Dynamics of Vortices in Superconductors*

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Abstract. We study the dynamics of vortices in type-II superconductors from the point of view of time-dependent Ginzburg-Landau equations. We outline a proof of existence, uniqueness and regularity of strong solutions for these equations. We then derive reduced systems of ODEs governing the motion of the vortices in the asymptotic limit of large Ginzburg-Landau parameter.

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1. Introduction

In this paper, we report some of our results on the study of time-dependent Ginzburg-Landau (TDGL) equations with application to the modeling of superconductors. The present paper contains two topics: the well-posedness of TDGL (including existence, uniqueness and regularity) and the derivation of dynamical equations governing the evolution of vortex solutions.

The basic phenomenology of superconductivity is described by the Ginzburg-Landau free energy density [5]:

$$f = \frac{1}{2m^*} \left| \left(\frac{\hbar}{i} \nabla - \frac{e^*}{c} A \right) \varphi \right|^2 + \alpha |\varphi|^2 + \frac{\beta}{2} |\varphi|^4 + \frac{B^2}{8\pi} \quad (1.1)$$

Here m^* and e^* are the effective mass and charge of the "Cooper pairs", φ is a complex order parameter, with $|\psi|^2$ representing the density of super-electrons, B is the induced magnetic field, A is the vector potential: $B = \nabla \times A$. In the absence of fields and gradients, f is reduced to the condensation energy: $f = \alpha |\varphi|^2 + \frac{\beta}{2} |\varphi|^4$, where the temperature-dependent constants $\alpha(T)$ and $\beta(T)$ satisfy:

$$\alpha(T) \sim \alpha_0 \left(\frac{T}{T_c} - 1 \right), \quad \beta(T) > 0, \alpha_0 > 0. \quad (1.2)$$

Therefore for $T > T_c$, the minimum free energy occurs at $|\psi|^2 = 0$ (normal state), whereas for $T < T_c$, the minimum free energy occurs at $|\psi|^2 = -\alpha/\beta$ (superconducting state). T_c is the transition temperature.

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(1.1) defines two length scales: the penetration depth $\lambda(T)$ which is the characteristic length scale associated with the variation of the magnetic field, and the coherence length $\xi(T)$ which is associated with the variation of the order parameter ψ :

$$\xi^2(T) = \frac{\hbar^2}{2m^*|\alpha(T)|}, \quad \lambda^2(T) = \frac{m^*c^2}{4\pi|\varphi|^2e^*2}. \quad (1.3)$$

These two length scales are typically temperature-dependent, and diverge as $T \rightarrow T_c$. However, their ratio

$$\kappa = \frac{\lambda(T)}{\xi(T)} \quad (1.4)$$

is a roughly temperature-independent quantity and characterizes the property of the material. κ is called the Ginzburg-Landau parameter. It is known that if $\kappa < 1/\sqrt{2}$, the surface energy of a normal-superconducting interface is positive, and the material is called a type-I superconductor. If $\kappa > 1/\sqrt{2}$, then the surface energy is negative and the material is called a type-II superconductor.

Type-II superconductors exhibit an additional phase, called the mixed state, besides the usual normal and superconducting phases. This occurs when the external field H is between the lower and the upper critical fields H_{c1} and H_{c2} . In the mixed state, a superconducting bulk sample is shredded by tiny tubes of normal states where magnetic field penetrates. These tiny tubes are called vortices since there is a screening current flowing around each of the vortex.

When a transport current is applied to the sample, the vortices will move because of the various forces acting on them. From a phenomenological view point, the vortices typically experience the Lorentz force, the Magnus force and the viscous drag force. It is established that the motion of vortices causes resistance. Therefore, there is major theoretical as well as practical interest in studying the dynamics of vortices.

In this paper we derive reduced system of ODEs which govern the dynamics of the vortices in the asymptotic limit $\kappa \gg 1$. We remark that typical high temperature superconductors such as YBCO and BiSCCO have rather large values of κ (between 100 and 200). Although the Ginzburg-Landau equations have to be modified to describe the phenomenology of high T_c materials, the techniques we develop are sufficiently general that we expect similar results can also be derived for the high T_c materials, using for example the layered models [2].

