit tracking and fitting of the shocks. Although we do know of a few exceptions (e.g., see Herrmann et al. (1970)), from numerical experiments that have accumulated over many years we can say that the vNR scheme usually exhibits the features of a good numerical method: accurate computations even in the presence of shocks, absence of noise, satisfaction of the shock - jump conditions, consistency, and stability with an appropriate timestep. Hydrocodes for shock computations which don't use shock fitting require a smoothing device that

prevents physically unrealistic results and converts kinetic energy into internal energy while increasing the entropy. Thus, the smoothing device is very important to the effectiveness of these codes.

Material dynamics codes should obey conservation laws (energy, mass, momentum, volume), the specified material law (e.g., Hooke's, ideal gas, Maxwell's, Mie - Grüneisen, et cetera), and the increasing entropy law associated with the specified material model. The conservation laws have many different representations; e.g., they may be represented in terms of integral equations, differential equations, difference equations, et cetera. However, we should not forget that the conservation laws are more fundamental than any of these representations.

The discrete analogs of the conservation laws chosen by von Neumann and Richtmyer (1950) are now presented.

Conservation of momentum:

$$(1) \quad \frac{u_j^{n+1/2} - u_j^{n-1/2}}{t^{n+1/2} - t^{n-1/2}} = - \frac{p_{j+1/2}^n - p_{j-1/2}^n}{\mu_{j+1/2} - \mu_{j-1/2}},$$

where μ is the material coordinate, t is the temporal coordinate, u is specific momentum, p is pressure, and we neglect, temporarily, q , the viscosity.

Conservation of volume:

$$(2) \quad \frac{V_{j+1/2}^{n+1} - V_{j+1/2}^n}{t^{n+1} - t^n} = \frac{u_{j+1}^{n+1/2} - u_j^{n+1/2}}{\mu_{j+1} - \mu_j},$$

where V is specific volume.

Material law: For purposes of stability analysis and phase error analysis it is convenient to introduce the *linear pV law*:

$$(3) \quad P_H(V) = p_{mc} - a_{mc}^2(V - V_{mc}),$$

which is a special case of Hooke's Law with a_{mc} being the acoustic impedance; the mc - subscript on a symbol indicates a material constant. Substitute (3) into (1) to get

$$(4) \quad \frac{u_j^{n+1/2} - u_j^{n-1/2}}{t^{n+1/2} - t^{n-1/2}} = a_{mc}^2 \frac{V_{j+1/2}^n - V_{j-1/2}^n}{\mu_{j+1/2} - \mu_{j-1/2}}.$$

If we denote the position at time t^n in the spatial reference frame of the material particle labeled μ_j by x_j^n , then we have the following vNR relations:

$$u_j^{n+1/2} = \frac{x_j^{n+1} - x_j^n}{t^{n+1} - t^n},$$

$$V_{j+1/2}^n = \frac{x_{j+1}^n - x_j^n}{\mu_{j+1} - \mu_j}.$$

If we assume uniform temporal and material increments, the last pair of equations into (4) yields

$$\frac{x_j^{n+1} - 2x_j^n + x_j^{n-1}}{\Delta t^2} = a_{mc}^2 \frac{x_{j+1}^n - 2x_j^n + x_{j-1}^n}{\Delta \mu^2},$$

or letting

$$C_{FL} = \frac{a_{mc} \Delta t}{\Delta \mu},$$

we get

$$x_j^{n+1} - 2x_j^n + x_j^{n-1} = C_{FL}^2 (x_{j+1}^n - 2x_j^n + x_{j-1}^n),$$

which is the Courant - Friedrichs - Lewy (1928) difference equation for the linear wave equation,

$$\partial^2 x / \partial t^2 = a_{mc}^2 \partial^2 x / \partial \mu^2,$$

where C_{FL} is the Courant - Friedrichs - Lewy number. As is well known, a necessary and sufficient condition for stability is

$$C_{FL} < 1.$$

We'll have more on stability analyses later; but first, let's consider artificial viscosity.

2.0 ARTIFICIAL VISCOSITY

In 1950 von Neumann and Richtmyer introduced their artificial viscosity in the following form

$$(5) \quad q = -c_2 \rho s_\Delta (\Delta_\mu u)^2, \text{ where } s_\Delta = \text{sign}(\Delta_\mu u), \text{ with } \Delta_\mu u_{j+1/2} = u_{j+1} - u_j,$$

further $\rho = 1/V$ is the mass density, furthermore $c_2 \doteq 1.0$ is a dimensionless constant called the quadratic coefficient of the artificial viscosity. Moreover, Landshoff (1955) introduced a linear term leading to

$$(6) \quad q = -\nu \Delta_\mu u, \text{ with } \nu = c_1 a + c_2 \rho |\Delta_\mu u|,$$

where $c_1 \doteq 0.1$ is a dimensionless constant called the linear coefficient of the artificial viscosity and we refer to ν simply as the coefficient of artificial viscosity. The form in (6) has become, with variations in the values of c_1 and c_2 , the standard form, more or less, for the artificial viscosity in vNR codes such as AFWL's PUFF, SNLA's WONDY and CHARTD, et cetera [11]. Also we should mention that the Rosenbluth modification (which turns the artificial viscosity off in expansion, see Richtmyer-Morton(1967)) is usually used.

The representation of the law of conservation of momentum in vNR codes is usually of the form

$$\frac{u_j^{n+1/2} - u_j^{n-1/2}}{t^{n+1/2} - t^{n-1/2}} = -\frac{\sigma_{j+1/2}^n - \sigma_{j-1/2}^n}{\mu_{j+1/2} - \mu_{j-1/2}},$$

with

$$\sigma^n = p^n + q^{n-},$$

$$q^{n-} = -\nu^{n-} \Delta_\mu u^{n-1/2},$$

$$\nu^{n-} = c_1 a^n + c_2 \rho^n |\Delta_\mu u^{n-1/2}|,$$

where the mixed time levels in ν^{n-} come about mainly as a matter of computational convenience.

Next let's consider the effect that viscosity has on the energy partition and the entropy production. The specific total energy, E , partitions into specific internal energy, e , and specific kinetic energy, $u^2/2$; i.e.,

$$E = e + u^2/2.$$

In the standard continuum model, the law of conservation of energy is represented by:

$$(7) \quad \frac{\partial E}{\partial t} = -\frac{\partial u(p+q)}{\partial \mu}.$$

The continuum evolution equation for the kinetic energy,

$$(8) \quad \frac{\partial u^2/2}{\partial t} = -\frac{u\partial(p+q)}{\partial\mu},$$

follows from the standard continuum representation for conservation of momentum:

$$(9) \quad \frac{\partial u}{\partial t} = -\frac{\partial(p+q)}{\partial\mu}.$$

Subtracting (8) from (7) yields the continuum evolution equation for the internal energy:

$$(10) \quad \frac{\partial e}{\partial t} = -(p+q)\frac{\partial u}{\partial\mu}.$$

For simplicity, assume a viscosity of the form

$$(11) \quad q = -\nu \frac{\partial u}{\partial\mu},$$

where $\nu \geq 0$ is a constant. From (10) and (11) we get

$$(12) \quad \frac{\partial e}{\partial t} = -p\frac{\partial u}{\partial\mu} + \nu \left(\frac{\partial u}{\partial\mu} \right)^2.$$

The term $\nu \left(\frac{\partial u}{\partial\mu} \right)^2 \geq 0$ acts as a heat source term which, in vNR's discrete analog of (12), can cause spikes in the internal energy at places in the material where colliding shocks temporarily cause huge velocity gradients. Apparently, some S.P.H. researchers have recently discovered this effect and started calling it 'wall heating' when a shock collision with a wall is equivalent to a symmetric shock collision. Many researchers in computational material dynamics have been aware of this effect for over a quarter of a century; e.g., see [9] and the references cited there.

From (7) we get

$$(13) \quad \frac{\partial E}{\partial t} = -\frac{\partial up}{\partial\mu} + \nu \frac{\partial^2}{\partial\mu^2} \left(\frac{u^2}{2} \right)$$

and we see a kinetic energy diffusion term, $\nu \frac{\partial^2}{\partial\mu^2} \left(\frac{u^2}{2} \right)$, produced by the artificial viscosity but, conspicuous by its absence, no internal energy diffusion term. Therefore, we see that an artificial viscosity can generate internal energy spikes that just sit there forever at the same point in the material because there is no internal energy diffusion mechanism in the vNR model. We shall illustrate this with some numerical experiments involving shock collisions and also show how to fix the problem using conservative smoothing on the internal energy which has the effect of an artificial heat conduction.

