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HYPERSINGULAR INTEGRALS AT A CORNER

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March 25, 1993

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- * This research was sponsored by the Applied Mathematical Sciences Research Program, Office of Energy Research, U.S. Department of Energy under contract DE-AC05-84OR21400 with the Martin Marietta Energy Systems, Inc.

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SEP 29 1993

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Abstract

For a smooth boundary, hypersingular integrals can be defined as a limit from the interior, the approach direction being taken, for convenience, normal to the surface. At a boundary corner, the limit process, with a necessarily non-normal approach direction, provides a valid definition of the hypersingular equation, as long as the same direction is employed for all integrations. The terms which are potentially singular in the limit are shown to cancel, provided the function approximations at the corner are consistent. The analytical formulas for the singular integrals are more complicated than for a smooth surface, but are easily obtained using symbolic computation. These techniques have been employed to accurately solve the 'L-shaped domain' potential problem considered by Jaswon and Symm.

Keywords: Hypersingular equation, hypersingular integral, corners, symbolic computation, potential theory.

1 Introduction

In many applications, a boundary integral analysis can be considerably improved, from the standpoint of simplicity and accuracy of the solution, by employing hypersingular integral equations.^{1,2} The simplest example of a hypersingular equation occurs in conjunction with the two-dimensional Laplace equation $\nabla^2 \phi = 0$. As is well known,³ the boundary integral formulation for this equation is

$$\phi(P) + \int_{\Gamma} \phi(Q) \frac{\partial}{\partial n} G(P, Q) d\Gamma(Q) = \int_{\Gamma} G(P, Q) \frac{\partial}{\partial n} \phi(Q) d\Gamma(Q), \quad (1)$$

where Γ is the boundary of the domain D and the point source potential

$$G(P, Q) = -\frac{1}{2\pi} \log \|Q - P\| \quad (2)$$

can be employed for the Green's function. For the moment, it is convenient to restrict the point P to be inside the domain D , and thus the coefficient 1 for the leading term in this equation. For $P \in D$, the integrands are not singular, and the derivative of Eq. (1) respect to P can be computed by interchanging the order of differentiation and integration. The resulting hypersingular boundary integral equation for the gradient of the potential (flux) can therefore be expressed as

$$(\nabla \phi \cdot \mathbf{D})(P) + \int_{\Gamma} \phi(Q) \nabla \frac{\partial}{\partial n} G(P, Q) \cdot \mathbf{D} d\Gamma(Q) = \int_{\Gamma} \frac{\partial}{\partial n} \phi(Q) \nabla G(P, Q) \cdot \mathbf{D} d\Gamma(Q), \quad (3)$$

where $\mathbf{D} = (D_1, D_2)$ is any specified direction vector, $\|\mathbf{D}\| = 1$.

Hypersingular equations have been primarily employed on crack surfaces,⁴ which are generally assumed to be smooth (particular applications in potential theory are Darcy flow⁵ and electroplating simulation⁶). Nevertheless, the ability to evaluate hypersingular integrals at a boundary corner point is also essential. In particular, the treatment of surface crack problems will require employing the hypersingular equation at a non-smooth boundary point. Moreover, if Dirichlet boundary conditions are supplied at a corner,⁶ or if 'interior corners' arise from subdividing the domain,⁷ then there is more than one derivative quantity (e.g., flux, traction) to be determined. By augmenting Eq. (1) with its derivative, Eq. (3), at the corner point, an equal number of unknowns and equations can be achieved.⁸

The traditional method for obtaining the desired boundary formulation for Eq. (1) is to deform the boundary around $P_0 \in \Gamma$ and define the singular integral involving