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2. Preliminaries

Our starting point for the dynamics of vortices is the TDGL equation:

$$\bar{\gamma} \left(\varphi_t + i \frac{e^*}{\hbar} V \varphi \right) = - \frac{\delta \int f d^3 x}{\delta \bar{\varphi}} \quad (2.1)$$

together with Maxwell's equation

$$\frac{c}{4\pi} \nabla \times B = J. \quad (2.2)$$

Here V is the electric potential, J is the total current. The time-derivative of φ has to enter in the combination $\varphi_t + i \frac{e^*}{\hbar} V \varphi$ because of the requirement of gauge invariance: If (φ, A, V) is a solution of (2.1)-(2.2), then (φ', A', V') should also be a solution where

$$\varphi' = \varphi e^{-i \frac{e^*}{\hbar c} \eta}, \quad A' = A - \nabla \eta, \quad V' = V + \frac{1}{c} \eta_t \quad (2.3)$$

and η is an arbitrary scalar function. The total current consists of two parts: the normal part J_n which obeys Ohm's law and a super-current J_s :

$$J = J_n + J_s = \sigma_n E + e^* |\varphi|^2 v_s \quad (2.4)$$

Here σ_n is the normal state conductivity, E is the electric field, v_s is the super-electron velocity:

$$E = -\frac{1}{c} A_t - \nabla V, \quad v_s = \frac{1}{m^*} \left(\hbar \nabla \theta - \frac{e^*}{c} A \right) \quad (2.5)$$

θ is the phase of the order parameter φ .

We nondimensionalize these equations by choosing new variables: $x \rightarrow \lambda x'$, $\varphi \rightarrow \varphi_0 \varphi'$, $A \rightarrow \lambda H_0 \sqrt{2} A'$, $t \rightarrow t_0 t'$, $V \rightarrow V_0 V'$, where $\varphi_0^2 = -\alpha/\beta$, $H_0 = (-4\pi\alpha)^{1/2} \varphi_0$, $t_0 = 4\pi\sigma_n \lambda^2 / c^2$, $V_0 = \lambda^2 H_0 \sqrt{2} / (t_0 c)$. The nondimensionalized equations are (omitting the primes)

$$\begin{cases} \gamma^* \left(\frac{1}{\kappa} \varphi_t + i V \varphi \right) = \left(\frac{1}{i\kappa} \nabla - A \right)^2 \varphi + (1 - |\varphi|^2) \varphi \\ A_t + \nabla V = -\nabla \wedge \nabla \times A + |\varphi|^2 \left(\frac{\nabla \theta}{\kappa} - A \right) \end{cases} \quad (2.6)$$

where

$$\gamma^* = - \frac{\bar{\gamma} e^* V_0}{\hbar \alpha}$$

γ^* depends on the properties of the material been studied.

(2.6) holds inside the sample Ω . Naturally, we associate (2.6) with the following boundary conditions on $\partial\Omega$:

$$\left(\frac{i}{\kappa} \nabla \varphi + A \varphi \right) \cdot n = 0 \quad (2.7)$$

$$\nabla \times A \times n = H \times n \quad (2.8)$$

$$E \cdot n = 0 \quad (2.9)$$

Here n is the outnormal on $\partial\Omega$, H is the external field. (2.7) is a consequence of (2.1) in nondimensionalized variables. (2.8) is a statement that there is no surface current present at the boundary. Therefore the tangential component of the magnetic field should be continuous. We should also assign an initial data to (2.6)-(2.9):

$$\varphi = \varphi_0(x), A = A_0(x). \quad (2.10)$$

Clearly the solutions of (2.6) ~ (2.10) cannot be unique because of the free choice of gauge. It is sometimes important to fix the gauge. It turns out that one convenient way of fixing the gauge, particularly for the purpose of numerical computations, is to make the electric potential zero.

In the rescaled variables, we have as $\kappa \rightarrow +\infty$,

$$H_{c_1} \sim \frac{\ell_n \kappa}{\kappa}, \quad H_{c_2} \sim \kappa. \quad (2.11)$$

Therefore under a wide range of magnetic fields, the material is in the mixed state. Before studying the dynamics of vortices in this state, we will prove the basic well-posedness theorems for the initial-boundary value problem (2.6)-(2.10).