To see how artificial viscosity increases entropy let's assume an equation of state of the form

$$e = \hat{e}(V, s),$$

where s is the specific entropy. From the chain rule we get

$$\frac{\partial e}{\partial t} = \left. \frac{\partial \hat{e}}{\partial V} \right|_s \frac{\partial V}{\partial t} + \left. \frac{\partial \hat{e}}{\partial s} \right|_V \frac{\partial s}{\partial t}.$$

Then making use of the thermodynamic identities

$$\left. \frac{\partial \hat{e}}{\partial V} \right|_s = -p, \quad \text{and} \quad \left. \frac{\partial \hat{e}}{\partial s} \right|_V = T,$$

where

$$T > 0$$

is the absolute temperature, we get the following familiar equation:

$$T \frac{\partial s}{\partial t} = \frac{\partial e}{\partial t} + p \frac{\partial V}{\partial t}.$$

Next from (12) follows

$$\frac{\partial e}{\partial t} + p \frac{\partial V}{\partial t} = \nu \left(\frac{\partial u}{\partial \mu} \right)^2.$$

The last two equations yield

$$\frac{\partial s}{\partial t} = \frac{\nu}{T} \left(\frac{\partial u}{\partial \mu} \right)^2.$$

Thus, the artificial viscosity yields an entropy increase at the rate

$$\frac{\nu}{T} \left(\frac{\partial u}{\partial \mu} \right)^2 \geq 0.$$

To see the effect of the artificial viscosity on the shape of a steady shock wave we follow the procedure introduced by von Neumann - Richtmyer (1950). Consider a steady shock wave propagating from left to right with left state

$$\mathbf{U}_* = (V_*, u_*, E_*)^T,$$

right state

$$\mathbf{U}_0 = (V_0, u_0, E_0)^T,$$

and shock impedance, z , (the shock propagation rate in the material reference frame, i.e., mass engulfed by the shock per cross sectional area per second)

$$z = (u_{sh} - u_0)\rho_0,$$

where u_{sh} is the velocity of the shock in the spatial reference frame and $\rho_0 = 1/V_0$. Let's change to shock - coordinates, i.e., $\omega = \mu - zt$, where the problem reduces to a system of ODEs for the steady shock:

$$\left(\frac{\partial \mathbf{U}}{\partial t} = -\frac{\partial \mathbf{F}}{\partial \mu}\right) \rightarrow \left(z \frac{d\mathbf{U}}{d\omega} = \frac{d\mathbf{F}}{d\omega}\right),$$

where

$$\mathbf{F} = (-u, \sigma, u\sigma)^T.$$

Also note that for the original von Neumann - Richtmyer artificial viscosity we have

$$q = -c_2 \rho (\Delta\mu)^2 \frac{\partial V}{\partial t} \left| \frac{\partial V}{\partial t} \right| \rightarrow c_2 \rho (\Delta\mu)^2 z^2 \left(\frac{dV}{d\omega} \right)^2.$$

Thus, we have a two point boundary value problem with boundary conditions:

$$\omega \rightarrow -\infty \Rightarrow \mathbf{U} \rightarrow \mathbf{U}_*$$

$$\omega \rightarrow +\infty \Rightarrow \mathbf{U} \rightarrow \mathbf{U}_0$$

Integrating the steady shock system of ODEs yields

$$z\mathbf{U} - \mathbf{F} = \mathbf{K},$$

where

$$\mathbf{K} = (K_1, K_2, K_3)^T$$

are constants of integration determined by the boundary conditions; i.e.,

$$\mathbf{K} = z\mathbf{U}_* - \mathbf{F}_*$$

and

$$\mathbf{K} = z\mathbf{U}_0 - \mathbf{F}_0$$

therefore,

$$(14) \quad z(\mathbf{U}_* - \mathbf{U}_0) = \mathbf{F}_* - \mathbf{F}_0.$$

Observe that if $q_* = q_0 = 0$, then $\sigma_* = p_*$, $\sigma_0 = p_0$ and therefore, in (14) we have just recovered the shock jump conditions. Note that

$$e + \sigma V + z^2 V^2 / 2 = K_4$$

and

$$z^2 V + \sigma = K_5$$

therefore,

$$(15) \quad e = z^2 V^2 / 2 - K_5 V + K_4.$$

Consider a material law of the Mie - Grüneisen form

$$e = pV/\Gamma + f(V),$$

where Γ is the Grüneisen number. Use the K_5 equation to get to

$$(16) \quad e = (K_5 - q - z^2 V)V/\Gamma + f(V).$$

Put (15) and (16) together to get

$$qV = g(V),$$

where

$$g(V) = \gamma K_5 V - z^2 V^2 (\gamma + 1)/2 + \Gamma f(V) - K_4 \Gamma,$$

with $\gamma = \Gamma + 1$. Note that if $f(V)$ is quadratic in V , then $g(V)$ is quadratic in V . Therefore, if $f(V)$ is quadratic in V , then the boundary conditions tell us that

$$(17) \quad qV = z^2 M_G^2 (V_0 - V)(V - V_*),$$

where M_G is a dimensionless number we call the Mie - Grüneisen number,

$$M_G^2 = (\gamma + 1)/2 + m_{g2},$$

with

$$\Gamma f(V) = -z^2 (m_{g2} V^2 + m_{g1} V + m_{g0}).$$

Recall that for the original von Neumann - Richtmyer q transformed to shock coordinates we have

$$(18) \quad qV = c_2 (\Delta\mu)^2 z^2 \left(\frac{dV}{d\omega} \right)^2.$$

Put (17) and (18) together to get the steady - shock - ODE,

$$c_2 (\Delta\mu)^2 \left(\frac{dV}{d\omega} \right)^2 = M_G^2 (V_0 - V)(V - V_*),$$

which, for

$$-\pi\omega_0 \leq 2\omega \leq +\pi\omega_0,$$

has solution

$$V(\omega) = \frac{V_0 + V_*}{2} + \frac{V_0 - V_*}{2} \sin \frac{\omega}{\omega_0},$$

where

$$\omega_0 = \frac{(c_2)^{1/2}}{M_G} \Delta\mu.$$

We can also solve for $u(\omega)$, $p(\omega)$, et cetera. For example, note that the specific momentum is given by

$$u(\omega) = \frac{u_0 + u_*}{2} + \frac{u_0 - u_*}{2} \sin \frac{\omega}{\omega_0},$$

the mass density satisfies

$$\rho(\omega) = 1/V(\omega),$$

the viscosity is specified by

$$q(\omega) = c_2(\Delta\mu)^2 z^2 \rho(\omega) \left(\frac{dV(\omega)}{d\omega} \right)^2,$$

the specific energy is determined by

$$e(\omega) = (K_5 - q(\omega) - z^2 V(\omega))V(\omega)/\Gamma + f(V(\omega)), \quad E(\omega) = e(\omega) + u^2(\omega)/2,$$

where the K_5 constant is

$$K_5 = z^2 \frac{V_0 + V_*}{2} + \frac{p_0 + p_*}{2},$$

and finally the pressure profile for this Mie - Grüneisen material law is seen to be

$$p(\omega) = \Gamma(e(\omega) - f(V(\omega)))\rho(\omega).$$

For $2\omega < -\pi\omega_0$ the solution is $\mathbf{U} = \mathbf{U}_*$ and for $+\pi\omega_0 < 2\omega$ the solution is $\mathbf{U} = \mathbf{U}_0$. Thus, we see that from the left state to the right state there is a finite transition region (i.e., $-\pi\omega_0/2 \leq \omega \leq +\pi\omega_0/2$) and the width of this region, ω_{sh} , which we call the *shock width* is

$$\omega_{sh} = \frac{\pi(c_2)^{1/2}}{M_G} \Delta\mu.$$

Note that the shock width is directly proportional to the square root of c_2 , the quadratic coefficient of the artificial viscosity. Also observe that the shock width does not depend on shock strength.