$\partial G/\partial n$ as a Cauchy Principal Value (CPV).¹ Hadamard's finite part method⁹⁻¹² and somewhat more complicated variations of the CPV regularization procedure have been employed to make sense of the hypersingular integral in Eq. (3) involving two derivatives of the Green's function (see the surveys by Krishnasamy et al.¹ and Lutz et al.¹³ for a complete discussion of this topic). An alternative method, successful for *both* equations,¹⁴⁻¹⁶ is to define the singular integrals as a limit to the boundary, $P \rightarrow P_0$. With this definition, Eq. (1) and Eq. (3) *remain valid* as they stand for $P \in \Gamma$. Note that the coefficient of $\phi(P)$ – interior boundary angle in two dimensions and interior solid angle in three dimensions – that is commonly present in writing Eq. (1) is automatically calculated within the limit process.¹⁶

The corner analysis presented earlier¹ was for three-dimensions, and involved a somewhat complicated interpolation algorithm. In addition, the discussion was further complicated by the specific choice of limit process used to evaluate the singular integrals. In this paper, the issues involved in evaluating a hypersingular integral at a corner point will be considered in the simpler context of two-dimensions. An alternative, more straightforward, limit procedure for evaluating the singular integrals is described in section 2, and analytic formulas for the singular integrals are derived. Moreover, the two dimensional setting considerably simplifies the discussion of the constraints on the boundary interpolations at the corner. These constraints arise from the analysis of the singular integrals, and it is essential that these conditions be implemented in the calculation. Another consequence of this analysis is the observation that the finite part method fails for a boundary corner. In section 3, these techniques will be used to examine the 'L-shaped domain' potential problem discussed in Jaswon and Symm's classic text.¹⁷ For this problem, traditional boundary integral methods do not yield accurate solutions in the vicinity of the corners.

2 Singular Integrals

In this section, the analytic formulas for the singular integrals in Eq. (3), at a boundary corner, will be derived. In addition to describing the limiting process, the principal goal will be to indicate the singular terms which arise at a corner point, and to discuss the interpolation constraints necessary to insure their eventual cancellation.

Denote the coordinates of the corner point P_0 by (x_0, y_0) , and assume that the integration path is a straight line segment connecting P_0 to the point $P_1 = (x_1, y_1)$.

It should be emphasized that the linear geometry is chosen solely to simplify the subsequent discussion. With some modifications, higher order curved elements can be handled by essentially the same analytic method described below.¹⁸ In parametric form, the element to be integrated over is therefore

$$Q(t) = (x_0, y_0) + t \{ (x_1, y_1) - (x_0, y_0) \} \quad (4)$$

where $t \in [0, 1]$. In two dimensions, a corner node belongs to two elements, and thus there are two singular integrations to be considered. The above assumption that the singular corner point P_0 is the initial node of the element will be true for one of these elements. However, the integration over the second element can be put into this form by reversing the element orientation. This requires only a sign change in the formulas.

Assuming $\nabla\phi$ is continuous on $D \cup \Gamma$, the limit to the boundary is computed by setting P in Eq. (3) to be $P_\epsilon = P_0 + \epsilon \mathbf{L}$, $\mathbf{L} = (L_1, L_2)$, $\|\mathbf{L}\| = 1$. As $\epsilon \rightarrow 0$, $P_\epsilon \rightarrow P_0$ along the approach direction $\mathbf{L}$, which is arbitrary as long as it is not tangent to the boundary. Either an interior or exterior limit can be employed, as both lead to the same equation. An interior limit, i.e., $\mathbf{L}$ is chosen such that $P_\epsilon \in \mathcal{D}$, has been assumed herein. Finally, note that while the complete singular integration involving P_0 is independent of the choice of $\mathbf{L}$, the integral over a particular element *does depend* upon $\mathbf{L}$. This will be reflected in the formulas developed below.