3. Existence, Uniqueness and Regularity

In this section we outline the argument and a priori estimates needed to establish existence, uniqueness and regularity of strong solutions for (2.6)-(2.10). The key to all of these is the observation that with the choice of gauge such that the scalar potential is zero, (2.6)-(2.9) can be written as

$$\begin{cases} A_t = -\frac{\delta F(A, \varphi)}{\delta A} \\ \frac{\gamma^* \varphi_t}{\kappa} = -\frac{\delta F(A, \varphi)}{\delta \bar{\varphi}} \end{cases} \quad (3.1)$$

where

$$F(A, \varphi) = \int \left\{ \left| \left(\frac{1}{i\kappa} \nabla - A \right) \varphi \right|^2 - |\varphi|^2 + \frac{1}{2} |\varphi|^4 + (\nabla \times A)^2 - 2(\nabla \times A) \cdot H \right\} d^3x \quad (3.2)$$

(3.1) implies that

$$\frac{d}{dt} \int F(A, \varphi) d^3x + \frac{\kappa}{\gamma^*} \int \left| \frac{\delta F}{\delta \bar{\varphi}} \right|^2 d^3x + \int \left| \frac{\delta F}{\delta A} \right|^2 d^3x = 0. \quad (3.3)$$

If $F(A_0, \varphi_0) < +\infty$, we obtain from (3.3)

$$\begin{aligned} F(A(\cdot, t), \varphi(\cdot, t)) + \int_0^t \int \left\{ \frac{\kappa}{\gamma^*} \left| \frac{\delta F}{\delta \varphi} \right|^2 + \left| \frac{\delta F}{\delta A} \right|^2 \right\} d^3x dt \\ \leq F(A_0, \varphi_0). \end{aligned} \quad (3.4)$$

This is the basic energy estimate.

Next let $\varphi = fe^{i\theta}$, $f > 0$. Using (2.6) we obtain an equation for f :

$$\begin{cases} \frac{\gamma^*}{\kappa} f_t = \frac{1}{\kappa^2} \Delta f + (1 - f^2)f - \left| \frac{\nabla \theta}{\kappa} - A \right|^2 f \\ \frac{\partial f}{\partial n} = 0 \quad \text{at } \partial\Omega \end{cases} \quad (3.5)$$

where Ω is the domain occupied by the sample. A standard application of the maximum principle gives: If $0 \leq f(x, 0) \leq 1$, then we have for $t > 0$,

$$0 \leq f(x, t) \leq 1. \quad (3.6)$$

(3.4) gives control on the L^2 -norm of $B = \nabla \times A$. To control A , we also need to estimate $\nabla \cdot A$. From (2.6), we have

$$(\nabla \cdot A)_t = -(|\varphi|^2 \nabla \cdot A + \nabla |\varphi|^2 \cdot A) - \frac{i}{2\kappa} (\bar{\varphi} \Delta \varphi - \varphi \Delta \bar{\varphi}) \quad (3.7)$$

This, together with (3.4) and (3.6), gives an additional estimate for $\nabla \cdot A$

$$\max_{0 \leq t \leq T} \int (\nabla \cdot A)^2 d^3x \leq C(T). \quad (3.8)$$

(3.4), (3.6) and (3.8) are the key a priori estimates needed to establish the existence and uniqueness of strong solutions. Since the rest of the details are more or less standard, we will omit them and refer the interested reader to the paper of Du [3] who has independently found these estimates and proved the following result:

Theorem Assume that $\varphi_0, A_0, H \in H^1(\Omega)$. Then exists a unique solution (φ, A) of (2.6)-(2.10) (with $V = 0$), such that for any $T > 0$, $\varphi \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$, $A \in L^\infty(0, T; H^1(\Omega))$, $\nabla \times A \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$.