If instead of a quadratic artificial viscosity, as in the original vNR, we use a linear artificial viscosity of the Landshoff (1955) form, i.e.,

$$q = -c_1 c_0 \rho \Delta \mu \frac{\partial V}{\partial t} \rightarrow c_1 c_0 \rho \Delta \mu \frac{dV}{d\omega} z$$

where $c_1 \doteq 0.1$ is the dimensionless constant we call the linear coefficient of artificial viscosity and c_0 is the sound speed in the right state, then the steady - shock - ODE becomes

$$c_1 c_0 \Delta \mu \frac{dV}{d\omega} = z M_G^2 (V_0 - V)(V - V_*).$$

If

$$k = -z M_G^2 / (c_1 c_0 \Delta \mu),$$

then the steady - shock - ODE is satisfied by

$$V(\omega) = \frac{V_* e^{k\omega V_0} + V_0 e^{k\omega V_*}}{e^{k\omega V_0} + e^{k\omega V_*}}.$$

In this equation we see that the boundary conditions are satisfied, i.e.,

$$\omega \rightarrow -\infty \Rightarrow V \rightarrow V_*$$

$$\omega \rightarrow +\infty \Rightarrow V \rightarrow V_0$$

but the shock does not have a finite transition region as in the quadratic viscosity case; i.e., the shock width is not finite in the linear viscosity case. Instead, the solution for the linear viscosity goes asymptotically to the left state as $\omega \rightarrow -\infty$ and goes asymptotically to the right state as $\omega \rightarrow +\infty$. To get an approximate measure of the region where the shock transitions most rapidly we can consider the width of the region centered on $\omega = 0$ where the change in ΔV is $(V_0 - V_*)/2$; we call this the *shock - half - transition - region*. The width of this region is approximately

$$\omega_{sh} = \frac{2c_1}{M_G^2 M_{sh}} \Delta \mu,$$

where the shock - Mach number,

$$M_{sh} = (u_* - u_0)/c_0,$$

measures the strength of the shock. We see that the stronger the shock the narrower the shock - half - transition - region. That is, ω_{sh} , the width of the shock - half - transition - region, is inversely proportional to M_{sh} for the linear artificial viscosity. Note that this contrasts with the quadratic artificial viscosity where the shock width is independent of shock strength.

Thus we see that for the linear viscosity, the thickness of the shock - half - transition - region varies with the shock strength, approaching zero as the shock strength goes to infinity and approaching infinity as the shock strength goes to zero.

Parts of the analysis above (of the structure of the shockwave with linear or quadratic viscosities) have appeared before, of course, and are scattered around in various places including: Becker (1922), von Neumann - Richtmyer (1950), Richtmyer - Morton (1967), and [11].

3.0 CONSERVATIVE SMOOTHING

We illustrate the fundamental idea in conservative smoothing using the von Neumann - Richtmyer discrete representation of conservation of volume:

$$(19) \quad V_t = u_\mu,$$

where

$$(V_t)_{j+1/2}^{n+1/2} = \frac{V_{j+1/2}^{n+1} - V_{j+1/2}^n}{t^{n+1} - t^n}$$

and

$$(u_\mu)_{j+1/2}^{n+1/2} = \frac{u_{j+1}^{n+1/2} - u_j^{n+1/2}}{\mu_{j+1} - \mu_j}.$$

For simplicity in the illustration, assume given a uniform (uniformity is not a necessity) material mesh

$$(20) \quad \mu_j = j\Delta\mu, \quad 0 \leq j \leq J$$

with boundaries

$$(21) \quad \mu_{\ell b} = \mu_0, \quad \mu_{rb} = \mu_J.$$

Remark: Although, for simplicity, we are discussing a one dimensional problem first, this seems to be a good place to observe that if we had J mass chunks, $0 \leq j \leq J-1$,

$$m_{j+1/2} = A(\mu_{j+1} - \mu_j),$$

(A is an area factor whose value we choose as unity for simplicity) with specific volumes, $V_{j+1/2}$, distributed throughout a two - or three - dimensional material, then this illustration of the fundamental idea in conservative smoothing would hold, mutatis mutandis, as well in three - or two - as it does in one - dimension(s). Caveat: Observe though, that implicit here is the assumption that a linear ordering has been introduced, (20), along with a tagging of the boundary points, (21). Further, to maintain local smoothing (i.e., to avoid 'action at a distance') suggests that there should be contiguity, in some sense, between $m_{j-1/2}$ and $m_{j+1/2}$. Furthermore, it seems that $m_{1/2}$ and $m_{J-1/2}$ should be boundary masses of the two - or three - dimensional problem.

Let's define the midpoint mesh by

$$\mu_{j+1/2} = (\mu_j + \mu_{j+1})/2, \quad \text{for } 0 \leq j \leq J-1.$$

Now let's do a conservative redefining of the $V_\bullet$ distribution on the original mesh to the $\hat{V}_\bullet$ distribution on the on the midpoint mesh by

$$\hat{V}_j = (V_{j-1/2} + V_{j+1/2})/2, \quad \text{for } 1 \leq j \leq J-1,$$

with boundary conditions

$$\hat{V}_0 = V_{1/2}, \quad \hat{V}_J = V_{J-1/2}$$

We say that this is a conservative redefining (a.k.a., conservative rezoning) because the total volume before and after is the same, i.e., the sum total volume before is given by

$$\Sigma_{V_{before}} = A \sum_{j=0}^{J-1} V_{j+1/2} (\mu_{j+1} - \mu_j)$$

and the sum total volume after is given by

$$\Sigma_{V_{after}} = A \left\{ \hat{V}_0 (\mu_{1/2} - \mu_0) + \sum_{j=1}^{J-1} \hat{V}_j (\mu_{j+1/2} - \mu_{j-1/2}) + \hat{V}_J (\mu_J - \mu_{J-1/2}) \right\}$$

and it is straightforward to verify that

$$\Sigma_{V_{before}} = \Sigma_{V_{after}}.$$

Next, we conservatively rezone the $\hat{V}_\bullet$ distribution on the midpoint mesh to the $\tilde{V}_\bullet$ distribution on the original mesh via

$$\tilde{V}_{j+1/2} = (\hat{V}_{j+1} + \hat{V}_j)/2, \text{ for } 0 \leq j \leq J-1,$$

which in terms of the original $V_\bullet$ distribution is

$$(22) \quad \tilde{V}_{j+1/2} = (V_{j+3/2} + 2V_{j+1/2} + V_{j-1/2})/4, \text{ for } 1 \leq j \leq J-2,$$

with boundary values

$$(23) \quad \tilde{V}_{1/2} = (V_{3/2} + 2V_{1/2} + V_{1/2})/4, \quad \tilde{V}_{J-1/2} = (V_{J-1/2} + 2V_{J-1/2} + V_{J-3/2})/4.$$

Note that if we define the 'phantom zone' values $V_{-1/2}$ and $V_{J+1/2}$ by

$$V_{-1/2} = V_{1/2}, \quad V_{J+1/2} = V_{J-1/2},$$

then the single recipe

$$\tilde{V}_{j+1/2} = (V_{j+3/2} + 2V_{j+1/2} + V_{j-1/2})/4, \text{ for } 0 \leq j \leq J-1$$

includes both the interior points, (22), and the boundary points, (23), recipes. Also observe that if we wish to leave the boundary values, $V_{+1/2}$ and $V_{J-1/2}$, invariant in this form of conservative smoothing, then we may accomplish this with the following recipes. For the interior values of $\tilde{V}_{j+1/2}$, $2 \leq j \leq J-3$,

$$(24) \quad \tilde{V}_{j+1/2} = (V_{j+3/2} + 2V_{j+1/2} + V_{j-1/2})/4$$

and for the 'near-boundary' values of $\tilde{V}_{j+1/2}$, $j = 1, J-2$,

$$(25) \quad \tilde{V}_{3/2} = (V_{5/2} + 3V_{3/2})/4, \quad \tilde{V}_{J-3/2} = (3V_{J-3/2} + V_{J-5/2})/4,$$

and for the boundary values of $\tilde{V}_{j+1/2}$, $j = 0, J - 1$

$$\tilde{V}_{1/2} = V_{1/2}, \quad \tilde{V}_{J-1/2} = V_{J-1/2}.$$

Note that we may write (24) as

$$\tilde{V} = V + c_s \Delta_\mu^2 V$$

where $c_s = 1/4$ and

$$(\Delta_\mu^2 V)_{j+1/2} = V_{j+3/2} - 2V_{j+1/2} + V_{j-1/2}.$$

Thus the effect of a conservative smoothing of the type here discussed on the von Neumann - Richtmyer representation of the conservation of volume, $V_t = u_\mu$, would be to transform it to

$$V_t = [ac_s \Delta \mu / C_{FL}] V_{\mu\mu} + u_\mu,$$

where

$$V_{\mu\mu} = \Delta_\mu^2 V / (\Delta \mu)^2.$$

We call this **conservative smoothing of type d** because it introduces a term, $[ac_s \Delta \mu / C_{FL}] V_{\mu\mu}$, which looks like a *diffusion* term. Consistency with the standard continuum model,

$$\partial V / \partial t = \partial u / \partial \mu,$$

would require $[ac_s \Delta \mu / C_{FL}] V_{\mu\mu}$ to go to zero as the material and temporal increments vanished.