Flux Integral

Differentiating Eq. (2) with respect to P , the kernel function for the $\partial\phi/\partial n$ integral in Eq. (3) is found to be

$$\frac{\partial}{\partial \mathbf{D}} G(P, Q) = \frac{1}{2\pi} \frac{\mathbf{R} \cdot \mathbf{D}}{r^2} \quad (5)$$

where $r = \|\mathbf{R}\| = \|Q - P\|$. The integral over a parametric element is therefore

$$\frac{a}{2\pi} \int_0^1 \frac{\partial\phi}{\partial n}(t) \frac{\mathbf{R} \cdot \mathbf{D}}{r^2} dt \quad (6)$$

where the Jacobian factor a is $a^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2$. The distance function in the denominator is given by $r^2(t) = a^2 t^2 - 2a(\mathbf{T} \cdot \mathbf{L})t + \epsilon^2$, where $\mathbf{T} \equiv (T_1, T_2)$ is the unit tangent vector $(x_1 - x_0, y_1 - y_0)/a$. The boundary flux and potential are, as usual, approximated by polynomials, but only the lower order terms will

give rise to potentially singular terms in the limit $\epsilon \rightarrow 0$. For Eq. (6), only the constant term $\partial\phi/\partial n(t) = \partial\phi/\partial n(P_0)$ need be considered. Although the normal derivative $\partial\phi/\partial n(P_0)$ is, for convenience, written as a single valued function, it is important to remember that there are *two* normal derivative values at P_0 , one on each side of the corner. Eq. (6) can therefore be formulated as

$$\mathcal{I}_f = \frac{a}{2\pi} \frac{\partial\phi}{\partial n}(P_0) \lim_{\epsilon \rightarrow 0} \int_0^1 \frac{(a\mathbf{T} - \epsilon\mathbf{L}) \cdot \mathbf{D}}{a^2 t^2 - 2a \cos(\theta)t + \epsilon^2} dt, \quad (7)$$

where $\cos(\theta) = \mathbf{T} \cdot \mathbf{L}$. Appropriate formulas for evaluating this integral can be found in most tables, but, as indicated above, it is much easier to employ symbolic computation¹⁸ to carry out the integrations *and* the limit process. As will be the case in all subsequent integrations, part of the integral is finite in the limit $\epsilon \rightarrow 0$, while there is also a remaining 'potentially singular' part which must be dealt with separately. The connotation 'potentially singular' is used to emphasize that, as shown below, all of these terms cancel out, and thus the limit to the boundary will exist. Nevertheless, it is important that these terms (hereafter simply referred to as singular) be examined, and not simply thrown away, as they place constraints on the interpolations at the corner.

For the above integral, the well-behaved component is

$$\mathcal{I}_f = \frac{1}{2\pi} \frac{\partial\phi}{\partial n}(P_0) \frac{\theta(\mathbf{D} \cdot \mathbf{L}) + \cos(\theta)(\mathbf{D} \cdot \mathbf{T})(\log(a) - \theta)}{\sin(\theta)}. \quad (8)$$

As indicated above, the approach direction cannot be along $\mathbf{T}$, and thus $\sin(\theta) \neq 0$. The singular component is logarithmic,

$$\mathcal{I}_{f_s} = -\frac{1}{2\pi} \frac{\partial\phi}{\partial n}(P_0)(\mathbf{D} \cdot \mathbf{T}) \log(\epsilon). \quad (9)$$

The presence of this term is not surprising: the Cauchy Principal Value is usually invoked to handle the corresponding $\partial G/\partial n$ singular integral in Eq. (1). Note however that in the hypersingular equation, the directional derivative of the Green's function is $\partial G/\partial \mathbf{D}$, where $\mathbf{D}$ is a *fixed* direction. That is, unlike n , $\mathbf{D}$ does not change when integrating over the two elements comprising the corner geometry. Thus, as distinct from the boundary element equation Eq. (1), this $\log(\epsilon)$ singularity *does not cancel* simply by summing the two flux integral contributions from both sides of the corner. (Note that for Eq. (1), the coefficient in Eq. (9) is not the normal derivative, but $\phi(P_0)$, which is the same on both sides of the corner.) For example, let P_0 be the origin, the corner being formed by the positive x and y axes. If $\mathbf{D} = (1, 0)$, then $\mathbf{D} \cdot \mathbf{T} = 1$ when integrating over the x axis, while

$\mathbf{D} \cdot \mathbf{T} = 0$ for the y axis. Thus, the two singular contributions, Eq. (9), do not cancel.