4. Dynamics of Interacting Vortices

In this section we derive reduced systems of ODEs which govern the evolution of interacting vortices in the asymptotic limit as $\kappa \rightarrow \infty$. One difficulty in this asymptotic analysis is the scaling of the different time scales. Since φ and A have different length scales, we expect that their typical time scales can also be very different. However, the relative time scale of φ and A is a material-dependent quantity, and it is difficult to decide how it should behave as $\kappa \rightarrow +\infty$. Therefore we will take the relative time scale of φ and A as another parameter δ and study the distinguished limit as δ and $\frac{1}{\kappa} \rightarrow 0$. Here we will only study the case when $\delta = O(1)$. We refer the interested reader to [4] for details of the other distinguished limits.

In this paper we will restrict ourselves to the following situation: an infinite cylindrical sample is placed in an external field parallel to the axis of the cylinder. In this case, the vortices are columnar and we only have to consider a two-dimensional problem in the plane perpendicular to the axis of the cylinder.

The basic phenomenology is the following. We have N vortices moving in the plane. Their core sizes are of $O(\frac{1}{\kappa})$, and they are $O(1)$ distance apart. Let $\epsilon = \frac{1}{\kappa}$, we write (2.6) as

$$\gamma\delta(\epsilon\varphi_t + iV\varphi) = (-i\epsilon\nabla - A)^2\varphi + (1 - |\varphi|^2)\varphi \quad (4.1)$$

$$A_t + \nabla V = -\nabla \times \nabla \times A + |\varphi|^2(\epsilon\nabla\theta - A) \quad (4.2)$$

where $\gamma = \gamma^*/\delta$.

We will restrict ourselves to the case when $\delta = 1$. Rescale time $t = \frac{t'}{\epsilon}\gamma$ and omit the primes we get

$$\begin{aligned} \epsilon^2\varphi_t + \gamma iV\varphi &= (-i\epsilon\nabla - A)^2\varphi + (1 - |\varphi|^2)\varphi \\ \frac{\epsilon}{\gamma}A_t + \nabla V &= -\nabla \times \nabla \times A + |\varphi|^2(\epsilon\nabla\theta - A). \end{aligned} \quad ((4.3))$$

We proceed to analyze (4.3) using matched asymptotics. For simplicity we choose the zero-electric potential gauge. It can be checked easily that the procedure we follow does not depend on the gauge.

Outer solutions:

It is well-known that the flux carried by each vortex is quantized: Let us estimate the magnitude of the flux quanta. Away from the vortex core, the super-electron velocity is essentially zero. Therefore we have

$$\oint v_s \cdot d\ell = 0 \quad (4.4)$$

where the line integral is evaluated sufficiently far from the vortex core. Since $v_s = \epsilon\nabla\theta - A$, we get

$$\oint A \cdot d\ell = \epsilon \oint \nabla\theta \cdot d\ell = 2\pi n\epsilon. \quad (4.5)$$

Here $n = \frac{1}{2\pi} \oint \nabla \theta d\ell$ is the Poincare index of the vortex. It is known that if $\kappa > \frac{1}{\sqrt{2}}$, then vortices with multiple indices ($|n| > 1$) are unstable. Therefore we will assume that the Poincare index of the vortices satisfy $|n| = 1$.

Denote by $\xi_j(t)$ the location of the j -th vortex and n_j its Poincare index. The total magnetic flux

$$\int_{\Omega} H d^2x = \oint_{\partial\Omega} A \cdot d\ell = 2\pi\epsilon \sum_j n_j \quad (4.6)$$

is of $O(\epsilon)$. Based on this information, we make the following ansatz:

$$\begin{aligned} \varphi(x, t) &= \varphi_0(x, t) + \epsilon \varphi_1(x, t) + \epsilon^2 \varphi_2(x, t) + \dots \\ A(x, t) &= \epsilon a_1(x, t) + \epsilon^2 a_2(x, t) + \dots \\ H(x, t) &= \epsilon h_1(x, t) + \epsilon^2 h_2(x, t) + \dots \end{aligned} \quad ((4.7))$$