In conservative smoothing the number of zones may increase, decrease, or stay the same. Consider the case with a fixed number, J , of zones and let $U_\bullet \in \mathbb{R}^J$ be the distribution of a quantity associated with a conservation law. Define the sum total of the quantity in the $U_\bullet$ distribution by

$$\Sigma_U = \sum_{j=1}^J U_{j-1/2} (\mu_j - \mu_{j-1}).$$

For example, if

$$U = A(1, V, u, E),$$

then

$$\Sigma_U = \text{sum total (mass, volume, momentum, energy)}.$$

Contemplate a linear transformation, $\Omega \in \mathbb{R}^{J \times J}$, of $\Delta \mu_\bullet \in \mathbb{R}^J$ and of $U_\bullet \in \mathbb{R}^J$ such that

$$\widetilde{\Delta \mu_\bullet} = \Omega \Delta \mu_\bullet,$$

or in indicial notation and in terms of the components, ω_{ij} , of Ω

$$\widetilde{\Delta\mu_{i-1/2}} = \sum_{j=1}^J \omega_{ij} \Delta\mu_{j-1/2}$$

and

$$\widetilde{U}_{i-1/2} \widetilde{\Delta\mu_{i-1/2}} = \sum_{j=1}^J \omega_{ij} U_{j-1/2} \Delta\mu_{j-1/2}$$

or in direct notation

$$\widetilde{U}_* = \mathbf{A} U_*,$$

where the components of $\mathbf{A} \in \mathbb{R}^{J \times J}$, α_{ij} , are given by

$$\alpha_{ij} = \omega_{ij} \Delta\mu_{j-1/2} / \widetilde{\Delta\mu_{i-1/2}}.$$

The linear transformation, Ω , is called a conservative smoothing operator provided the ω_{ij} satisfy the following constraints.

Conservative Smoothing Constraints: for $1 \leq i, j \leq J$,

CoSmC1 (symmetric) $\Omega^T = \Omega$

CoSmC2 (row - normalized) $\sum_{j=1}^J \omega_{ij} = 1$

CoSmC3 (positive) $\omega_{ij} \geq 0$

CoSmC4 (weak diagonal dominance) $\omega_{ii} \geq 1/2$

CoSmC5 (γ_{cs} - smoothing length) $|i - j| \geq \gamma_{cs} \Rightarrow \omega_{ij} = 0$

Remarks: Observe that if the material increments are uniform, then $\mathbf{A} = \Omega$. We now state several other fundamental facts about conservative smoothing that follow from CoSmC1 - 5. The proofs of these facts are usually rather straightforward. If they are not very straightforward or are not presented here, then the proofs may be found in [6] or [4].

Fact I. If Ω is a conservative smoothing operator and

$$\widetilde{\Sigma}_U = \sum_{j=1}^J \widetilde{U}_{j-1/2} \widetilde{\Delta\mu_{j-1/2}},$$

then

$$(26) \quad \widetilde{\Sigma}_U = \Sigma_U.$$

Remark: This is why we call Ω a conservative operator.

Fact II. If Ω is a conservative smoothing operator, then

$$\begin{aligned} \min \Delta\mu_{j-1/2} &\leq \min \widetilde{\Delta\mu}_{j-1/2} \leq \max \widetilde{\Delta\mu}_{j-1/2} \leq \max \Delta\mu_{j-1/2} \\ &\text{and} \\ \min U_{j-1/2} &\leq \min \widetilde{U}_{j-1/2} \leq \max \widetilde{U}_{j-1/2} \leq \max U_{j-1/2}. \end{aligned}$$

Remark: This is one of the reasons why we call Ω a smoothing operator.

Fact III. If $p \geq 1$, then

$$|\widetilde{U}_{i-1/2}|^p \widetilde{\Delta\mu}_{i-1/2} \leq \sum_{j=1}^J \omega_{ij} |U_{j-1/2}|^p \Delta\mu_{j-1/2}.$$

Fact IV. If $p \geq 1$, then

$$\sum_{i=1}^J |\widetilde{U}_{i-1/2}|^p \widetilde{\Delta\mu}_{i-1/2} \leq \sum_{j=1}^J |U_{j-1/2}|^p \Delta\mu_{j-1/2}.$$

Fact V. If $p \geq 1$, then, the ℓ_p - variation of the motion decreases (i.e., is nonincreasing); in other symbols

$$\sum_{i=1}^J \left| \frac{\widetilde{\Delta\mu}_x}{\Delta\mu_{i-1/2}} \right|^p \widetilde{\Delta\mu}_{i-1/2} \leq \sum_{j=1}^J \left| \frac{\Delta\mu_x}{\Delta\mu_{j-1/2}} \right|^p \Delta\mu_{j-1/2}.$$

Fact VI. If a conservative smoothing operator is applied to the distributions $\Delta\mu_\bullet$ and $U_\bullet$ for $U = (V, u, E)$, and $\tilde{E} = \tilde{e} + \tilde{u}^2/2$, then the quantities (mass, volume, momentum, energy) are conserved, the total kinetic energy decreases (i.e., is nonincreasing), and the total internal energy increases (i.e., is nondecreasing).

Fact VII. For the ideal gas law where the specific entropy, s , is related to the specific internal energy and specific volume by

$$s = s_{mc} + C_V \ln[(e/e_{mc})(V/V_{mc})^\Gamma],$$

where C_V , $\Gamma = \gamma - 1$, s_{mc} , e_{mc} , and V_{mc} are all positive material constants, conservative smoothing increases (i.e., either strictly increases or leaves constant) the total entropy.

Proof sketch: This follows because the logarithm function is monotone increasing and concave downward. Therefore,

$$\tilde{\alpha} \geq \alpha > 0 < \beta \leq \tilde{\beta}, \quad 0 \leq \theta \leq 1 \Rightarrow \theta \ln \alpha + (1 - \theta) \ln \beta \leq \ln[\theta \tilde{\alpha} + (1 - \theta) \tilde{\beta}].$$

Fact VIII. If the material law

$$e = \hat{e}(V, s)$$

satisfies the thermodynamic identities

$$\frac{\partial \hat{e}}{\partial V} = -p,$$

and

$$\frac{\partial \hat{e}}{\partial s} = T,$$

where p is the pressure and $T > 0$ is the absolute temperature, then conservative smoothing increases (i.e., either strictly increases or leaves constant) the total entropy.

Proof sketch: By the continuum chain rule we have

$$de = -pdV + Tds;$$

the discrete analog is

$$\Delta e = \frac{\Delta_V \hat{e}}{\Delta V} \Delta V + \frac{\Delta_s \hat{e}}{\Delta s} \Delta s.$$

From this we can derive

$$\bar{T} \Delta s = \Delta e + \bar{p} \Delta V.$$

It follows that

$$\bar{T} \Delta \Sigma_s = \Delta \Sigma_e + \bar{p} \Delta \Sigma_V,$$

where $\Delta \Sigma_U = \tilde{\Sigma}_U - \Sigma_U$ and then the desired result, i.e., $\Delta \Sigma_s \geq 0$, follows from $\Delta \Sigma_e \geq 0$, $\bar{T} > 0$ and $\Delta \Sigma_V = 0$.