Note that cancellation would occur if, instead of a corner, the surface were smooth at P_0 . Thus this difficulty with the singular $\log(\epsilon)$ term only arises for the hypersingular equation at a nonsmooth boundary point. Clearly, if the hypersingular equation is to have a sensible value at the corner point P_0 , then this singular term must eventually cancel with some other contribution. The only possible source for the counterbalancing term is the hypersingular integral, and thus resolving this difficulty depends upon the evaluation of this integral.

Hypersingular Integral

The kernel function for the hypersingular integral is given by

$$\frac{\partial}{\partial \mathbf{D}} \frac{\partial}{\partial \mathbf{n}} G(P, Q) = \frac{1}{2\pi} \left[\frac{\mathbf{n} \cdot \mathbf{D}}{r^2} - 2 \frac{(\mathbf{n} \cdot \mathbf{R})(\mathbf{D} \cdot \mathbf{R})}{r^4} \right] . \quad (10)$$

and thus the integral to be evaluated is

$$\frac{a}{2\pi} \int_0^1 \phi(Q(t)) \left[\frac{\mathbf{n} \cdot \mathbf{D}}{r^2} - 2 \frac{(\mathbf{n} \cdot \mathbf{R})(\mathbf{D} \cdot \mathbf{R})}{r^4} \right] dt . \quad (11)$$

In this case, the constant and linear terms from $\phi(Q(t))$ will yield potentially singular contributions. Note that

$$\frac{d\phi(Q(t))}{dt} = \nabla \phi \cdot Q'(t) = J \nabla \phi \cdot \mathbf{T} = a \nabla \phi \cdot \mathbf{T} . \quad (12)$$

where J is the Jacobian (in this case equal to a), and therefore the potential function *should* be written as

$$\phi(Q(t)) = \phi(P_0) + a \nabla \phi \cdot \mathbf{T}(P_0) t . \quad (13)$$

The integrals can be carried out directly, once again most easily with symbolic computation. The constant terms yields a finite quantity at $\epsilon = 0$,

$$\mathcal{I}_{h_0} = -\frac{1}{2\pi} \phi(P_0) \left(\mathbf{T} \wedge \mathbf{L} \frac{\cos(\theta) \mathbf{D} \cdot \mathbf{T} - \mathbf{D} \cdot \mathbf{L}}{a \sin^2(\theta)} \right) , \quad (14)$$

plus the singular ϵ^{-1} term of the form

$$S = -\frac{1}{2\pi \epsilon} \phi(P_0) \frac{\mathbf{T} \wedge \mathbf{L} \{ \mathbf{D} \cdot \mathbf{T} - \cos(\theta) (\mathbf{D} \cdot \mathbf{L}) \}}{\sin^2(\theta)} . \quad (15)$$

Figure 1: Right angle corner direction vectors.

where $\mathbf{T} \wedge \mathbf{L} \equiv T_1 L_2 - T_2 L_1$. Although not immediately apparent, $\mathcal{S}$ depends only on $\mathbf{D}$ and $\mathbf{L}$, and not $\mathbf{T}$. Thus, as expected, the ϵ^{-1} contributions from the two elements forming the corner will cancel. The details of this computation can be found in the appendix.