(4.7) should hold outside the cores of the vortices. Substituting (4.7) into (4.3) and collecting equal powers of ϵ , we get a hierarchy of equations. The $O(\epsilon^0)$ equation is:

$$(1 - |\varphi_0|^2)\varphi_0 = 0 \quad (4.8)$$

(4.8) implies that

$$|\varphi_0| = 1, \quad \varphi_0 = e^{i\theta_0(x, t)}. \quad (4.9)$$

The $O(\epsilon)$ equations are:

$$\text{Re}(\varphi_0 \bar{\varphi}_1) = 0 \quad (4.10)$$

$$-\nabla \times \nabla \times a_1 - a_1 = 0 \quad (4.11)$$

or

$$\Delta h_1 - h_1 = 0 \quad (4.12)$$

(4.12) holds in the punctuated region outside the vortex cores. The boundary condition for h_1 near these cores are obtained from the inner solutions.

Inner solutions:

To study the inner solution in the core of the j -th vortex, we introduce the stretched coordinates $X = \frac{x - \xi_j(t)}{\epsilon}$, and expand the variables as follows:

$$\begin{aligned} \varphi &= \Phi_0(X, t) + \epsilon \Phi_1(X, t) + \dots \\ A &= \epsilon A_1(X, t) + \epsilon^2 A_2(X, t) + \dots \\ H &= \epsilon H_1(X, t) + \epsilon^2 H_2(X, t) + \dots \end{aligned} \quad (4.13)$$

We immediately see that A_1 should be zero since otherwise it will contribute to $O(1)$ terms in H . Substituting (4.13) into (4.3) and collecting equal powers of ϵ , we get, at $O(\epsilon^0)$:

$$\Delta \Phi_0 + \Phi_0(1 - |\Phi_0|^2) = 0 \quad (4.14)$$

$$-\nabla \times \nabla \times A_2 - \frac{i}{2} [\bar{\Phi}_0 \nabla \Phi_0 - \Phi_0 \nabla \bar{\Phi}_0] = 0 \quad (4.15)$$

or

$$\Delta H_1 - \frac{i}{2} \nabla \times [\bar{\Phi}_0 \nabla \Phi_0 - \Phi_0 \nabla \bar{\Phi}_0] = 0. \quad (4.16)$$

These equations describe the leading order core structure. We look for solutions which take the form:

$$\Phi_0(X, t) = f_0(R) e^{in_j \theta}, H_1(X, t) = H_1(R, t) \quad (4.17)$$

where (R, θ) is the polar coordinate of X , $n_j = \pm 1$. (4.14) and (4.16) become:

$$f_0'' + \frac{1}{R} f_0' + \left(1 - f_0^2 - \frac{1}{R^2}\right) f_0 = 0 \quad (4.18)$$

$$H_1'' + \frac{1}{R} H_1' + n_j \frac{2f_0 f_0'}{R} = 0. \quad (4.19)$$

Since the order parameter should vanish at the center of the core, and match to the outer solution outside the core, f_0 satisfies the boundary conditions:

$$f_0(0) = 0, \quad f_0(+\infty) = 1. \quad (4.20)$$

It can be proved [1] that such solution exists and satisfies

$$1 - f_0^2(R) - \frac{1}{R^2} = O\left(\frac{1}{R^4}\right), \quad \text{for } R \gg 1. \quad (4.21)$$

Since

$$\int_{R^2} \frac{2f_0 f_0'}{R} d^2 X = 2\pi \int_0^{+\infty} 2f_0 f_0' dR = 2\pi \quad (4.22)$$

we have as $R \rightarrow +\infty$,

$$H_1(R) = -n_j \log R + C^* + o(1) \quad (4.23)$$

where C^* is a constant determined from matching to the outer solution.