Fact IX. Assume a uniform material increment, $\Delta\mu$, and map the material interval onto $[0, 2\pi]$, i.e., $[\mu_{\ell b}, \mu_{rb}] = [0, 2\pi]$, also let $J\Delta\mu = 2\pi$, where J is an even integer, and $\mu_j = j\Delta\mu$, $0 \leq j \leq J$; let the minimum possible wavelength be denoted $\ell_{\min} = 2\Delta\mu$ and the maximum possible wavenumber be denoted $k_{\max} = \pi/\Delta\mu$. Embed U into an even, 2π - periodic function so that

$$U_j = \sum_{k=-k_{\max}}^{k_{\max}} v_k \exp(ik\mu_j),$$

for $0 \leq j \leq J$. By the periodicity, $U_{J+j} = U_j$ for all integer j . Assume $\gamma_{cs} = 2$,

$$\tilde{U}_j = \omega_{jj-1} U_{j-1} + \omega_{jj} U_j + \omega_{jj+1} U_{j+1},$$

for $1 \leq j \leq J$ and let $\tilde{U}_{J+j} = \tilde{U}_j$ for all integer j . If the smoothing is uniform in the following sense: $\omega_{jj} = \omega$ and

$$\omega_{jj-1} = \omega_{jj+1},$$

then

$$\tilde{U}_j = \sum_{k=-k_{\max}}^{k_{\max}} \tilde{v}_k \exp(ik\mu_j),$$

where

$$\tilde{v}_k = A_{cs}(k)v_k$$

with

$$A_{cs}(k) = \omega + (1 - \omega) \cos(k\Delta\mu).$$

In particular if $\omega = 1/2$, then

$$A_{cs}(k) = [1 + \cos(k\Delta\mu)]/2$$

and

$$\tilde{v}_{\pm k_{\max}} = 0.$$

Remark: The last equation says that this particular version of conservative smoothing, i.e.,

$$\tilde{U}_j = [U_{j-1} + 2U_j + U_{j+1}]/4$$

completely kills the highest possible frequency (which is usually where most of the numerical noise is). Further, note that the amplitude factor recipe, $A_{cs}(k) = [1 + \cos(k\Delta\mu)]/2$, says that the amplitudes of the higher frequencies get damped much more than the lower frequencies with no damping at all at $k = 0$, i.e., $A_{cs}(0) = 1.0$, about seven percent damping at $k_{\max}/6$, i.e., $A_{cs}(k_{\max}/6) \doteq .93$, about fifteen percent damping at $k_{\max}/4$, i.e., $A_{cs}(k_{\max}/4) \doteq .85$, and with $A_{cs}(k)$ continuing to decrease monotonely as $|k|$ increases and finally going to zero at $|k| = k_{\max}$.

4.0 Artificial Viscosity as a Special Case of Conservative Smoothing

When conservative smoothing is done only on the momentum it is equivalent to an artificial viscosity as we now show. Consider the J zones (μ_{j-1}, μ_j) with zone boundaries μ_{j-1}, μ_j for $1 \leq j \leq J$ and for simplicity let's assume that this original mesh is uniform. Let the original momentum of zone (μ_{j-1}, μ_j) be $Mom_{j-1/2}$ and let

$$u_{j-1/2} = Mom_{j-1/2} / \Delta\mu.$$

Let's generate a new zoning by splitting each interior zone boundary, μ_j , $1 \leq j \leq J-1$ of the original mesh into two new zone boundaries $\mu_j - \phi_j \Delta\mu$ and $\mu_j + \phi_j \Delta\mu$, where

$$\phi_j \in [0, 1/2]$$

is the *zone boundary fraction* at μ_j . Note that if $\phi_j = 0$, for all j , then the new mesh is the same as the old mesh and, at the other extreme, if $\phi_j = 1/2$, for all j , then the new mesh is just the midpoint mesh. If $\phi_j \in (0, 1/2)$, then the new mesh has $2J - 1$ zones.

Conservatively define the momentum distribution on the new mesh.

The new momentum of the leftmost zone,

$$(\mu_0, \mu_1 - \phi_1 \Delta\mu),$$

is

$$Mom_{j-1/2}(1 - \phi_1).$$

For $1 \leq j \leq J-1$, the new momentum of the zone centered on μ_j ,

$$(\mu_j - \phi_j \Delta\mu, \mu_j + \phi_j \Delta\mu),$$

is

$$(Mom_{j-1/2} + Mom_{j+1/2})\phi_j.$$

For $2 \leq j \leq J$, the new momentum of the zone between μ_{j-1} and μ_j ,

$$(\mu_{j-1} + \phi_{j-1} \Delta\mu, \mu_j - \phi_j \Delta\mu),$$

is

$$Mom_{j-1/2}(1 - \phi_{j-1} - \phi_j).$$

The new momentum of the rightmost zone

$$(\mu_{J-1} + \phi_{J-1} \Delta\mu, \mu_J)$$

is

$$Mom_{J-1/2}(1 - \phi_{J-1}).$$

Now from the new mesh definitions of the momentum distribution conservatively redefine the momentum distribution back on the old mesh to get:

For the leftmost zone:

$$\widetilde{Mom}_{1/2} = Mom_{1/2} + \phi_1/2(Mom_{3/2} - Mom_{1/2}).$$

For the interior zones, $1 \leq j \leq J - 2$:

$$\widetilde{Mom}_{j+1/2} = Mom_{j+1/2} + \phi_{j+1}/2(Mom_{j+3/2} - Mom_{j+1/2}) - \phi_j/2(Mom_{j+1/2} - Mom_{j-1/2}).$$

For the rightmost zone:

$$\widetilde{Mom}_{J-1/2} = Mom_{J-1/2} - \phi_{J-1}/2(Mom_{J-1/2} - Mom_{J-3/2}).$$

Combine this conservative rezone with a timestep advance to get the following evolution equation for the momentum, $1 \leq j \leq J - 2$:

$$\begin{aligned} Mom_{j+1/2}^{n+1} = & Mom_{j+1/2}^n + \phi_{j+1}/2(Mom_{j+3/2}^n - Mom_{j+1/2}^n) \\ & - \phi_j/2(Mom_{j+1/2}^n - Mom_{j-1/2}^n) - \Delta t \Delta_\mu p_{j+1/2}^{n+1/2}. \end{aligned}$$

Compare this with the momentum evolution equation with artificial viscosity, $1 \leq j \leq J - 2$:

$$Mom_{j+1/2}^{n+1} = Mom_{j+1/2}^n - \Delta t \Delta_\mu q_{j+1/2}^n - \Delta t \Delta_\mu p_{j+1/2}^{n+1/2}.$$

Matching terms we get

$$\Delta t q_j = -\phi_j/2(Mom_{j+1/2} - Mom_{j-1/2})$$

and matching this with

$$q_j = -\nu_j a_j (u_{j+1/2} - u_{j-1/2})$$

results in the following relationship between dimensionless coefficients

$$\phi = 2\nu C_{FL}.$$

5.0 Analysis

As we observed earlier, the von Neumann - Richtmyer scheme with Hooke's Law and zero viscosity, i.e.,

$$p = p_{mc} - a^2(V - V_{mc}), \quad q = 0,$$

and uniform increments reduces to the Courant - Friedrichs - Lewy (1928) scheme,

$$(27) \quad \frac{x_j^{n+1} - 2x_j^n + x_j^{n-1}}{\Delta t^2} = a^2 \frac{x_{j+1}^n - 2x_j^n + x_{j-1}^n}{\Delta \mu^2},$$

for the linear wave equation:

$$(28) \quad \frac{\partial^2 x}{\partial t^2} = a^2 \frac{\partial^2 x}{\partial \mu^2}.$$

We can show that the solutions to (27) converge to the solutions to (28) by showing consistency and stability and appealing to the Lax - Richtmyer Equivalence Theorem [see Richtmyer - Morton (1967)]. Recall that the theorem says that, within the context of linear, well posed initial value problems, convergence is equivalent to consistency and stability.