For the integration of the linear term in Eq. (13), the finite term is now of the form

$$\mathcal{I}_{h_1} = -\frac{1}{2\pi} \nabla \phi \cdot \mathbf{T}(P_0) [\theta \mathbf{D} \cdot \mathbf{T} + \mathbf{T} \wedge \mathbf{D} \log(a) + \sin(\theta)] , \quad (16)$$

and the singular part is once again logarithmic,

$$\mathcal{I}_{h_1} = \frac{1}{2\pi} \nabla \phi \cdot \mathbf{T}(P_0) \mathbf{T} \wedge \mathbf{D} \log(\epsilon) . \quad (17)$$

Logarithmic singularity

Note that there are in general four $\log(\epsilon)$ terms present. The coefficients of these terms relating to ϕ are all different: the two contributions from the flux integral involve the two different values of $\partial\phi/\partial n$ on either side of the corner. The same is true for the hypersingular contributions, only now the coefficients depend upon the tangential derivatives $\partial\phi/\partial \mathbf{T}$. However, these four values are necessarily related, and this relationship results in the cancellation of the singular terms.

The basic idea is again most easily illustrated using the simple example of a right

angle corner, Figure 1. As indicated in the figure, the boundary is oriented counterclockwise with an outward normal. The two elements are labeled A and B . Ignoring the $\log(\epsilon)/2\pi$ factor present in all terms, Eq. (9) implies that the flux integral over element A yields a coefficient of $\phi_x D_2$ (the x subscript denoting partial derivative). Note the sign change due to the fact that the corner point is not the initial node of the element. This term is exactly matched by the hypersingular integral over element B , Eq. (17). Similarly, the flux integral over B and the hypersingular over A both yield coefficients of $\phi_y D_1$, therefore also canceling from Eq. (3).

It is a straightforward calculation to verify that this cancellation occurs for an arbitrary corner angle. As in Figure 1, let A and B denote the two sides forming the corner P_0 , and let $\mathbf{T}^A = (T_1^A, T_2^A)$, $\mathbf{n}^A = (n_1^A, n_2^A)$ denote the 'A element' unit tangent and normal vectors at P_0 , $\phi_n(A)$ and $\phi_T(A)$ the associated directional derivatives. Similar notation is used for the corresponding B element vectors and derivatives. Taking into account the sign change needed for the A element contributions, the difference of the flux and hypersingular singular contributions, Eq. (9) and Eq. (17), is (again ignoring the common factor of $\log(\epsilon)/2\pi$)

$$(\mathbf{D} \cdot \mathbf{T}^A) \phi_n(A) - (\mathbf{D} \cdot \mathbf{T}^B) \phi_n(B) + (\mathbf{T}^A \wedge \mathbf{D}) \phi_T(A) - (\mathbf{T}^B \wedge \mathbf{D}) \phi_T(B). \quad (18)$$

As $\{\mathbf{T}^A, \mathbf{n}^A\}$ and $\{\mathbf{T}^B, \mathbf{n}^B\}$ are orthonormal coordinate systems,

$$\begin{aligned} \mathbf{n}^B &= \alpha \mathbf{n}^A + \beta \mathbf{T}^A & \mathbf{n}^A &= \alpha \mathbf{n}^B - \beta \mathbf{T}^B \\ \mathbf{T}^B &= -\beta \mathbf{n}^A + \alpha \mathbf{T}^A & \mathbf{T}^A &= \beta \mathbf{n}^B + \alpha \mathbf{T}^B \end{aligned} \quad (19)$$

where $\alpha = \mathbf{n}^A \cdot \mathbf{n}^B = \mathbf{T}^A \cdot \mathbf{T}^B$ and $\beta = \mathbf{T}^A \cdot \mathbf{n}^B = -\mathbf{n}^A \cdot \mathbf{T}^B$. It thus follows that $\{\phi_n(B), \phi_T(B)\}$ can be expressed in terms of $\{\phi_n(A), \phi_T(A)\}$,

$$\begin{aligned} \phi_n(B) &= \alpha \phi_n(A) + \beta \phi_T(A) \\ \phi_T(B) &= -\beta \phi_n(A) + \alpha \phi_T(A), \end{aligned} \quad (20)$$

and the singular term, Eq. (18), becomes

$$\begin{aligned} \phi_n(A) [\mathbf{D} \cdot \mathbf{T}^A - \alpha \mathbf{D} \cdot \mathbf{T}^B + \beta \mathbf{T}^B \wedge \mathbf{D}] + \\ \phi_T(A) [\mathbf{T}^A \wedge \mathbf{D} - \beta \mathbf{D} \cdot \mathbf{T}^B - \alpha \mathbf{T}^B \wedge \mathbf{D}]. \end{aligned} \quad (21)$$

Using Eq. (19) and observing that $\mathbf{T}^A \wedge \mathbf{D} = -\mathbf{D} \cdot \mathbf{n}^A$, it is easily seen that the coefficients of $\phi_n(A)$ and $\phi_T(A)$ are zero.