We now turn to the $O(\epsilon)$ equations:

$$-\xi_j \cdot \nabla \Phi_0 = \Delta \Phi_1 + \Phi_1 (1 - 2|\Phi_0|^2) - \Phi_0^2 \bar{\Phi}_1 \quad (4.24)$$

$$-\nabla \times \nabla \times A_3 - \frac{i}{2} [\bar{\Phi}_1 \nabla \Phi_0 + \bar{\Phi}_0 \nabla \Phi_1 - (\Phi_1 \nabla \bar{\Phi}_0 + \Phi_0 \nabla \bar{\Phi}_1)] = 0. \quad (4.25)$$

Let $\Phi_1 = f_1 e^{in_j \theta}$. Then (4.24) becomes

$$\begin{aligned} & -\xi_1(f_0' \cos \theta - in_j \frac{f_0}{R} \sin \theta) - \xi_2(f_0' \sin \theta + in_j \frac{f_0}{R} \cos \theta) \\ & = \Delta f_1 + in_j \nabla f_1 \cdot \nabla \theta - \frac{1}{R^2} f_1 + (1 - 2f_0^2) f_1 - f_0^2 \bar{f}_1. \end{aligned} \quad (4.26)$$

Let $f_1 = A(R) \cos \theta + B(R) \sin \theta$ and separate real and imaginary parts:

$$A(R) = A_r(R) + iA_i(R), \quad B(R) = B_r(R) + iB_i(R),$$

we obtain the following equations for the real functions $A_r(R)$, $B_r(R)$, $A_i(R)$, and $B_i(R)$:

$$-\dot{\xi}_1 f'_0 = A''_r + \frac{1}{R} A'_r + \left(1 - 3f_0^2 - \frac{2}{R^2}\right) A_r - n_j \frac{2B_i}{R^2} \quad (4.27)$$

$$-\dot{\xi}_2 f'_0 = B''_r + \frac{1}{R} B'_r + \left(1 - 3f_0^2 - \frac{2}{R^2}\right) B_r + n_j \frac{2A_i}{R^2} \quad (4.28)$$

$$-n\dot{\xi}_2 \frac{f_0}{R} = A''_i + \frac{1}{R} A'_i + \left(1 - f_0^2 - \frac{2}{R^2}\right) A_i + n_j \frac{2B_r}{R^2} \quad (4.29)$$

$$n\dot{\xi}_1 \frac{f_0}{R} = B''_i + \frac{1}{R} B'_i + \left(1 - f_0^2 - \frac{2}{R^2}\right) B_i - n_j \frac{2A_r}{R^2}. \quad (4.30)$$

Here we have adopted a slight abuse of notation: $\xi_j = (\xi_1, \xi_2)$.

The solutions of these equations can be obtained through:

$$\begin{aligned} A_r &= \dot{\xi}_1 Z, & A_i &= \dot{\xi}_2 n_j W \\ B_r &= \dot{\xi}_2 Z, & B_i &= -\dot{\xi}_1 n_j W \end{aligned} \quad (4.31)$$

where Z and W satisfy

$$-f'_0 = Z'' + \frac{1}{R} Z' + \left(1 - 3f_0^2 - \frac{2}{R^2}\right) Z + \frac{2W}{R^2} \quad (4.32)$$

$$-\frac{f_0}{R} = W'' + \frac{1}{R} W' + \left(1 - f_0^2 - \frac{2}{R^2}\right) W + \frac{2Z}{R^2} \quad (4.33)$$

As $R \rightarrow +\infty$, the asymptotic behavior of W and Z is given by

$$W = -\frac{1}{2} R \log R + C_0 R + O(\log R) \quad (4.34)$$

$$Z = -\frac{1}{2R} \log R + O\left(\frac{1}{R}\right) \quad (4.35)$$

Here C_0 is a fixed constant which can be determined numerically from the profile of f_0 .

Turning now to (4.25), we have

$$\begin{aligned} &\bar{\Phi}_1 \nabla \Phi_0 + \bar{\Phi}_0 \nabla \Phi_1 - (\Phi_1 \nabla \bar{\Phi}_0 + \Phi_0 \nabla \bar{\Phi}_1) \\ &= (\bar{f}_1 - f_1) f'_0 \nabla R + 2in_j f_0 \nabla \theta (\bar{f}_1 + f_1) + f_0 \nabla (f_1 - \bar{f}_1) \end{aligned} \quad (4.36)$$

Therefore, we can write (4.25) as

$$\begin{aligned} \Delta H_2 + \frac{2}{R} \cos \theta &\left[f'_0 B_i + n_j f'_0 A_r + n_j f_0 A'_r \right] \\ &+ \frac{2}{R} \sin \theta \left[-f'_0 A_i + n_j f'_0 B_r + n_j f_0 B'_r \right] = 0 \end{aligned} \quad (4.37)$$