Consistency is easy to establish via a Taylor expansion: i.e., if $f(\cdot)$ is sufficiently differentiable, then

$$\frac{f(y + \Delta y) - 2f(y) + f(y - \Delta y)}{\Delta y^2} = \frac{\partial^2 f}{\partial y^2} + O(\Delta y^2).$$

To establish stability, let

$$x_j^n = A^n B^j, \quad A = \exp(\pm i k a_k \Delta t), \quad B = \exp(i k \Delta \mu).$$

Substitute this into (27) to produce

$$A - 2 + A^{-1} = C_{FL}^2 (B - 2 + B^{-1})$$

and observe that

$$B - 2 + B^{-1} = (B^{1/2} - B^{-1/2})^2 = -4 \sin^2(k \Delta \mu / 2)$$

to get the stability polynomial

$$A^2 - 2bA + 1 = 0,$$

where

$$b = 1 - 2C_{FL}^2 \sin^2(k \Delta \mu / 2).$$

Recall that the solutions to the three level difference scheme

$$f^{n+1} - 2bf^n + cf^{n-1} = 0$$

fall into one of two cases:

case (i) distinct roots:

$$d = (b^2 - c)^{1/2} \neq 0, r_{\pm} = b \pm d, f^n = \alpha(r_+)^n + \beta(r_-)^n.$$

case (ii) equal roots:

$$b^2 = c, r = b, f^n = (\alpha + \beta n)(r)^n.$$

Absolute stability requires:

case (i) distinct roots:

$$\max |r_{\pm}|^n \leq 1 \Leftrightarrow \max |r_{\pm}| \leq 1.$$

case (ii) equal roots:

$$\max n |r|^n \leq 1 \Leftrightarrow |r| \leq \left(\frac{1}{3}\right)^{\frac{1}{3}}.$$

Consider the stability polynomial $A^2 - 2bA + 1 = 0$ in case (i) distinct roots:

$$|A| \leq 1 \Leftrightarrow C_{FL} \leq 1.$$

This follows from

Lemma 1A [7]. $b, c \in \mathbb{R}$; $r = \max |b \pm \sqrt{b^2 - c}|$.

Case(A): $1 < b^2 \Rightarrow r > 1$.

Case(B): $1 \geq b^2 \geq c \Rightarrow [r \leq 1 \Leftrightarrow 2|b| \leq c + 1]$.

Case(C): $1 \geq b^2 < c \Rightarrow [r \leq 1 \Leftrightarrow c \leq 1]$.

Moreover, this also holds when the $\leq$ signs inside the square brackets are replaced by $<$ or $=$ or $>$.

Observe that $C_{FL} < 1$ implies that the roots are distinct. It follows that the necessary and sufficient condition for absolute stability is $C_{FL} < 1$. An unstable solution for $C_{FL} = 1$ is $x_j^n = n(-1)^n(-1)^j$.

To see how far apart the solutions to the discrete and continuum linear wave equations may get we consider the phase error. It is easy to see that

$$x(\mu, t) = \alpha_k \exp[ik(\mu \pm at)]$$

are solutions of

$$\frac{\partial^2 x}{\partial t^2} = a^2 \frac{\partial^2 x}{\partial \mu^2}.$$

Similarly, the solutions to

$$\frac{x_j^{n+1} - 2x_j^n + x_j^{n-1}}{\Delta t^2} = a^2 \frac{x_{j+1}^n - 2x_j^n + x_{j-1}^n}{\Delta \mu^2}$$

may be written in the form

$$x_j^n = \alpha_k \exp[ik(\mu_j \pm a_k t^n)].$$

To see this recall

$$x_j^n = A^n B^j, \quad A = \exp(\pm ika_k \Delta t), \quad B = \exp(ik\Delta \mu)$$

and

$$A - 2 + A^{-1} = C_{FL}^2 (B - 2 + B^{-1}).$$

Note that this may be written

$$(A^{1/2} - A^{-1/2})^2 = C_{FL}^2 (B^{1/2} - B^{-1/2})^2$$

which reduces to

$$\sin(ka_k \Delta t/2) = C_{FL} \sin(k\Delta \mu/2).$$

From here it is but a few steps to the phase relationship

$$\frac{a_k}{a} = [\sin^{-1}(C_{FL} \sin(\theta_k))]/(C_{FL} \theta_k),$$

where

$$0 \leq \theta_k = k\Delta \mu/2 \leq \pi/2.$$

Hence, if $C_{FL} = 1$, then the discrete and continuum wavespeeds agree. However, if $C_{FL} < 1$, then a_k/a is strictly monotone decreasing in k and at $k_{\max}$, where $\theta_{k_{\max}} = \pi/2$, we have

$$a_{k_{\max}}/a = [\sin^{-1}(C_{FL})]/(C_{FL}\pi/2).$$

Therefore,

$$a_{k_{\max}}/a \rightarrow 2/\pi, \quad \text{as } C_{FL} \rightarrow 0.$$

Thus, we see that discrete wavespeeds for large wave numbers and small C_{FL} numbers can be as low as 64 percent of the continuum wavespeed. In other words, for small C_{FL}

numbers the discrete wavespeeds for large wave numbers can be as much as 36 percent slower than the continuum wavespeed.

If we introduce an artificial viscosity of the form

$$q = -\nu a \Delta_\mu u,$$

then we get

$$\frac{x_j^{n+1} - 2x_j^n + x_j^{n-1}}{\Delta t^2} = a^2 \frac{x_{j+1}^n - 2x_j^n + x_{j-1}^n}{\Delta \mu^2} + \nu a \frac{u_{j+1}^{n-1/2} - 2u_j^{n-1/2} + u_{j-1}^{n-1/2}}{\Delta \mu},$$

where

$$\frac{u_{j+1}^{n-1/2} - 2u_j^{n-1/2} + u_{j-1}^{n-1/2}}{\Delta \mu} = \left[\frac{x_{j+1}^n - 2x_j^n + x_{j-1}^n}{\Delta \mu} - \frac{x_{j+1}^{n-1} - 2x_j^{n-1} + x_{j-1}^{n-1}}{\Delta \mu} \right] / \Delta t.$$

The stability relation then becomes

$$A - 2 + A^{-1} = (B - 2 + B^{-1})[C_{FL}^2 + \nu C_{FL}(1 - A^{-1})]$$

or

$$A - 2 + A^{-1} + 4 \sin^2(k \Delta \mu / 2) [C_{FL}^2 + \nu C_{FL}(1 - A^{-1})] = 0$$

This yields the stability polynomial

$$A^2 - 2bA + c = 0,$$

where

$$-2b = -2 + 4[C_{FL}^2 + \nu C_{FL}] \sin^2(k \Delta \mu / 2)$$

and

$$c = 1 - 4\nu C_{FL} \sin^2(k \Delta \mu / 2).$$

By a stability analysis similar to the one above we find that the stability condition is

$$\hat{C}_{FL} < 1,$$

where

$$\hat{C}_{FL} = \frac{\hat{a} \Delta t}{\Delta \mu},$$

with

$$\hat{a} = a[\nu + \sqrt{1 + \nu^2}].$$

Before we begin the stability analysis of vNR with conservative smoothing we wish to make a point which may seem rather obvious. The seemingly 'obvious' point is that the sequencing of the computation *does* make a difference. Consider the following computational operations on the system

$$U_j^{n+1} = U_j^n + \Delta t A \left(\frac{\Delta_\mu U}{\Delta \mu} \right)_j^n.$$

$\mathcal{R}$: compute rates, i.e., compute $\frac{\Delta_\mu U}{\Delta \mu}$.

$\mathcal{S}$: smooth conservatively, i.e., compute $\widetilde{U}_j^n$.

$\mathcal{T}$: time advance, i.e., compute U_j^{n+1} .

To do an $(\mathcal{R}, \mathcal{S}, \mathcal{T})$ computation (i.e., first, compute rates; second, smooth; third, time advance) would result in the scheme

$$(29) \quad U_j^{n+1} = \widetilde{U}_j^n + \Delta t A \left(\frac{\Delta_\mu U}{\Delta \mu} \right)_j^n.$$

To do an $(\mathcal{S}, \mathcal{R}, \mathcal{T})$ computation (i.e., first, smooth; second, compute rates; third, time advance) would result in the scheme

$$(30) \quad U_j^{n+1} = \widetilde{U}_j^n + \Delta t A \left(\frac{\Delta_\mu \widetilde{U}}{\Delta \mu} \right)_j^n.$$

Note that (29) and (30) are different schemes. Although this point may seem obvious, we've learned that it is important not to overlook the obvious.