Note that the cancellation of the $\log(\epsilon)$ terms has occurred precisely because the linear term in the expansion for ϕ at P_0 , Eq. (13), was chosen consistent with the

L-Shape Domain

Figure 2: Jaswon and Symm's L-shape domain problem.

normal derivative values at the corner, Eq. (19). Typically, a boundary element approximation *does not* satisfy this requirement. In general, the tangential derivative is merely a byproduct of the chosen interpolation for ϕ , and is independent of the corner normal derivatives. This constraint is analogous to the differentiability required for evaluating hypersingular integrals on a smooth surface.¹⁹ Violation of the C^1 condition can lead to incorrect solutions, so it is not surprising that ignoring the corner constraints can be equally perilous.⁸

3 Calculations

The problem posed by Jaswon and Symm¹⁷ is a simple Dirichlet boundary value for the L-shape domain shown in Figure 2. The applied boundary condition is $\phi(x, y) = x^2 - y^2$, and thus the normal derivative to be calculated everywhere on the boundary is $(2x, -2y) \cdot n$. The boundary arcs are parallel to the axes, and thus the exact solution is constant on each boundary segment. At the boundary corners, there are two normal derivative values to be determined, and the solution vector is discontinuous through the corners (except at the origin, where both normal

Figure 3: Exact vs. calculated normal derivative for the L-shape domain.

derivatives are zero).

A straightforward boundary element formulation of this problem, one without any special consideration of the corners, will yield an inaccurate solution.¹⁷ Figure 3 displays the results of such a calculation employing 160 equally spaced nodes and an Overhauser spline approximation.²⁰ Except for the vicinity of the corners (nodes 1, 41, 61, 81, 101, 121), the boundary element solution is virtually the same as the exact solution (the solid horizontal lines), and thus only the calculated values near the corners are plotted. Note that the boundary element solution is similar to a Fourier series approximation to a discontinuous function. At the corners, the calculated value is nearly the average of the two correct values on either side of the corner, while the solution oscillates (Gibbs phenomenon) in the neighborhood of these points.

As described previously,⁸ the extra unknown at the corner can be handled by writing an additional equation at these points. The boundary integral equation Eq. (1) can be combined with Eq. (3) or, as demonstrated herein, two hypersingular equations can be employed. If N^A and N^B are the two normals at the corner, the

equations

$$\begin{aligned}\nabla\phi \cdot \mathbf{N}^A &= \int_{\Gamma} \left[\frac{\partial}{\partial n} \phi(Q) \nabla G(P, Q) - \phi(Q) \nabla \frac{\partial}{\partial n} G(P, Q) \right] \cdot \mathbf{N}^A d\Gamma \\ \nabla\phi \cdot \mathbf{N}^B &= \int_{\Gamma} \left[\frac{\partial}{\partial n} \phi(Q) \nabla G(P, Q) - \phi(Q) \nabla \frac{\partial}{\partial n} G(P, Q) \right] \cdot \mathbf{N}^B d\Gamma \quad (22)\end{aligned}$$

are independent and will effectively determine the corner values.

As indicated above, the limit direction $\mathbf{L}$ used for evaluating the singular integrals can be chosen arbitrary. As an additional check of the algorithm, each calculation was performed twice, the only difference being the value of $\mathbf{L}$. In each case, there was essentially no change in the solution. In the results reported below, $\mathbf{L}$ is along the bisector of the interior angle at the corner.