Hence we have

$$H_2 = G_1(R) \cos \theta + G_2(R) \sin \theta, \quad (4.38)$$

where G_1 and G_2 satisfy

$$G_1'' + \frac{1}{R}G_1' - \frac{1}{R^2}G_1 + \frac{2}{R}[f_0' B_i + n_j f_0' A_r + n_j f_0 A_r'] \quad (4.39)$$

$$G_2'' + \frac{1}{R}G_2' - \frac{1}{R^2}G_2 + \frac{2}{R}[-f_0' A_i + n_j f_0' B_r + n_j f_0 B_r'] = 0 \quad (4.40)$$

Using (4.31), we obtain

$$G_1 = n_j \dot{\xi}_1 G, \quad G_2 = n_j \dot{\xi}_2 G \quad (4.41)$$

where G satisfies

$$G'' + \frac{1}{R}G' - \frac{1}{R^2}G + \frac{2}{R}[-f_0' W + f_0' Z + f_0 Z'] = 0 \quad (4.42)$$

As $R \rightarrow +\infty$, we have

$$G = m_1 R + O(\log R) \quad (4.43)$$

where m_1 is a constant which can be determined from the functions f_0, W and Z .

In summary, we have obtained the following inner solution for H in the core of the j -th vortex:

$$H = \epsilon(-n_j \log R + C^*) + \epsilon^2 n_j m_1 \dot{\xi}_j \cdot X + \dots \quad (4.44)$$

Outer solution revisited:

Coming back to the outer solution, we let $r_j = |x - \xi_j(t)|$. From (4.23) we have

$$h_1(x) \rightarrow -n_j \log r_j, \quad \text{as } r_j \rightarrow 0. \quad (4.45)$$

The solution of (4.12) and (4.45) is given by

$$h_1(x) = -2\pi \sum_{j=1}^N n_j K_0(|x - \xi_j(t)|) \quad (4.46)$$

Here $K_0(r)$ is the modified Bessel function.

The asymptotic behavior of $K_0(r)$ is given by:

$$K_0(r) \sim \begin{cases} \frac{1}{2\pi} \ln r, & r \rightarrow 0 \\ -\frac{1}{2\pi} \left(\frac{\pi}{2r}\right)^{1/2} e^{-r}, & r \rightarrow +\infty \end{cases} \quad (4.47)$$

For $r_j \ll 1$, we have locally:

$$H_1(x) = -n_j \log r_j + C_1 - \left(\sum_{i \neq j} n_i \nabla K_0(|\xi_j - \xi_i|) \right) \cdot (x - \xi_j(t)) + O(r_j^2) \quad (4.48)$$

where $C_1 = \sum_{i \neq j} K_0(|\xi_j - \xi_i|)$.

Matching:

Write (4.44) as

$$H = \epsilon(-n_j \log r_j + n_j \log \epsilon + C^* + n_j m_1 \dot{\xi}_j \cdot (x - \xi_j(t))) + \dots \quad (4.49)$$

Matching of (4.48) and (4.49) gives:

$$C^* = C_1 - n_j \log \epsilon \quad (4.50)$$

$$n_j m_1 \dot{\xi}_j = \sum_{i \neq j} n_i \nabla K_0(|\xi_j - \xi_i|) \quad (4.51)$$

Ideally we should have included an order $\epsilon \log \epsilon$ term in the inner expansion. But this is only a matter of formality.

(4.51) is the equation we are looking for. We can rewrite it as

$$m_1 \dot{\xi}_j = -\nabla_{\xi_j} \mathcal{H}(\xi_1, \dots, \xi_N) \quad (4.52)$$

where

$$\mathcal{H}(\xi_1, \dots, \xi_N) = \sum_{i \neq j} n_i n_j K_0(|\xi_i - \xi_j|) \quad (4.53)$$

An immediate consequence is that the interaction between a pair of vortices is repulsive if their indices have the same sign, and attractive if their indices have different sign.

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