To show the stability analysis of vNR with conservative smoothing it is convenient to cast the problem in the following form:

$$(31) \quad \begin{aligned} V_{j+1/2}^{n+1} &= V_{j+1/2}^n + \Delta t \left(\frac{\Delta_\mu u}{\Delta \mu} \right)_{j+1/2}^{n+1/2} \\ u_j^{n+1/2} &= u_j^{n-1/2} + a^2 \Delta t \left(\frac{\Delta_\mu V}{\Delta \mu} \right)_j^n \end{aligned}$$

Let

$$E_k = \exp(ik\Delta\mu)$$

and make the transformation

$$V_{j+1/2}^n \rightarrow \hat{V}_k^n (E_k)^{j+1/2}, \quad u_j^{n-1/2} \rightarrow \hat{u}_k^n (E_k)^j$$

so that (31) transforms to

$$\hat{V}_k^{n+1} = \hat{V}_k^n + \frac{\Delta t}{\Delta \mu} \hat{u}_k^{n+1} (E_k^{1/2} - E_k^{-1/2})$$

$$\hat{u}_k^{n+1} = \hat{u}_k^n + a^2 \frac{\Delta t}{\Delta \mu} \hat{V}_k^n (E_k^{1/2} - E_k^{-1/2})$$

or letting

$$\alpha_k = \frac{\Delta t}{\Delta \mu} (E_k^{1/2} - E_k^{-1/2})$$

we get

$$\hat{V}_k^{n+1} = \hat{V}_k^n + \alpha_k (\hat{u}_k^n + a^2 \hat{V}_k^n \alpha_k)$$

$$\hat{u}_k^{n+1} = \hat{u}_k^n + a^2 \hat{V}_k^n \alpha_k$$

Letting $\mathbf{U}_j^n = (V_{j+1/2}^n, u_j^{n-1/2})^T$ and $\widehat{\mathbf{U}}_k^n = (\hat{V}_k^n, \hat{u}_k^n)^T$, then we may write this in matrix form as

$$\widehat{\mathbf{U}}_k^{n+1} = \mathbf{B}_k \widehat{\mathbf{U}}_k^n,$$

where

$$\mathbf{B}_k = \begin{bmatrix} 1 + a^2 \alpha_k^2 & \alpha_k \\ a^2 \alpha_k & 1 \end{bmatrix}.$$

Note that the stability polynomial for this two by two amplification matrix is

$$\det(\mathbf{B}_k - \lambda \mathbf{I}) = \lambda^2 - 2b\lambda + 1,$$

where

$$b = 1 - 2C_{FL}^2 \sin^2(k\Delta\mu/2).$$

Let

$$\mathbf{U}_j^n = (V_{j+1/2}^n, u_j^{n-1/2})^T$$

so that we may write conservative smoothing of type d as

$$(32) \quad \widetilde{\mathbf{U}}_j^n = \mathbf{U}_j^n + c_s (\Delta_\mu^2 \mathbf{U})_j^n$$

and note that the transform,

$$\mathbf{U}_j^n \rightarrow \widehat{\mathbf{U}}_k^n (E_k)^j,$$

yields

$$\widehat{\widetilde{\mathbf{U}}}_k^n = \mathbf{A}_k \widehat{\mathbf{U}}_k^n,$$

where

$$\mathbf{A}_k = \begin{bmatrix} A_{cs}(k) & 0 \\ 0 & A_{cs}(k) \end{bmatrix},$$

with

$$A_{cs}(k) = 1 - 2c_s(1 - \cos(k\Delta\mu)).$$

If we do the computational sequence in the $(\mathcal{S}, \mathcal{R}, \mathcal{T})$ order, then we have the following scheme:

$$(33) \quad \begin{aligned} V_{j+1/2}^{n+1} &= \tilde{V}_{j+1/2}^n + \Delta t \left(\frac{\Delta_\mu \tilde{u}}{\Delta\mu} \right)_{j+1/2}^{n+1/2} \\ u_j^{n+1/2} &= \tilde{u}_j^{n-1/2} + a^2 \Delta t \left(\frac{\Delta_\mu \tilde{V}}{\Delta\mu} \right)_j^n. \end{aligned}$$

Fact XI. Assume that we have a scheme with a two by two amplification matrix, $\mathbf{B}_k$, as follows

$$\widehat{\mathbf{U}}_k^{n+1} = \mathbf{B}_k \widehat{\mathbf{U}}_k^n,$$

and the stability polynomial equation is

$$\lambda^2 - 2b\lambda + c = 0.$$

If we apply conservative smoothing of type d as in (32) and do the computational sequence in the $(\mathcal{S}, \mathcal{R}, \mathcal{T})$ order, then the stability polynomial equation becomes

$$\lambda^2 - 2\tilde{b}\lambda + \tilde{c} = 0,$$

where

$$\tilde{b} = A_{cs}(k)b$$

$$\tilde{c} = A_{cs}^2(k)c.$$

Proof sketch: Without loss of generality, assume $\mathbf{B}_k$ is in Jordan canonical form and consider $\mathbf{B}_k \mathbf{A}_k$, i.e.,

$$\begin{bmatrix} \lambda_1 & \delta \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} A_{cs} & 0 \\ 0 & A_{cs} \end{bmatrix} = \begin{bmatrix} A_{cs}\lambda_1 & A_{cs}\delta \\ 0 & A_{cs}\lambda_2 \end{bmatrix}$$

where δ is zero or one and note that the characteristic polynomial is

$$(\lambda - \lambda_1 A_{cs})(\lambda - \lambda_2 A_{cs}) = \lambda^2 - 2bA_{cs}\lambda + cA_{cs}^2.$$

Q.E.D.

It follows that the stability polynomial equation for the vNR scheme with conservative smoothing of type d (in the $(\mathcal{S}, \mathcal{R}, \mathcal{T})$ order) is

$$\lambda^2 - 2\tilde{b}\lambda + \tilde{c} = 0,$$

where

$$\tilde{b} = (1 - 2C_{FL}^2 \sin^2(k\Delta\mu/2))A_{cs}(k)$$

$$\tilde{c} = A_{cs}^2(k).$$

From this we can deduce that the necessary and sufficient condition for stability for all values of c_s in the interval $[0, 1/4]$ is

$$C_{FL} < 1.$$

Contrast this with the stable timestep constraint for artificial viscosity, i.e.,

$$C_{FL}[\nu + \sqrt{1 + \nu^2}] < 1$$

and we see that conservative smoothing of type d allows a larger timestep than does artificial viscosity. This is summarized in the following proposition:

Fact XII. The vNR scheme with conservative smoothing of type d (in the $(S, \mathcal{R}, T)$ order) allows a larger stable timestep than that allowed by the von Neumann - Richtmyer - Landshoff form of the artificial viscosity.

It seems to be an under appreciated fact that: Even if the timestep satisfies the stability constraints it may still be too large for other important reasons; e.g., from the perspectives of accuracy and physical acceptability a stable timestep may be too big. A specific example: It is possible in a stable timestep for the volume to go negative. To prevent such physically unacceptable computations we introduce further timestep restrictions on the volume increase or decrease as follows:

$$\left| \frac{\Delta_t V}{V} \right| \leq \theta_V < 1,$$

where $\theta_V > 0$ is a dimensionless number called the volume growth factor. Similarly, we introduce further timestep restrictions on the internal energy increase or decrease:

$$\left| \frac{\Delta_t e}{e} \right| \leq \theta_e < 1,$$

where $\theta_e > 0$ is a dimensionless number called the internal energy growth factor.

6.0 Results of Numerical Experiments

TIMESTEP PARAMETERS: The following calculations were run with the timestep restrictions:

$$\theta_V = .15 = \theta_e.$$

These values of θ_V and θ_e were decided by numerical experimentation. The volume growth restriction controlled the time step in the calculations reported here. Specifically, the zone being most compressed by the shock controlled the timestep.

SMOOTHING RECIPES: In these calculations $\Delta\mu$ is uniform and all conservative smoothing is of the following form:

$$\tilde{U}_j = U_j + C_U^S(j+1/2)(U_{j+1} - U_j) - C_U^S(j-1/2)(U_j - U_{j-1}).$$

Specifically, for specific momentum:

$$C_u^S(j+1/2) = C_{FL} \left(c_1 + c_2 \left| \frac{\Delta_\mu u}{c} \right| \right), \quad c_1 = 0.1, \quad c_2 = 1.0.$$

For the other quantities smoothed (i.e., V , e , E) we used the following procedure. To determine where there are shocks or incipient shocks in a zone we compute $|q/p|$ and if this nondimensional number exceeds a certain threshold value, θ_σ , then we say that the zone is in shock (meaning that there is already a well established shock in the zone or that there is a shock in the process of forming in the zone). If the zone is in shock, then we already have strong conservative smoothing on the specific momentum as shown above and we do not wish to also smooth V , e , nor E across a zone in shock. Therefore, we have the following recipe which turns off smoothing on V , e , and E for zones in shock and tapers off smoothing for zones approaching shock:

$$C_U^S(j+1/2) = \begin{cases} 0 & \text{for } |q/p|_{j+1/2} \geq \theta_\sigma^U \\ C_{Ucon}^S(\cos(\pi |q/p|_{j+1/2} / \theta_\sigma^U) + 1)/2 & \text{for } |q/p|_{j+1/2} < \theta_\sigma^U \end{cases}.$$

For specific volume, specific internal energy, specific energy:

$$(C_{Vcon}^S, \theta_\sigma^V) = (.01, 79), \quad (C_{econ}^S, \theta_\sigma^e) = (.0798, 5400), \quad (C_{Econ}^S, \theta_\sigma^E) = (.002, 5400).$$

These numbers were determined with numerical experiments.