The calculations reported below employ an approximation²¹ based upon Hermite interpolation (this type of approximation for boundary integral analysis was first investigated by Watson,²² and later by Rudolphi,²³ and Rudolphi and Muci-Küchler²⁴). This algorithm incorporates the tangential derivative of the potential along the boundary, and these quantities are determined by Eq. (3) with $\mathbf{D} = \mathbf{T}$. Note that at a corner, the corner constraints specify the tangential derivatives in terms of the normal derivatives. Thus, at the corners, tangential hypersingular equations are not employed (note that these equations and Eq. (22) would be dependent, and thus the Hermite approximation forces the use of additional information at a corner). There are two advantages to the Hermite approximation for this problem. In the first place, a linear geometry can be employed in conjunction with a high order (cubic) approximation for ϕ . As indicated above, the linear element simplifies the task of evaluating the singular integrals. More importantly, as discussed above, the interpolation of the potential at the corner, specifically the resulting approximation of the tangential derivatives, must conform with normal derivatives. The tangential derivatives are explicitly included in the Hermite approach, and thus the constraints are easily incorporated into the calculation. This is accomplished by expressing a tangent vector as a linear combination of the corner normal vectors.

The primary concern herein is the treatment of the corner, so one other non-standard aspect of the calculation will only be mentioned in passing: the boundary integral equation Eq. (1) has not been used at all. Instead, the hypersingular equation is employed everywhere on the boundary. The rationale for this approach is discussed elsewhere,²⁵ where it is shown that a hypersingular formulation is particularly effective for Dirichlet boundary value problems.

Figure 4: Error in the computed normal derivative for the L-shape domain problem using the hypersingular formulation.

Figure 4 plots the error (scaled by the factor 10^{12}) in the computed normal derivative, for the same discretization employed in the first calculation, Figure 3. However, to account for the two solution values, two nodes are employed at the corners, and thus there are now 166 nodes in the model (the added double nodes at the corners are numbered 161 – 166). Note that the actual geometry and the element approximation are linear. Similarly, the exact potential and normal derivative functions on the boundary are respectively quadratic and piecewise constant. The Hermite interpolation employs a cubic polynomial for ϕ and a linear approximation for the flux, and thus neither the geometry nor the function interpolations introduce any error in the calculation. The only source of error is therefore the numerical quadrature (a 4 point Gauss rule) used for the nonsingular integrals, and it is therefore not surprising that the computed solution is essentially exact (maximum error = 0.585×10^{-12}) everywhere on the boundary. One cannot, in general, expect results this accurate. However, as desired, this calculation clearly demonstrates that the hypersingular equations at the corners are being evaluated correctly.

Figure 5: Error in the computed normal derivative for the hypersingular formulation without the corner constraint equations.

As discussed above, the constraints are included in the Hermite approximation by specifically setting the tangential derivatives equal to the appropriate linear combination of the corner normal derivatives. Note that if a simple linear approximation had been employed, the tangential derivatives would have been automatically specified by a simple finite difference formula involving the boundary value of ϕ at the corner node and a neighboring node. It is interesting to observe the consequence of abandoning the corner constraints in favor of this finite difference approximation. While the solution in the vicinity of the corners is significantly worse, Figure 5, the error is roughly the same order of magnitude as the error in the finite difference approximation to the tangential derivative. The computational results in Gray and Lutz⁸ would indicate that three dimensional problems are more sensitive to the treatment of the corner interpolation. A possible explanation is that in three dimensions corners are not, for the most part, isolated points, e.g., an edge. The effect of treating a sequence of neighboring points incorrectly is likely to be more severe.

It is conceivable that the accuracy of the calculation in Figure 4 is fortuitous, a

Figure 6: Error in the computed normal derivative for the triangular domain problem using the hypersingular formulation.

consequence of having only simple 90° corners, in which case the constraints are simple equations. The same boundary value problem was therefore examined for the triangular domain with vertices at $(0,0)$, $(-1,0)$, and $(0,4)$. The calculation was again highly accurate, Figure 6, the maximum error in this case being 0.441×10^{-8} .