ERROR MEASURES: Let U_{excon} be the exact continuum solution and U_{dcomp} be the digital computer solution, where U may be specific volume, specific momentum, specific energy, et cetera. Let $U_{\max}$ be the maximum magnitude of U_{excon} and

$$\Delta_{err} U \equiv \sum_{j=1}^{n_{zones}} |U_{excon}(j) - U_{dcomp}(j)| / U_{\max}.$$

ERROR COMPARISONS: The $\Delta_{err}U$ numbers in the following table for the *PUFF* (a *vNR* code using $c_1 = 0.25$, $c_2 = 1.8$) and *LAXWEN* (a Lax - Wendroff - Richtmyer code, see Richtmyer - Morton (1967) and [9]) hydrocodes compared. The *vNRAV* hydrocode is the *vNR* code with artificial viscosity coefficients $c_1 = 0.1$, $c_2 = 1.0$. The *vNRCS* hydrocode is the *vNR* code with conservative smoothing as described above. It was started with an initial timestep of 3×10^{-7} seconds and it ran for 985 cycles on 200 zones. Further details on *PUFF* and *LAXWEN* and their behavior on this and other test problems may be found in [9]. Further details and a listing of the *vNRAV* and *vNRCS* hydrocodes may be found in [4].

Comparison of $\Delta_{err}U$ Values :					
Quantity	<i>PUFF</i>	<i>LAXWEN</i>	<i>vNRAV</i>	<i>vNRCS</i>	worst/best
u	2.4	3.7	1.8	1.3	2.8
ρ	7.4	8.1	4.6	1.1	7.4
e	8.8	5.4	4.1	.75	11.7
p	1.6	6.1	2.5	1.2	5.1
V	N.A.	N.A.	3.2	.99	3.2

Observe that in all cases *vNRCS*, i.e., *vNR* with conservative smoothing, gives the least error. The last column in the table above is the ratio of the worst (i.e., the largest) to the best (i.e., the least) error; this provides a quantitative measure of how much improvement conservative smoothing can make. For example, in the e row we have the worst error occurring in the *PUFF* code with $\Delta_{err}e \doteq 8.8$ and the best error occurring in the *vNRCS* code with $\Delta_{err}e \doteq .75$; the ratio of worst to best here is approximately 11.7. Thus, by this measure, the *vNRCS* code, a *vNR* code with conservative smoothing, did more than eleven times better than the *PUFF* code, a *vNR* code with artificial viscosity.

GRAPHICAL PRESENTATIONS: Now we present the results in graphical form. The next table provides a 'table of contents' for the graphs which follow.

<i>FIGURE</i>	<i>DESCRIPTION</i>
6.1 top	u_{excon} at $t = 0$
6.1 bottom	u_{excon} after collision
6.2 top	p_{excon} at $t = 0$
6.2 bottom	p_{excon} after collision
6.3 top	e_{excon} at $t = 0$
6.3 bottom	e_{excon} after collision
6.4 top	V_{excon} at $t = 0$
6.4 bottom	V_{excon} after collision
6.5 top	u_{dcomp} after collision, $vNRCS$
6.5 bottom	$\dots \dots \dots$, $vNRAV$
6.6 top	V_{dcomp} after collision, $vNRCS$
6.6 bottom	$\dots \dots \dots$, $vNRAV$
6.7 top	p_{dcomp} after collision, $vNRCS$
6.7 bottom	$\dots \dots \dots$, $vNRAV$
6.8 top	e_{dcomp} after collision, $vNRCS$
6.8 bottom	$\dots \dots \dots$, $vNRAV$

Specific Momentum (cm/sec)
(initial data and exact solution after collision)

Figure 6.1

Pressure (dynes/cm²)
(initial data and exact solution after collision)

Figure 6.2

**Specific Internal Energy (ergs/gm)
(Initial data and exact solution after collision)**

Figure 6.3

Specific Volume ($1.0 \times 10^5 \text{cm}^3/\text{gm}$)
(initial data and exact solution after collision)

Figure 6.4

**Specific Momentum (cm/sec)
(after collision)**

Figure 6.5

Specific Volume ($1.0 \times 10^5 \text{cm}^3/\text{gm}$)
(after collision)

Figure 6.6

**Pressure (dynes/cm²)
(after collision)**

Figure 6.7

Specific Internal Energy (ergs/gm)
(after collision)

Figure 6.8

7.0 Summary

The results in this report show that conservative smoothing, properly used, can produce significantly more accurate solutions than the von Neumann - Richtmyer - Landshoff artificial viscosity. This is illustrated using the vNR scheme on a test problem with known exact solution involving a shock collision in an ideal gas.

REFERENCES

- [1] J. E. Barnes, L. E. Hernquist(1993), "Computer models of colliding Galaxies", **Physics Today**, **V46**, N3, pp54-61.
- [2] R. Becker (1922), "Stosswelle und Detonation", **Z. Physik**, **V8**, p321.
- [3] R. Courant, K. Friedrichs, H. Lewy(1928), 'Über die partiellen Differenzengleichungen der mathematischen Physik', **Math. Ann.**, **V100**, p.32.
- [4] C. Guenther (1994), "Comparison of artificial viscosity and conservative smoothing for computational hydrodynamics", **Michigan Technological University, Department of Mathematical Sciences, Master's Thesis** done under the supervision of D. L. Hicks, 84p.
- [5] W. Herrmann, D. L. Hicks, E.G. Young (1970), "Attenuation of elastic-plastic stress waves", **Shockwaves and the Mechanical Properties of Solids**, eds., J. Burke, V. Weiss, Syracuse U. Press, p23-63.
- [6] D. L. Hicks (1969), "The Convergence of Numerical Solutions of Hydrodynamic Shock Problems", **Air Force Weapons Labs. Report, A.F.W.L.-T.R.-69-20**, 120p.
- [7] D. L. Hicks(1978), "Stability Analysis of WONDY (a Hydrocode based on the Artificial Viscosity Method of von Neumann and Richtmyer) for a Special Case of Maxwell's Material Law", **Mathematics of Computation**, **V32**, N144, p1123-1130.
- [8] D. L. Hicks, K. L. Kuttler, L.M.Liebrock, C. E. Southwell(1993), "Answering von Neumann's conjecture on the convergence of averages and analyzing the von Neumann - Richtmyer scheme", **Applied Mathematics and Computation**, **V55**, N2&3, p265-295.
- [9] D. L. Hicks, R. Pelzl(1968), "Comparison between a von Neumann-Richtmyer hydrocode (AFWL'S PUFF) and a Lax - Wendroff hydrocode", **Air Force Weapons Labs. Report, AFWL-TR-68-112**, 152p.
- [10] D. L. Hicks, J. W. Swegle, S. W. Attaway (1994), 'Conservative Smoothing Stabilizes Discrete - Numerical Instabilities in S.P.H. for Solid Material Dynamics', submitted.
- [11] D. L. Hicks, R. T. Walsh (1976), "Numerical and computational analysis of the partial differential equations in hydrocodes and wavecodes", **S.N.L.A. SAND-75-0448**, 141p.
- [12] R. Landshoff(1955), 'A numerical method for treating fluid flow in the presence of shocks,' **Los Alamos National Lab. Report, LA-1930**.
- [13] J. von Neumann, R. Richtmyer (1950), 'A Method for the Numerical Calculation of Hydrodynamic Shocks,' **J. Applied Physics**, **V21**, N3, p.232-237.
- [14] R. Richtmyer, K. Morton (1967), **Difference Methods for Initial Value Problems**, Interscience Publishers.
- [15] J. W. Swegle, D. L. Hicks, S. W. Attaway (1994), 'Stability Analysis of S.P.H.', submitted
- [16] T. Tezduyar, S. Aliabadi, M. Behr, A. Johnson, S. Mittal (1993), 'Parallel Finite - Element Computation of 3D Flows', **IEEE Computer**, **V26**, N10, pp27-36

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