4 Conclusions

As shown previously in three-dimensions,⁸ hypersingular equations can be employed at a boundary corner. Although the two singular integrals in this equation can be considered separately for a smooth surface, the individual integrals are *not finite* at a corner point. In this case, the logarithmic singularities only drop out by examining the difference of the integrals. Effecting this cancellation within a calculation places constraints on the approximation to ϕ , namely, the surface tangential derivatives implied by the interpolation function must be consistent with the corner normal derivatives.

Assuming $\nabla\phi$ is continuous on $D \cup \Gamma$, a direct approach defined by the vector $\mathbf{L}$ provides a valid 'limit to the boundary' definition of the singular integrals at a corner. Compared to a previous method,⁸ this technique is simpler and, moreover, leads to analytic formulas valid for both sides of the corner. These formulas are more complicated than the corresponding smooth surface relations, but are nevertheless easily derived using symbolic computation.

In contrast, the 'finite part' definition for the singular integrals is not a valid procedure for a corner element. For a smooth surface, finite parts is successful, being equivalent to approaching normal to the boundary.²⁶ A finite part evaluation at a corner would therefore, in effect, assume different approach directions for the two corner elements. As the integrals depend upon $\mathbf{L}$ (Eq. (8), Eq. (14)), this cannot, in general, lead to correct results. A finite part analysis which considers the difference of the hypersingular and flux integrals *over the entire neighborhood of the corner* could be successful. However, deriving appropriate integration formulas in this case is likely to be very difficult, especially in three dimensions. As the 'limit to the boundary' technique is successful in two and three dimensions, and is moreover, readily computable, there doesn't seem to be any compelling reason to pursue this program.

Appendix

In order for the ϵ^{-1} terms, Eq. (15), to cancel, the expression S must be independent of $\mathbf{T}$, and this will now be demonstrated. First, note that $\mathbf{L}$ and $\mathbf{T}$ being unit vectors implies that

$$\sin^2(\theta) = 1 - (\mathbf{T} \cdot \mathbf{L})^2 = (T_1 L_2 - T_2 L_1)^2. \quad (23)$$

If it is stipulated that θ is measured from $\mathbf{T}$ to $\mathbf{L}$, then $(T_1 L_2 - T_2 L_1) = \sin(\theta)$ and thus, ignoring constants, Eq. (15) can be written as as

$$S = \frac{\mathbf{D} \cdot \mathbf{T} - \cos(\theta)(\mathbf{D} \cdot \mathbf{L})}{\sin(\theta)}. \quad (24)$$

Define $\mathbf{D} \cdot \mathbf{T} = \cos(\psi)$, and note that $\mathbf{D} \cdot \mathbf{L}$ is therefore $\cos(\psi - \theta)$. Substituting into the above expression,

$$S = \frac{\cos(\psi) - \cos(\theta) [\cos(\psi) \cos(\theta) + \sin(\psi) \sin(\theta)]}{\sin(\theta)}$$

$$\begin{aligned}
&= \frac{\cos(\psi) - \cos(\psi) (1 - \sin^2(\theta)) - \cos(\theta) \sin(\psi) \sin(\theta)}{\sin(\theta)} \\
&= -\sin(\psi - \theta)
\end{aligned} \tag{25}$$

Thus, S only depends upon the angle between $\mathbf{D}$ and $\mathbf{L}$. As $\mathbf{D}$ and $\mathbf{L}$ are the same for integrating over both elements comprising the corner, the change of sign (required for reversing the orientation for one of the elements) produces the desired cancellation of the ϵ^{-1} terms.

Acknowledgment

This work was supported by the Applied Mathematical Sciences Subprogram, Office of Basic Energy Sciences, U.S. Department of Energy, under contract DE-AC05-84OR21400 with Martin Marietta Energy Systems Inc.

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